A NOTE ON SPLIT EDGE DOMINATION NUMBER OF A GRAPH

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Abstract
A set \(F\) of a graph \(G(V,E)\) is an edge dominating set if every edge in \(E - F\) is adjacent to some edge in \(F\). An edge domination number \(\gamma'(G)\) of \(G\) is the minimum cardinality of an edge dominating set. An edge dominating set \(F\) is called a split edge dominating set if the induced subgraph \(\langle E - F \rangle\) is disconnected. The minimum cardinality of the split edge dominating set in \(G\) is its domination number and is denoted by \(\gamma_s'(G)\). We investigate several properties of split edge dominating sets and give some bounds on the split edge domination number.

1. Introduction
Let \(G(V,E)\) be a graph with \(p = |V|\) and \(q = |E|\) denoting the number of vertices and

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edges respectively. All the graphs considered here are finite, non-trivial, undirected and connected without loops or multiple edges.

The degree of a vertex \( u \) is denoted by \( d(u) \). The degree of an edge \( e = uv \) of a graph \( G \) is the number defined by \( \text{deg}(e) = d(u) + d(v) - 2 \). An edge \( e = uv \) is called an universal edge if \( d(e) = q - 1 \). The minimum (maximum) degree of an edge is denoted by \( \delta'(\Delta') \).

The induced subgraph of \( X \subseteq E \) is denoted by \( \langle X \rangle \). For a real number \( x \), \( \lfloor x \rfloor \) denotes the greatest integer less than or equal to \( x \) and \( \lceil x \rceil \) denotes the smallest integer greater than or equal to \( x \). The edge independence number \( \beta_1(G) \) is defined to be the number of edges in a maximum independent set of edges of \( G \). A vertex of degree one is called a pendant vertex.

Let \( G(V, E) \) be a connected graph. A subset \( S \) of \( V \) is called a dominating set of \( G \) if every vertex in \( V - S \) is adjacent to at least one vertex in \( S \). The concept of edge domination was introduced by Mitchell and Hedetniemi ([5], [6]). A subset \( F \) of \( E \) is called an edge dominating set of \( G \) if every edge not in \( F \) is adjacent to some edge in \( F \). The minimum cardinality of an edge dominating set of \( G \) is called an edge domination number and is denoted by \( \gamma'(G) \).

A dominating set \( D \) of \( G \) is a split dominating set if the induced subgraph \( \langle V - D \rangle \) is disconnected. The split domination number \( \gamma_s(G) \) of \( G \) is the minimum cardinality of a split dominating set. This concept was introduced by Kulli, Janakiram in [8]. Any undefined term or notation in this paper can be found in Harary ([4], [9]). We need the following theorems.

**Theorem 1.1** [10] : For any graph \( G \) with an end edge, \( \gamma'_s = \gamma'(G) \). Furthermore, there exists a \( \gamma'_s \)-set of \( G \) containing all edges adjacent to end-edges.

**Theorem 1.2** [2] : For any connected graph \( G \) of even order \( p \), \( \gamma'(G) = p/2 \) if and only if \( G \) is isomorphic to \( K_p \) or \( K_{p/2,p/2} \).

**Theorem 1.3** [3] : For every \( n \), \( \gamma'_s n = n \).

In this paper we study the split edge domination number of a graph characterizing the problem for certain class of graphs.

### 2. Split Edge Domination Number of a Graph

**Definition 2.1** : A set \( F \subseteq E(G) \) is said to be split edge dominating set if \( F \) is an edge dominating set and induced subgraph \( \langle E - F \rangle \) is disconnected. The minimum
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Cardinality of split edge dominating set in $G$ is the split edge domination number and is denoted by $\gamma'_s(G)$ of $G$. This concept was introduced by K. M. Yogeesha, N. D. Soner, and Sirous Ghobadi [10] and later extended by V. R. Kulli and Radha Rajamani Iyer [7].

Now we study on split edge domination of subdivision of some standard graphs.

**Theorem 2.2** [10]: For any path $P_p$ with $p \geq 5$, the split edge domination number,

$$\gamma'_s(P_p) = \begin{cases} 
\lfloor \frac{p}{3} \rfloor & \text{if } p \equiv 0 \pmod{3}, \\
\lfloor \frac{p}{3} \rfloor + 1 & \text{if } p \equiv 1 \pmod{3}, \\
\lfloor \frac{p}{3} \rfloor + 1 & \text{if } p \equiv 2 \pmod{3}.
\end{cases}$$

**Proof**: Let $P_p = \{v_1, v_2, ..., v_p\}$ be any path and let $e_i = v_iv_{i+1}$ be an edge on $P_p$. Let

$$S_1 = \begin{cases} 
S & \text{if } p \equiv 0 \text{ or } 1 \pmod{3}, \\
S \cup \{e_{p-1}\} & \text{if } p \equiv 2 \pmod{3}.
\end{cases}$$

be an edge set on $G$ where $S = \{e_j : j = 3k + 2 \text{ for } 0 \leq k \leq \lfloor \frac{p}{3} \rfloor - 1\}$. Clearly $S_1$ is a split edge dominating set and $|S_1|$ will be the split edge domination number with minimum cardinality. Hence the proof.

