UNIQUENESS AND VALUE DISTRIBUTION FOR q-SHIFT DIFFERENCE POLYNOMIALS

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Abstract
We investigate the zero distribution of q-shift difference polynomials of entire and meromorphic functions with zero order and obtain some results that extend previous results of Liu et al. [18]

1. Introduction and Main Results
Let $f(z)$ and $g(z)$ be two non constant meromorphic functions in the complex plane. By $S(r, f)$, we denote any quantity satisfying $S(r, f) = o(T(r, f))$, as $r \to \infty$, possibly outside a set of $r$ with finite linear measure. The meromorphic function $\alpha$ is called a small function of $f(z)$, if $T(r, \alpha) = S(r, f)$. If $f(z) - \alpha$ and $g(z) - \alpha$ have same zeros, counting multiplicity (ignoring multiplicity), then we say that $f(z)$ and $g(z)$ share the

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small function $\alpha$ CM(IM). (see, e.g., [1, 2]) The logarithmic density of a set $F_n$ is defined as follows:

$$\limsup_{r \to \infty} \frac{1}{\log r} \int_{[1, r] \cap F_n} \frac{1}{t} dt$$

In recent years, there has been an increasing interest in studying the uniqueness problems related to meromorphic functions and their shifts or difference operators (see, e.g., [3-15]). Our aim in this article is to investigate the value distribution for q-shift polynomials of transcendental meromorphic and entire functions with zero order. Liu et al. [13] considered uniqueness of difference polynomials of meromorphic functions, corresponding to uniqueness theorems of meromorphic functions sharing values (see, e.g., [16]) and obtained the following results.

**Theorem A:** Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions with finite order. Suppose that $c$ is a non-zero complex constant and $n$ is an integer. If $n \geq 14$ and $f^n(z) f(z + c)$ and $g^n(z) g(z + c)$ share 1 CM, then $f(z) \equiv t g(z)$ or $f(z) g(z) = t$, where $t^{n+1} = 1$.

**Theorem B:** Under the conditions of Theorem A, if $n \geq 26$ and $f^n(z) f(z + c)$ and $g^n(z) g(z + c)$ share 1 IM, then $f(z) \equiv t g(z)$ or $f(z) g(z) = t$, where $t^{n+1} = 1$.

Recently, Liu et al. [18], considered the case of q-shift difference polynomials and extended the Theorem A as follows:

**Theorem C:** Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions with $\rho(f) = \rho(g) = 0$. Suppose that $q$ and $c$ are two non-zero complex constants and $n$ is an integer. If $n \geq 14$ and $f^n(z) f(qz + c)$ and $g^n(z) g(qz + c)$ share 1 CM, then $f(z) \equiv t g(z)$ or $f(z) g(z) = t$, where $t^{n+1} = 1$.

**Theorem D:** Under the conditions of Theorem C, if $n \geq 26$ and $f^n(z) f(qz + c)$ and $g^n(z) g(qz + c)$ share 1 IM, then $f(z) \equiv t g(z)$ or $f(z) g(z) = t$, where $t^{n+1} = 1$.

**Theorem E:** Let $f(z)$ and $g(z)$ be two transcendental entire functions with $\rho(f) = \rho(g) = 0$, let $q$ and $c$ be two non-zero complex constants, let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$ be a non-zero polynomial, where $a_n(\neq 0)$, $a_{n-1}$, ..., $a_0$, are complex constants and $k$ denotes the number of the distinct zero of $P(z)$. If $n > 2k+1$ and $P(f(z)) f(qz + c)$ and $P(g(z)) g(qz + c)$ share 1 CM, then one of the following results holds:
(1) $f(z) \equiv tg(z)$ for a constant $t$ such that $t^d = 1$, where $d = \text{GCD}\{\lambda_0, \lambda_1, ..., \lambda_n\}$ and

$$\lambda_j = \begin{cases} n + 1, & a_j = 0, \\ j + 1, & a_j \neq 0, \end{cases} \quad j = 0, 1, ..., n;$$