**Theorem 2.3** [10]: For any cycle $C_p$ with $p \geq 5$, the split edge domination number,

$$\gamma'_s(C_p) = \begin{cases} 
\lfloor \frac{p}{3} \rfloor & \text{if } p \equiv 0 \pmod{3}, \\
\lfloor \frac{p}{3} \rfloor + 1 & \text{if } p \equiv 1 \text{ or } 2 \pmod{3}.
\end{cases}$$

**Proof**: Let $C_p = \{v_1, v_2, ..., v_p\}$ be any cycle and let $e_i = v_iv_{i+1}$ be an edge on $C_p$. Let $S = \{e_j : j = 3k + 1 \text{ for } 0 \leq k \leq \lceil \frac{p}{3} \rceil - 1\}$ be any edge set on $C_p$. Clearly $S$ is a split edge dominating set and $|S|$ will be its split edge domination number with minimum cardinality. Hence the proof.

**Theorem 2.4**: For any path $P_p$, $\gamma'_s(P_p) + \gamma'_s(S(P_p)) \leq p$. Equality holds for $p \equiv 0$ or $2 \pmod{3}$.

**Proof**: Let $S$ be the $\gamma'_s$ set of $P_p$. Therefore by Theorem 2.2,

$$\gamma'_s(P_p) = |S|$$

(1)

If $S'$ is the $\gamma'_s$ set of subdivision of $P_p$, then by Theorem 2.2 we get,
\[ \gamma_s'(S(P_p)) = |S'| \]

that is

\[ \gamma_s'(S(P_p)) = \begin{cases} 
\frac{2p}{3} & \text{if } p \equiv 0 \pmod{3}, \\
\left\lfloor \frac{2p}{3} \right\rfloor + 2 & \text{if } p \equiv 1 \pmod{3}, \\
\left\lfloor \frac{2p}{3} \right\rfloor + 1 & \text{if } p \equiv 2 \pmod{3}.
\end{cases} \quad (2) \]

Consider a particular case where \( p \equiv 1 \pmod{3} \). Adding (1) and (2), we get

\[
\gamma_s'(P_p) + \gamma_s'(S(P_p)) = |S| + |S'|,
\]

\[
= \left\lceil \frac{p}{3} \right\rceil + \left\lfloor \frac{2p}{3} \right\rfloor + 4,
\]

\[
\leq p + 3.
\]

The other two cases are obvious. Hence the proof.

**Theorem 2.5**: For any cycle \( C_p \), \( \gamma_s'(C_p) + \gamma_s'(S(C_p)) \leq p + 1 \). Equality holds for \( p \equiv 1 \) or \( 2 \pmod{3} \).

**Proof**: Let \( S \) be the \( \gamma_s' \) set of \( C_p \). Therefore by Theorem 2.3,

\[ \gamma_s'(C_p) = |S| \quad (3) \]

If \( S' \) is the \( \gamma_s' \) set of subdivision of \( C_p \), then by Theorem 2.3 we have

\[ \gamma_s'(S(C_p)) = \begin{cases} 
\frac{2p}{3} & \text{if } p \equiv 0 \pmod{3}, \\
\left\lfloor \frac{2p}{3} \right\rfloor + 1 & \text{if } p \equiv 1 \text{ or } 2 \pmod{3}.
\end{cases} \quad (4) \]

Consider a particular case where \( p \equiv 1 \pmod{3} \). Adding (3) and (4), we get

\[
\gamma_s'(C_p) + \gamma_s'(S(C_p)) = |S| + |S'|,
\]

\[
= \left\lceil \frac{p}{3} \right\rceil + \left\lfloor \frac{2p}{3} \right\rfloor + 1,
\]

\[
\leq p + 1.
\]

The other two cases are obvious. Hence the proof.

3. Bounds on the Split Edge Domination Number

**Theorem 3.1**: For any connected graph \( G \) with \( q \geq 3 \), \( \gamma_s'(G) \geq 1 \). Equality holds if and only if there exists only one cutset \( e \) in \( G \) with degree \( q - 1 \).

**Proof**: Suppose \( \gamma_s'(G) = 1 \) and \( S = \{e\} \) is a cut-set of \( G \). Clearly \( E - S \) is disconnected and \( e \) dominates all the other edges of \( G \). Hence \( e \) is a cut-set with degree \( q - 1 \). Suppose
there exists another cut-set $e_1$ of degree $q - 1$. Then $e_1$ is adjacent to all the remaining edges of $G$. In this case $\langle E - S \rangle$ is connected, a contradiction to $S = \{e\}$ is a $\gamma'_s$-set of $G$.

Converse part is obvious. Hence the proof. \hfill \Box

**Theorem 3.2**: Let $G$ be any graph with $\delta(G) > 1$ and $e$ be an edge in a graph $G$ with degree $k$ such that $\langle N(e) \rangle$ is disconnected. Then split edge domination number, $\gamma'_s(G) \leq q - k$.

**Proof**: If $e$ is an edge of degree $k$ and $\langle N(e) \rangle$ is disconnected, then $E - N(e)$ is a split edge dominating set. Therefore $|E - N(e)| > \gamma'_s(G)$. Hence the proof. \hfill \Box

**Theorem 3.3**: If $F$ is a $\gamma'_s$-set of a graph $G$, then $E - F$ is a dominating set of $G$ and hence $\gamma'(G) + \gamma'_s(G) \leq q$.

**Proof**: Suppose $E - F$ is not a dominating set of $G$, then there exists an edge $e$ in $F$ which is not adjacent to any of the edges in $E - F$. Thus by Theorem 3.2 $F - \{e\}$ is a split edge dominating set of $G$, a contradiction to the minimality of $F$. Further $E - F$ is a dominating set of $G$ and so $|E - F| \geq \gamma'(G)$. Hence the proof. \hfill \Box

**Theorem 3.4**: For any path $P_p$ with $p \geq 6$, $\gamma'_s(P_p) = 2(p - 4)$.

**Proof**: Consider an edge $e = uv$ with minimum degree in $P_p$. Clearly the set $F = N(e)$ is an edge dominating set of $P_p$. Also $\langle E(P_p) - F \rangle$ is a disconnected graph with two components $K_2$ and $G_{p-2}$. Thus $F$ itself is a split edge dominating set of $P_p$. Therefore

$$\gamma'_s(P_p) = |F|,$$

$$= |N(e)|,$$

$$= d(u) + d(v) - 2,$$

$$= 2p - 8.$$

Hence the proof. \hfill \Box

**Theorem 3.5**: For any Cycle $C_p$ with $p \geq 6$, $\gamma'_s(C_p) = 2(p - 4)$.

Proof of this Theorem is similar to the above one.

**Theorem 3.6**: If $F$ is an independent edge dominating set of a graph $G$ with $|F| > 1$, then $E - F$ is a split edge dominating set of $G$. In particular $\gamma'_s(G) + \beta_1(G) \leq q$.

**Proof**: Since $F$ is an independent edge dominating set of $G$, $E - F$ is a edge dominating set of $G$. Further $\langle F \rangle = \langle E - (E - F) \rangle$ is disconnected. Hence $E - F$ is a split edge dominating set of $G$. This gives that $|E - F| \geq \gamma'_s(G)$. In particular, if $|F| = \beta_1(G)$,
Theorem 4.2: Let $p$ be a split edge dominating set with minimum cardinality. Therefore $F$ is a split edge dominating set where the induced subgraph $an edge dominating set. Let $K$ be a connected graph with $p$ be the vertex set of $K_{1,p_2}$. Let $S = \{u,v : \forall u \in G\}$ be an edge set in the corona $G_1 \circ K_{1,p_2}$. Clearly $S$ is an edge dominating set. Let $E_1 = \{e_1, e_2, \ldots \}$ be the set of edges incident on a vertex $v_1$ where $v_1$ is one among the vertex with minimum degree in $G$. Then $F = S \cup E_1$ is a split edge dominating set where the induced subgraph $E - F$ is disconnected. Hence $F$ is a split edge dominating set with minimum cardinality. Therefore $\gamma_s'(G \circ K_{1,p_2}) = p_1 + \delta(G)$. Hence the proof.

\[ \gamma_s'(G \circ K_{1,p_2}) \leq p_1 + \left\lceil \frac{p_2}{2} \right\rceil + \delta(G_1). \]

Hence the proof.

4. Split Edge Domination Number on Corona and Join of Graphs

Here we discuss the results on split edge domination number of corona and join of two graphs. The corona $G = H \circ K_1$ is a graph constructed from a copy of $H$, where for each vertex $v \in V(H)$, a new vertex $v'$ and a pendant edge $vv'$ are added. The following theorem gives a sharp bound for the cototal edge domination number of $(G_{p_1} \circ G_{p_2})$.