(2) $f(z)$ and $g(z)$ satisfy the algebraic equation $R(f(z), g(z)) = 0$, where

$$R(w_1, w_2) = P(w_1)w_1(qz + c) - P(w_2)w_2(qz + c)$$

In this paper, we deal with value distribution for $q$-shift difference polynomials of transcendental meromorphic and entire functions of the form $f^n(z)P_m(f(qz + c))f'(z)$ in Theorems C, D, E and prove the following theorems:

**Theorem 1:** Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions with $\rho(f) = \rho(g) = 0$. Let $q$ and $c$ be two non-zero complex constants, $n$ an integer and $P_m(z) = a_m z^m + a_{m-1} z^{m-1} + ... + a_1 z + a_0$. If $n \geq 5m + 19$ and $f^n(z)P_m(f(qz + c))f'(z)$ and $g^n(z)P_m(g(qz + c))g'(z)$ share 1 CM, then $f(z) \equiv tg(z)$ or $f(z)g(z) = t$, where $t^d = 1, d = \text{GCD}(n + m + 1, n + m, ..., n + m + 1 - i, ..., n + 1), a_{m-i} \neq 0$, for some $i = 0, 1, ..., m$.

**Theorem 2:** Under the conditions of Theorem 1, if $n \geq 11m + 31$, $f^n(z)P_m(f(qz + c))f'(z)$ and $g^n(z)P_m(g(qz + c))g'(z)$ share 1 IM, then conclusion of Theorem 1 still holds.

**Theorem 3:** Let $f(z)$ and $g(z)$ be two transcendental entire functions with $\rho(f) = \rho(g) = 0$, $q$ and $c$ are two non-zero complex constants and $k$ denote the number of distinct zeros of $P_m(z)$. If $m > n + 2k + 4$, $f^n(z)P_m(f(qz + c))f'(z)$ and $g^n(z)P_m(g(qz + c))g'(z)$ share 1 CM, then one of the following results holds:

1. $f(z) \equiv tg(z)$ for a constant $t$ such that $t^d = 1$, where

$$d = \text{GCD}(n + m + 1, n + m, ..., n + m + 1 - i, ..., n + 1).$$

$a_{m-i} \neq 0$, for some $i = 0, 1, ..., m$.

2. $f(z)$ and $g(z)$ satisfy the algebraic equation $R(f, g) \equiv 0$ where

$$R(w_1, w_2) = \begin{cases} w_1^{n+1} \left[ \frac{a_m w_1^m}{n + m + 1} + \frac{a_{m-1} w_1^{m-1}}{n + m} + ... + \frac{a_0}{n + 1} \right] \\ -w_2^{n+1} \left[ \frac{a_m w_2^m}{n + m + 1} + \frac{a_{m-1} w_2^{m-1}}{n + m} + ... + \frac{a_0}{n + 1} \right] \end{cases}$$
2. Preliminary Lemmas
The following lemma is a $q$-difference analogue of the logarithmic derivative lemma.

**Lemma 2.1 (see [14])** : Let $f(z)$ be a meromorphic function of zero order, let $c$ and $q$ be two non-zero complex numbers, then

$$m \left( r, \frac{f(qz + c)}{f(z)} \right) = S(r, f)$$
on a set of logarithmic density 1.

**Lemma 2.2 (see [7])** : If $T : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing function such that,

$$\limsup_{r \to \infty} \frac{\log T(r)}{\log r} = 0$$

then, the set

$$E := \{ r : T(C_1 r) \geq C_2 T(r) \}$$

has the logarithmic density 0 for all $C_1 > 1$ and $C_2 > 1$.