Theorem 4.1: Let $G$ be a connected graph with $p_1$ vertices and $K_{1,p_2}$ be any star. The split edge domination number of the corona of $G$ and $K_{1,p_2}$, $\gamma_s'(G \circ K_{1,p_2}) = p_1 + \delta(G)$. Hence the proof.

\[ \gamma_s'(G \circ K_{1,p_2}) \leq p_1 + \left\lceil \frac{p_2}{2} \right\rceil + \delta(G_1). \]

Hence the proof.
Theorem 4.3 : Let \( G \) be a graph with \( \delta(G) = 1 \). Then split edge domination number, \( \gamma_{\delta}(G) = p - 1 \) if and only if \( G = H \circ K_1 \) where \( H \) is a complete graph.

Proof : Consider an edge \( e = uv \) where \( u \in K_p \) and \( v \in K_1 \) of \( G \). Clearly \( F = N(e) \) is an edge dominating set of \( G = H \circ K_1 \). Also \( G - F \) is a disconnected graph with two components. Thus \( F \) itself is a split edge dominating set of \( G \) with minimum cardinality. Therefore 

\[
\gamma_{\delta}(G) = |F|,
\]

\[
= |N(e)| = d(u),
\]

\[
= p - 1.
\]

Hence the proof. \( \square \)

The forthcoming Theorems gives a result on the join of some standard graphs.

Now we define the join of two graphs. For disjoint graphs \( G_1 \) and \( G_2 \), the join \( G = G_1 + G_2 \) is the graph with \( V(G) = V(G_1) \cup V(G_2) \) and \( E(G) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1) \wedge v \in V(G_2)\} \). The following Theorems gives the result on the join of some standard graphs.

Theorem 4.4 : Let \( C_{p_1} \) and \( C_{p_2} \) be any two cycles of order \( p_1 \) and \( p_2 \) respectively. Then the split edge domination number, \( \gamma_{\delta}[C_{p_1} + C_{p_2}] = p_1 + p_2 + 2 \).

Proof : Let \( C_{p_1} \) and \( C_{p_2} \) be the two cycles labelled in order as \( u_1e_1u_2e_2\cdots u_{p_1} e_{p_1} u_1 \) and \( v_1e'_1v_2e'_2\cdots v_{p_2} e'_{p_2} v_1 \) respectively. Let \( e = u_i v_j \) be an edge in \( E(G) \). Let \( E_1 = \{uv : u \neq u_i \text{ and } v \neq v_j\} \) be the set of edges in \( E(G) \). Then \( E - \{E_1 \cup \{e\}\} \) forms a split edge dominating set in the join of the two cycles \( C_{p_1} \) and \( C_{p_2} \) respectively. Thus

\[
\gamma_{\delta}[C_{p_1} + C_{p_2}] \leq |E| - \left[ |E_1| + 1 \right],
\]

\[
\leq p_1 + 1 + p_2 + 1,
\]

\[
\leq p_1 + p_2 + 2.
\]

Equality is obvious. Hence the proof. \( \square \)

Theorem 4.5 : Let \( P_{p_1} \) and \( P_{p_2} \) be any two paths of order \( p_1 \) and \( p_2 \) respectively. Then the split edge domination number, \( \gamma_{\delta}[P_{p_1} + P_{p_2}] = p_1 + p_2 \).

Proof : Let \( P_{p_1} \) and \( P_{p_2} \) be the two paths labelled in order as \( u_1 e_1 u_2 e_2 \cdots e_{p-1} u_p \) and \( v_1 e'_1 v_2 e'_2 \cdots e'_{p-1} v_p \) respectively. Let \( e = u_i v_j \) be an edge in \( E(G) \) where \( u_i \) and \( v_j \) are the pendant vertices of \( P_{p_1} \) and \( P_{p_2} \) respectively. Let \( E_1 = \{uv : u \neq u_i \text{ and } v \neq v_j\} \) be the set of edges in \( E(G) \). Then \( E - \{E_1 \cup \{e\}\} \) forms a split edge dominating set in the join of the two paths \( P_{p_1} \) and \( P_{p_2} \) respectively. Thus

\[
\gamma_{\delta}[P_{p_1} + P_{p_2}] \leq |E| - \left[ |E_1| + 1 \right],
\]

\[
\leq p_1 + 1 + p_2 + 1,
\]

\[
\leq p_1 + p_2 + 2.
\]

Equality is obvious. Hence the proof. \( \square \)
or $v \neq u_i$ and $u$ or $v \neq v_j$} be the set of edges in $E(G)$. Then $E - \{E_1 \cup \{e\}\}$ forms a split edge dominating set in the join of the two paths $P_{p_1}$ and $P_{p_2}$. Thus

$$
\gamma'_s[P_{p_1} + P_{p_2}] \leq |E| - \left|E_1\right| + 1,
$$

$$
\leq p_1 + p_2.
$$

Equality is obvious. Hence the proof.

**Theorem 4.6** : Let $P_{p_1}$ be a path and $C_{p_2}$ be a cycle of order $p_1$ and $p_2$ respectively. Then the split edge domination number, $\gamma'_s[P_{p_1} + C_{p_2}] = p_1 + p_2 + 1$.

**Proof** : Let $P_{p_1}$ and $C_{p_2}$ be a path and a cycle labelled in order as $u_1e_1u_2e_2 \cdots e_{p-1}u_1$ and $v_1e_1'v_2e_2' \cdots e_{p-1}'v_{p}v_{2}'$ respectively. Let $e = u_iv_j$ be an edge in $E(G)$ where $u_i$ is a pendant vertex of the path $P_{p_1}$ and $v_j$ be any vertex in the cycle $C_{p_2}$. Let $E_1 = \{uv : u \neq u_i \text{ and } u \text{ or } v \neq v_j\}$ be the set of edges in $E(G)$. Then $E - \{E_1 \cup \{e\}\}$ forms a split edge dominating set in the join of a path $P_{p_1}$ and a cycle $C_{p_2}$. Thus

$$
\gamma'_s[P_{p_1} + C_{p_2}] \leq |E| - \left|E_1\right| + 1,
$$

$$
\leq p_1 + p_2 + 1.
$$

Equality is obvious. Hence the proof.

5. Adding an End Edge

In this section we observe some properties of graphs obtained by adding $K_2$ to a cycle $C_p$. If $e = uv$ is an edge of a graph $G$ with $\deg(u) = 1$ and $\deg(v) > 1$, then $e$ is called an end edge and $u$ an end vertex.

**Theorem 5.1** : Let $G'$ be the graph obtained by adding $k$ end edges $u_1v_j$ for $j = 1, 2, \cdots k$ to a cycle $C_p$ where $u_1 \in C_p$ and $\{v_1, v_2, \cdots v_k\} \notin C_p$. Then the split edge domination number, $\gamma'_s(G') = \lceil \frac{p}{3} \rceil$.

**Proof** : Let $C_p = \{u_1, u_2, \cdots, u_p\}$ be a cycle with $p$ vertices and $G'$ be the graph obtained by adding $k$ end edges $\{u_1v_1, u_1v_2, \cdots u_1v_k\}$ such that $u_1 \in C_p$ and $\{v_1, v_2, \cdots v_k\} \notin C_p$. Let $e_i = u_iu_{i+1}$ be an edge on cycle.

Let $S = \{e_j : j = 3l + 1 \text{ for } 0 \leq l \leq \lfloor \frac{p}{3} \rfloor - 1\}$ and

$$
S_1 = \begin{cases} 
S & \text{if } p \equiv 0 \text{ or } 2(\text{mod}3), \\
S \cup \{e_{p-1}\} & \text{if } p \equiv 1(\text{mod}3). 
\end{cases}
$$
be an edge set on $S$. Then $S_1$ is an edge dominating set and the induced subgraph $\langle E - S_1 \rangle$ is disconnected. Therefore $S_1$ is a split edge dominating set and $|S_1|$ will be the split edge domination number for the graph $G'$.

The converse part of the Theorem is obvious. Hence the proof. □

**Theorem 5.2**: Let $G'$ be the graph obtained by adding $k$ end edges $u_iv_j$ to a cycle $C_p$ with $d(u_i) \geq 3$ where $u_i \in C_p$ for $i = 1, 2, \ldots, p$ and $v_j \notin C_p$ for $j = 1, 2, \ldots, k$. Then the split edge domination number, $\gamma'_s(G') = \lceil \frac{p}{2} \rceil$.

**Proof**: Let $C_p = \{u_1, u_2, \ldots, u_p\}$ be a cycle and $G'$ be the graph obtained by adding $k$ end edges $u_iv_j$ where $u_i \in C_p$ for $i = 1, 2, \ldots, p$ and $v_j \notin C_p$ for $j = 1, 2, \ldots, k$. Let $e_i = u_iu_{i+1}$ be an edge of $G'$.