**Lemma 2.3 (see [12])** : Let $f(z)$ be a meromorphic function of finite order, let $c \neq 0$ be fixed. Then

$$N(r, f(z + c)) \leq N(r, f(z)) + S(r, f),$$

$$N(r, 1/f(z + c)) \leq N(r, 1/f(z)) + S(r, f),$$

$$N(r, 1/f(z)) \leq N(r, f(z)) + S(r, f),$$

$$N(r, f(z + c)) \leq N(r, f(z)) + S(r, f).$$

**Lemma 2.4 (see [18])** : Let $f(z)$ be a meromorphic function with $\rho(f) = 0$, let $c$ and $q$ be two non-zero complex numbers, then

$$N(r, f(qz + c)) \leq N(r, f(z)) + S(r, f),$$

$$N(r, 1/(f(qz + c))) \leq N(r, 1/(f(z))) + S(r, f),$$

$$N(r, f(qz + c)) \leq N(r, f(z)) + S(r, f),$$

$$N(r, 1/(f(qz + c))) \leq N(r, 1/(f(z))) + S(r, f).$$

**Lemma 2.5 (see [18])** : Let $f$ be a non-constant meromorphic function of zero order, let $c$ and $q$ be two non-zero complex numbers, then

$$T(r, f(qz + c)) \leq T(r, f(z)) + S(r, f)$$
on a set of logarithmic density 1.

**Lemma 2.6** (see [18]). Let $f(z)$ be an entire function with $\rho(f) = 0$, let $c$ and $q$ be two fixed non-zero complex constants, let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$ be a non-zero polynomial, where $a_n(\neq 0), a_{n-1}, \ldots, a_0$, are complex constants, then

$$T(r, P(f(z))f(qz + c)) = T(r, P(f(z))f(z)) + S(r, f).$$

**Lemma 2.7** (see [17]): Let $F$ and $G$ be two non-constant meromorphic functions. If $F$ and $G$ share 1 CM, then one of the following three cases holds:

(i) $\max\{T(r, F), T(r, G)\} \leq N_2 \left(r, \frac{1}{F}\right) + N_2(r, F) + N_2 \left(r, \frac{1}{G}\right) + N_2(r, G) + S(r, F) + S(r, G)$

(ii) $F = G,$

(iii) $FG \equiv 1,$

where $N_2 \left(r, \frac{1}{F}\right)$ denotes the counting function of zero of $F$, such that simple zero are counted once and multiple zeros are counted twice.

**Lemma 2.8** (see [16]): Let $F$ and $G$ be two non-constant meromorphic functions. Let $F$ and $G$ share 1 IM and

$$H = \frac{F''}{F} - 2 \frac{F'}{F-1} - \frac{G''}{G} + 2 \frac{G'}{G-1}.$$

If $H \neq 0$, then

$$T(r, F) + T(r, G) \leq 2 \left(N_2 \left(r, \frac{1}{F}\right) + N_2(r, F) + N_2 \left(r, \frac{1}{G}\right) + N_2(r, G)\right) + 3 \left(N(r, F) + N(r, G) + N \left(r, \frac{1}{F}\right) + N \left(r, \frac{1}{G}\right)\right) + S(r, F) + S(r, G).$$

3. **Proof of Theorem 1**

Let

$$F(z) = f^n(z)P_m(f(qz + c))f'(z)$$

$$G(z) = g^n(z)P_m(g(qz + c))g'(z)$$
Thus, $F$ and $G$ share 1 CM. Combining the first main theorem with Lemma 2.5, we obtain

$$(n + 2)T(r, f(z)) \leq T(r, f^m(z)f^2(qz + c)) + O(1)$$

Hence, we obtain

$$(n - m - 2)T(r, f(z)) \leq T(r, F(z)) + S(r, f) \quad (1)$$
Similarly, $$(n - m - 2)T(r, g(z)) \leq T(r, G(z)) + S(r, g) \quad (2)$$

From Lemma 2.5, we have

$$T(r, F) \leq (n + m + 2)T(r, f) + S(r, f) \quad (3)$$
$$T(r, G) \leq (n + m + 2)T(r, g) + S(r, g) \quad (4)$$