Let $S = \{e_j : j = 2l + 1 \text{ for } 0 \leq l \leq \lfloor \frac{p}{2} \rfloor - 1\}$ and

$$S_1 = \begin{cases} S & \text{if } p \equiv 0 \text{ (mod } 2), \\ S \cup \{u_pv_1\} & \text{if } p \equiv 1 \text{ (mod } 2). \end{cases}$$

be an edge set of $G'$. Then $S_1$ is an edge dominating set and the induced subgraph $\langle E - S_1 \rangle$ is disconnected. Therefore $S_1$ is a split edge dominating set and $|S_1|$ will be the split edge domination number for the graph $G'$.

The converse part of the Theorem is obvious. Hence the proof. □

**Corollary 5.3**: Let $G'$ be the graph obtained by adding $k$ end edges $u_iv_j$ to a cycle $C_p$ of order $p \geq 3$ in any manner. Then from the above Theorems $\lceil \frac{p}{3} \rceil \leq \gamma'_s(G') \leq \lceil \frac{p}{2} \rceil$.

6. Cartesian Product of Split Edge Domination Number of a Graph

In this section we define “Independent split edge domination number” a new parameter of a graph. An edge dominating set $F$ is called an independent edge dominating set if no two edges of $F$ are adjacent [1]. The Independent edge domination number $\gamma'_i(G)$ of $G$ is the minimum cardinality taken over all independent edge dominating sets of $G$.

The split edge dominating set is said to be an independent split edge dominating set if the induced subgraph $\langle F \rangle$ is an independent edge set.

The Cartesian product of $G$ and $H$, denoted $G \times H$, has vertex set $V(G) \times V(H)$. Two vertices $(u, v), (u', v')$ in $V(G) \times V(H)$ are adjacent if either $u = u'$ and $vv' \in E(H)$, or $v = v'$ and $uu' \in E(G)$. The graph $(P_n \times P_m)$ has $m$ copy of the graph $P_n$ in $m$ columns. Let $\gamma'_{is}(P_n \times P_m)$ denotes the size of minimum independent split edge dominating set of
two paths \((P_n \times P_m)\) where \(n \leq m\). In the sequel (list) we give the values of \(\gamma'_{is}(P_n \times P_m)\)
and cartesian product for few other graphs.

**Theorem 6.1** : Let \(P_3\) be a path of length 2 and \(P_m\) be any path with \(m \geq 3\).
Then Independent split edge dominating number of the cartesian product of two paths,
\(\gamma'_{is}[P_3 \times P_m] = m + 1\).

**Proof** : Consider an independent edge set
\[ S_s = \{(i,1), (i,2) / i = 1, 2, 3, \ldots m\} \].
Also
\[ S_i = \{(1,4+3k), (1,5+3k), (2,3+3k), (2,4+3k), (3,4+3k), (3,5+3k) : k = 0, 1, 2 \cdots \left(\lfloor \frac{m}{3} \right) - 2\} \], see Figure(1).

![Figure 1: Cartesian product of \(P_3 \times P_m\)](image)

Let us discuss the following cases:

**Case(i)** Let \(m \equiv 0\) (mod 3).
Let \(F = S_s \cup S_i \cup \{(2, m - 1), (2, m)\}\) is an independent edge dominating set
and the induced subgraph \(\langle E - F \rangle\) is disconnected. By Theorem 1.3 \(F\) is an
independent split edge dominating set of \(P_3 \times P_m\) with minimum cardinality.
Therefore \(|F| = |S_s| + |S_i| + 1\).

**Case(ii)** Let \(m \equiv 1\) (mod 3).
Let \(F_1 = S_s \cup S_i \cup \{(1, m - 1), (2, m - 1), (2, m), (3, m)\}\) is an independent edge
dominating set and the induced subgraph \(\langle E - F_1 \rangle\) is disconnected. By Theorem
1.3 \(F_1\) is an independent split edge dominating set of \(P_3 \times P_m\) with minimum
cardinality. Therefore \(|F_1| = |S_s| + |S_i| + 2\).

**Case(iii)** Let \(m \equiv 2\) (mod 3).
Let \(F_2 = S_s \cup S_i \cup \{(1, m - 1), (1, m), (2, m - 2), (2, m - 1), (3, m - 1), (3, m)\}\)
is an independent edge dominating set and the induced subgraph \(\langle E - F_2 \rangle\) is
disconnected. By Theorem 1.3 $F_2$ is an independent split edge dominating set of $P_3 \times P_m$ with minimum cardinality. Therefore $|F_2| = |S_s| + |S_i|$.

Thus $\gamma'_s[P_3 \times P_m] = m + 1$. Hence the proof. \qed

**Theorem 6.2**: Let $P_n$ be a path of length $n - 1$ and $P_m$ be any path with $m \geq 3$. Then independent split edge domination number of the cartesian product of two paths, $\gamma'_s[P_n \times P_m] \leq \begin{cases} \frac{n(m-2)}{3} + n & \text{if } m \equiv 2 \text{ (mod 3)}, \\ \left\lceil \frac{n(m-2)}{3} \right\rceil + n + 1 & \text{if } m \equiv 0 \text{ or } 1 \text{ (mod 3)}. \end{cases}$

Equality holds for $m \equiv 2 \text{ (mod 3)}$.

**Proof**: Consider an independent edge set $S_s = \{(i, 1), (i, 2) \}/i = 1, 2, 3, \cdots n$. Also, if $n$ is odd, then

$$S_1 = \begin{cases} \bigcup_{i=1,3,\cdots n} \{(i, 3k + 4), (i, 3k + 5) : k = 0, 1, \cdots \left\lfloor \frac{m}{3} \right\rfloor - 2, \\ \bigcup_{i=2,4,\cdots n-1} \{(i, 3k + 3), (i, 3k + 4) : k = 0, 1, \cdots \left\lfloor \frac{m}{3} \right\rfloor - 1 - 2. \end{cases}$$

If $n$ is even, then

$$S_1 = \begin{cases} \bigcup_{i=1,3,\cdots n-1} \{(i, 3k + 4), (i, 3k + 5) : k = 0, 1, \cdots \left\lfloor \frac{m}{3} \right\rfloor - 2, \\ \bigcup_{i=2,4,\cdots n} \{(i, 3k + 3), (i, 3k + 4) : k = 0, 1, \cdots \left\lfloor \frac{m}{3} \right\rfloor - 2. \end{cases}$$

See Figure(2)

![Figure 2](image)

Let us discuss the following cases:

**Case(i)** If $m \equiv 2 \text{ (mod 3)}$,

Let, if $n$ is odd, then

$$S_2 = \begin{cases} \bigcup_{i=1,3,\cdots n} \{(i, m - 1), (i, m)\}, \\ \bigcup_{i=2,4,\cdots n-1} \{(i, m - 2), (i, m - 1)\}. \end{cases}$$
If \( n \) is even, then
\[
S_2 = \bigcup_{i=1,3,\ldots,n-1} \{(i,m-1),(i,m)\}, \\
\bigcup_{i=2,4,\ldots,n} \{(i,m-2),(i,m-1)\}.
\]

Then \( F = S_1 \cup S_2 \cup S_s \) forms an independent edge dominating set and the induced subgraph \( (E - F) \) of \( P_n \times P_m \) is disconnected. Thus \( F \) is an independent split edge dominating set of \( P_n \times P_m \).

\[
\gamma'_s [P_n \times P_m] \leq |F|, \\
\leq |S_1| + |S_2| + |S_s|.
\]

**Case(ii)** If \( m \equiv 0(mod3) \), Let

\[
S_3 = \left\{ \begin{array}{l}
\{(1,m),(2,m)\}, \{(3,m),(4,m)\}, \ldots \{(n-2,m),(n-1,m)\} \quad \text{if } n \text{ is odd,} \\
\{(1,m),(2,m)\}, \{(3,m),(4,m)\}, \ldots \{(n-1,m),(n,m)\} \quad \text{if } n \text{ is even.}
\end{array} \right.
\]

Then \( F_1 = S_1 \cup S_3 \cup S_s \) is an independent edge dominating set and the induced subgraph \( (E - F_1) \) of \( P_n \times P_m \) is disconnected. Thus \( |F_1| \leq |S_1| + |S_3| + |S_s| \). Therefore \( F_1 \) is an independent split edge dominating set of \( P_n \times P_m \).

**Case(iii)** Let \( m \equiv 1(mod3) \).

For \( n \geq 4 \), we can partition the set of \( m \) columns of \( P_n \times P_m \) in such a way that two columns at the \( B_i \) blocks for beginning, \( B_i, (i = 1,2,\ldots\lfloor \frac{m}{3} \rfloor - 1) \) at the middle and two columns at the end. The set \( S_s \cup S_1 \) will dominate the first two columns and \( B_i \) blocks. In addition we can determine a set isomorphic to \( S_R \) which dominates \( m \) and \( m - 1 \) columns by a set isomorphic to \( S_R \). Let \( n = 4q + l: 1 \leq q \leq \lfloor \frac{n}{4} \rfloor, 0 \leq l \leq 3 \). Consider the following two cases to find \( S_R \) as shown in Figure (3).
Figure 3

i) If \( q = 1 \) then \( S_R = \{R_l : 0 \leq l \leq 3\} \).

ii) If \( q > 1 \) then \( S_R = \{(\lfloor \frac{n}{q}\rfloor - 1)R_0 + R_l : 0 \leq l \leq 3\} \).