By Second main theorem, Lemma 2.5 and (4), we obtain

$$T(r, F) \leq \overline{N}(r, F) + \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F - 1}\right) + S(r, F)$$
$$\leq \overline{N}(r, f) + \overline{N}(r, P_m(f(qz + c))) + \overline{N}(r, f')$$
$$+ \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{P_m(f(qz + c))}\right) + \overline{N}\left(r, \frac{1}{f'}\right) + \overline{N}\left(r, \frac{1}{G - 1}\right) + S(r, f)$$
$$T(r, F) \leq (7 + m)T(r, f) + (n + m + 2)T(r, g) + S(r, f) + S(r, g) \quad (5)$$

Hence (1) and (5) imply that

$$(n - 2m - 9)T(r, f) \leq (n + m + 2)T(r, g) + S(r, f) + S(r, g) \quad (6)$$

Similarly, we have

$$(n - 2m - 9)T(r, g) \leq (n + m + 2)T(r, f) + S(r, f) + S(r, g) \quad (7)$$

Equations (6) and (7) imply that $S(r, f) = S(r, g)$.

Together the definition of $F$ with Lemma 2.5, we have

$$N_2\left(r, \frac{1}{F}\right) \leq 2\overline{N}\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{P_m(f(qz + c))}\right) + N\left(r, \frac{1}{f'}\right) + S(r, f)$$
$$\leq (4 + m)T(r, f) + S(r, f) \quad (8)$$
Similarly,
\begin{align*}
N_2 \left( r, \frac{1}{G} \right) & \leq (4 + m) T(r, g) + S(r, g), \\
N_2(r, F) & \leq (4 + m) T(r, f) + S(r, f), \\
N_2(r, G) & \leq (4 + m) T(r, g) + S(r, g).
\end{align*}

Combining Lemma 2.7 with (8)-(9), we obtain
\begin{align*}
T(r, F) + T(r, G) & \leq 2N_2 \left( r, \frac{1}{F} \right) + 2N_2(r, F) + 2N_2(r, G) + S(r, f) + S(r, g) \\
& \leq (16 + 4m) (T(r, f) + T(r, g)) + S(r, f) + S(r, g) \\
& \leq (16 + 4m) (T(r, f) + T(r, g)) + S(r, f) + S(r, g) \\
& \leq (16 + 4m) (T(r, f) + T(r, g)) + S(r, f) + S(r, g)
\end{align*}

Then, by (1) (2) and (10), we obtain
\begin{align*}
(n - 5m - 18) [T(r, f) + T(r, g)] & \leq S(r, f) + S(r, g)
\end{align*}

which is a contradiction, since \( n \geq 5m + 19 \).

By Lemma 2.7, we have \( F \equiv G \) or \( FG \equiv 1 \).

If \( F \equiv G \),
that is,
\begin{align*}
f^n(z) P_m(f(qz + c)) f'(z) & \equiv g^n(z) P_m(g(qz + c)) g'(z)
\end{align*}

Set \( H(z) = f(z)/g(z) \)
Suppose that \( H(z) \) is not a constant. Then, we obtain
\begin{align*}
\frac{f^n(z) P_m(f(qz + c)) f'(z)}{g^n(z) P_m(g(qz + c)) g'(z)} & = 1 \\
H^n(z) P_m(H(qz + c)) H'(z) & = 1
\end{align*}

From Lemma 2.5 and (12), we get
\begin{align*}
n T(r, H) = T \left( r, \frac{1}{P_m(H(qz + c))H'(z)} \right) & \leq T(r, P_m(H(qz + c)) H'(z)) + S(r, H) \\
& \leq (m + 2) T(r, H(z)) + S(r, H)
\end{align*}

Hence, \( H(z) \) must be non-zero constant, since \( n \geq 5m + 19 \).

Set \( H(z) = t \)
By (12), we have \( t^d = 1 \).
Thus \( f(z) = tg(z) \), where \( d = GCD(n + m + 1, n + m, ..., n + m + 1 - i, ..., n + 1) \), \( a_{m-i} \neq 0 \) for some \( i = 0, 1, 2, ...m + n \)

If \( FG \equiv 1 \), that is,
\[
f^n(z)P_m(f(qz + c))f'(z)g^n(z)P_m(g(qz + c))g'(z) = 1
\]
Let \( L(z) = f(z), g(z) \). Using similar method as above, we obtain that \( L(z) \) must also be a non-zero constant. Thus we have \( fg = t \), where \( t^d = 1 \), \( d = GCD(n + m + 1, n + m, ..., n + m + 1 - i, ..., n + 1) \), \( a_{m-i} \neq 0 \) for some \( i = 0, 1, 2, ...m + n \).

4. Proof of Theorem 2

Let
\[
F(z) = f^n(z)P_m(f(qz + c))f'(z) \\
G(z) = g^n(z)P_m(g(qz + c))g'(z)
\]
and \( H \) be defined as in Lemma 2.8. Using the same arguments as in Theorem 1, we prove that (1)-(9) holds.

By Lemma 2.5, we obtain
\[
N(r, F(z)) \leq N(r, f(z)) + N(r, P_m(f(qz + c))) + N(r, f'(z)) + S(r, f)
\]
\[
\leq (m + 2)T(r, f) + S(r, f),
\]

Similarly, \( \overline{N} \left(r, \frac{1}{F(z)}\right) \leq (m + 2)T(r, f) + S(r, f) \),
\[
N(r, G(z)) \leq (m + 2)T(r, g) + S(r, g),
\]
\[
\overline{N} \left(r, \frac{1}{G(z)}\right) \leq (m + 2)T(r, g) + S(r, g).
\]

Together Lemma 2.8 with (8), (9) and (14), we have
\[
T(r, F) + T(r, G) \leq (10m + 28)(T(r, f) + T(r, g)) + S(r, f) + S(r, g)
\]

By (1), (2) and (15)
\[
(n - m - 2)(T(r, f) + T(r, g)) \leq (10m + 28)(T(r, f) + T(r, g)) + S(r, f) + S(r, g)
\]
which is impossible, since $n \geq 11m + 31$. Hence, we have $H \equiv 0$.

By integrating $H$ twice, we have

$$F = \frac{(b + 1)G + (a - b - 1)}{bG + (a - b)}$$

(17)

which yields $T(r, F) = T(r, G) + O(1)$.

From (1)-(4), we obtain

$$(n - m - 2)T(r, f) \leq (n + m + 2)T(r, g) + S(r, f) + S(r, g)$$

(18)

$$(n - m - 2)T(r, g) \leq (n + m + 2)T(r, f) + S(r, f) + S(r, g)$$

(19)

Next, we will prove that $F \equiv G$ or $FG \equiv 1$

Case 1 : $(b \neq 0, -1)$. If $a - b - 1 \neq 0$, by (17), we obtain

$$\mathcal{N}\left( r, \frac{1}{F}\right) = \mathcal{N}\left( r, \frac{1}{G - (a - b - 1)/(b + 1)}\right)$$

(20)

Combining the Nevanlinna second main theorem with Lemma 2.5, (1),(4) and (19), we obtain

$$\begin{align*}
(n - m - 2)T(r, g) &\leq T(r, G) + S(r, g) \\
&\leq \mathcal{N}\left( r, \frac{1}{G}\right) + \mathcal{N}(r, G) + \mathcal{N}\left( r, \frac{1}{G - (a - b - 1)/(b + 1)}\right) + S(r, g) \\
&\leq \mathcal{N}\left( r, \frac{1}{G}\right) + \mathcal{N}(r, G) + \mathcal{N}\left( r, \frac{1}{F}\right) + S(r, g) \\
&\leq \mathcal{N}\left( r, \frac{1}{g}\right) + \mathcal{N}\left( r, \frac{1}{P_m(g(qz + c))}\right) + \mathcal{N}\left( r, \frac{1}{g'}\right) + \mathcal{N}(r, g) + \mathcal{N}(r, P_m(g(qz + c))) + \mathcal{N}(r, g') \\
&+ \mathcal{N}\left( r, \frac{1}{f}\right) + \mathcal{N}\left( r, \frac{1}{P_m(f(qz + c))}\right) + \mathcal{N}\left( r, \frac{1}{f'}\right) + S(r, g) \\
&\leq (7 + m)T(r, g) + (m + 4)T(r, f) + S(r, g)
\end{align*}$$

(21)

By simple calculation, we get contradiction, since $n \geq 11m + 31$. Hence we obtain, $a - b - 1 = 0$, so

$$F = \frac{(b + 1)G}{bG + 1}$$

(22)
Using the similar method as above, we obtain

\[(n - m - 2)T(r, g) \leq T(r, G) + S(r, g) \]

\[\leq \mathcal{N}\left(r, \frac{1}{G}\right) + \mathcal{N}(r, G) + \mathcal{N}\left(r, \frac{1}{G + 1/b}\right) + S(r, g) \]

\[\leq \mathcal{N}\left(r, \frac{1}{G}\right) + \mathcal{N}(r, G) + \mathcal{N}\left(r, \frac{1}{F}\right) + S(r, g) \]

\[\leq (7 + m)T(r, g) + (m + 4)T(r, f) + S(r, g) \]

which is impossible.

**Case 2**: If \(b = -1\) and \(a = -1\), then \(FG \equiv 1\) follows trivially. Therefore, consider \(b = -1\) and \(a \neq -1\).

By (17), we have

\[F = \frac{a}{a + 1 - G} \]

(23)

Similarly, as above we get contradiction.

**Case 3**: If \(b = 0\), \(a = 1\), then \(F \equiv G\) follows trivially. Therefore, consider \(b = 0\) and \(a \neq 1\). By (17), we have

\[F = \frac{G + a - 1}{a} \]

(24)

Similarly, as above we get contradiction.

5. **Proof of Theorem 3**

Let \(f(z)\) and \(g(z)\) be two transcendental entire functions. Since \(f^n(z)P_m(f(qz + c))f'(z)\) and \(g^n(z)P_m(g(qz + c))g'(z)\) share 1 CM, we have

\[\frac{f^n(z)P_m(f(qz + c))f'(z) - 1}{g^n(z)P_m(g(qz + c))g'(z) - 1} = e^{l(z)} \]

(25)

where \(l(z)\) is an entire function, by \(\rho(f) = 0\) and \(\rho(g) = 0\), we have \(e^{l(z)} \equiv \eta\) a constant.

Rewriting (25),

\[\eta g^n(z)P_m(g(qz + c))g'(z) = f^n(z)P_m(f(qz + c))f'(z) + \eta - 1 \]

(26)
If \( \eta \neq 1 \), by the first main theorem, the second main theorem and Lemma 2.5, we have

\[
T(r, f^n(z)P_m(f(qz + c))f'(z)) \leq N(r, f^n(z)P_m(f(qz + c))f'(z))
\]

\[
+ N \left( r, \frac{1}{f^n(z)P_m(f(qz + c))f'(z)} \right)
\]

\[
+ N \left( r, \frac{1}{f^n(z)P_m(f(qz + c))f'(z)} - 1 \right)
\]

\[
\leq (n + k + 3)T(r, f) + N \left( r, \frac{1}{g^n(z)P_m(g(qz + c))g'(z)} \right)
\]

\[
+ S(r, f) + S(r, g)
\]

\[
\leq (n + k + 3)T(r, f) + (n + k + 3)T(r, g) + S(r, f) + S(r, g)
\]

(27)

By Lemma 2.6 and (27), we have

\[
(n + m + 2)T(r, f) = T(r, f^n(z)P_m(f(qz + c))f'(z))
\]

\[
\leq (n + k + 3)T(r, f) + (n + k + 3)T(r, g) + S(r, f) + S(r, g)
\]

(28)

Similarly, \((m - k - 1)T(r, g) \leq (n + k + 3)T(r, f) + S(r, f) + S(r, g)\) (29)

Equations (28) and (29) imply that

\[
(m - 2k - 4 - n)(T(r, f) + T(r, g)) \leq S(r, f) + S(r, g)
\]

(30)

which is impossible, since \( m > n + 2k + 4 \).

Hence we have \( \eta = 1 \). Rewriting (25),

\[
g^n(z)P_m(g(qz + c))g'(z) = f^n(z)P_m(f(qz + c))f'(z)
\]

(31)

Set \( h(z) = f(z)/g(z) \)

**Case 1:** Suppose that \( h(z) \) is a constant.

Integrating (31), we get

\[
f^{n+1} \left[ \frac{a_m f^m(qz + c)}{m + n + 1} + \frac{a_{m-1} f^{m-1}(qz + c)}{m + n} + ... + \frac{a_0}{n + 1} \right]
\]

\[
= g^{n+1} \left[ \frac{a_m g^m(qz + c)}{m + n + 1} + \frac{a_{m-1} g^{m-1}(qz + c)}{m + n} + ... + \frac{a_0}{n + 1} \right]
\]

(32)
By substituting $f = gh$ in (32), we obtain
\[
\begin{align*}
g^{n+1}h^{n+1} & \left[ \frac{a_m g^m (qz + c) h^m}{m + n + 1} + \frac{a_{m-1} g^{m-1} (qz + c) h^{m-1}}{m + n} + \ldots + \frac{a_0}{n + 1} \right] \\
= g^{n+1} & \left[ \frac{a_m g^m (qz + c)}{m + n + 1} + \frac{a_{m-1} g^{m-1} (qz + c)}{m + n} + \ldots + \frac{a_0}{n + 1} \right] \\
\Rightarrow g^{n+1} & \left[ \frac{a_m g^m (qz + c)}{m + n + 1} (h^{m+n+1} - 1) + \frac{a_{m-1} g^{m-1} (qz + c)}{m + n + 1} (h^{m+n} - 1) \right] \\
& + \ldots + \frac{a_0}{n + 1} (h^{n+1} - 1) \equiv 0
\end{align*}
\]

Since $g$ is a transcendental entire function, we have $g^{n+1}(z) \neq 0$. Hence, we obtain
\[
\begin{align*}
\frac{a_m g^m (qz + c)}{m + n + 1} (h^{m+n+1} - 1) + \frac{a_{m-1} g^{m-1} (qz + c)}{m + n + 1} (h^{m+n} - 1) + \ldots + \frac{a_0}{n + 1} (h^{n+1} - 1) \equiv 0
\end{align*}
\]

Equation (33) implies that $h^d = 1$, where $d = \gcd(n + m + 1, n + m, \ldots, n + m + 1 - i, \ldots, n + 1)$, $a_{m-i} \neq 0$, for some $i = 0, 1, \ldots m$.

Thus $f = tg$ for a constant $t$, such that $t^d = 1$, where $d = \gcd(n + m + 1, n + m, \ldots, n + m + 1 - i, \ldots, n + 1)$, $a_{m-i} \neq 0$, for some $i = 0, 1, \ldots m$.

**Case 2**: Suppose that $h(z)$ is not a constant, then by (33) $f(z)$ and $g(z)$ satisfy the algebraic equation $R(f, g) \equiv 0$, where
\[
R(w_1, w_2) = w_1^{n+1} \left[ \frac{a_m w_1^m}{n + m + 1} + \frac{a_{m-1} w_1^{m-1}}{n + m} + \ldots + \frac{a_0}{n + 1} \right] \\
- w_2^{n+1} \left[ \frac{a_m w_2^m}{n + m + 1} + \frac{a_{m-1} w_2^{m-1}}{n + m} + \ldots + \frac{a_0}{n + 1} \right]
\]

**References**