Therefore \( S_4 = S_1 \cup S_s \cup S_R \) is an independent split edge dominating set and the induced subgraph \( \langle E - S_4 \rangle \) of \( P_n \times P_m \) is disconnected. Thus \( S_4 \) is an independent split edge dominating set with minimum cardinality. Therefore \( |S_4| \leq |S_1| + |S_R| + |S_s| \).

Hence the proof.

**Theorem 6.3**: Let \( C_3 \) be a cycle of order 3 and \( C_m \) be any cycle with \( m \geq 3 \). Then split edge domination number of the cartesian product of two cycles,

\[
\gamma'_s[C_3 \times C_m] \leq \begin{cases} 
m + 3 & \text{if } m \text{ is even}, 
m + 4 & \text{if } m \text{ is odd}. 
\end{cases}
\]

**Proof**: Consider an edge set

\( S_s = \{(1,1), (1,2), (1,1), (3,1), (1,1), (1,m), (2,1), (2,2), (2,1), (3,1), (2,1), (2,m)\} \). Also let us discuss the following cases:

**Case(i)** If \( m \) be an even number.

Let \( S_1 = \\{((1,2k+4),(2,2k+4)) : k = 0,1,\cdots,\lfloor \frac{m}{2}\rfloor - 2\} \cup \\{((2,2k+3),(3,2k+3)) : k = 0,1,\cdots,\lfloor \frac{m}{2}\rfloor - 2\} \).

Then \( F = S_1 \cup S_s \) is an edge dominating set and the induced subgraph \( \langle E - F \rangle \) of \( C_3 \times C_m \) is disconnected. Thus \( F \) is a split edge dominating set of \( C_3 \times C_m \).

\[
\gamma'_s[C_3 \times C_m] \leq |F|,
\gamma'_s[C_3 \times C_m] \leq |S_1| + |S_s|.
\]

**Case(ii)** If \( m \) be an odd number.

Let \( S_2 = \\{((1,2k+4),(2,2k+4)) : k = 0,1,\cdots,\lfloor \frac{m}{2}\rfloor - 1\} \cup \\{((2,2k+3),(3,2k+3)) : k = 0,1,\cdots,\lfloor \frac{m}{2}\rfloor \} \).

Then \( F_1 = S_2 \cup S_s \) is an edge dominating set and the induced subgraph \( \langle E - F_1 \rangle \) of \( C_3 \times C_m \) is disconnected. Thus \( F_1 \) is a split edge dominating set of \( C_3 \times C_m \).

\[
\gamma'_s[C_3 \times C_m] \leq |F_1|,
\gamma'_s[C_3 \times C_m] \leq |S_2| + |S_s|.
\]
Hence the proof.

**Theorem 6.4** : Let $C_4$ be a cycle of order 4 and $C_m$ be any cycle with $m \geq 4$. Then split edge domination number of the cartesian product of two cycles, 
\[ \gamma'_s[C_4 \times C_m] \leq m + \left\lceil \frac{m}{2} \right\rceil + 3. \]

**Proof** : Consider an edge set 
\[ S_s = \left\{ \{(1,1),(1,2)\}, \{(1,1),(4,1)\}, \{(1,1),(1,m)\}, \{(2,1),(2,2)\}, \{(2,1),(3,1)\}, \{(2,1),(2,m)\} \right\}. \]

Let 
\[ S_1 = \left\{ \{(1,2k+2),(2,2k+2)\} : k = 1, 2, \cdots \left\lfloor \frac{m}{2} \right\rfloor - 2 \right\} \]
\[ \cup \left\{ \{(2,2k+1),(3,2k+1)\} : k = 1, 2, \cdots \left\lfloor \frac{m}{2} \right\rfloor - 1 \right\}. \]

Also let us discuss the following cases:

**Case(i)** If $m$ be an odd number.

Let 
\[ S_2 = \left\{ \{(4,2k),(4,2k+1)\} : k = 1, 2, \cdots \left\lfloor \frac{m}{2} \right\rfloor \right\}. \]

Then $F = S_1 \cup S_2 \cup S_s$ is an edge dominating set and the induced subgraph $\langle E - F \rangle$ of $C_4 \times C_m$ is disconnected. Thus $F$ is a split edge dominating set of $C_4 \times C_m$.

\[ \gamma'_s[C_4 \times C_m] \leq |F|, \]
\[ \gamma'_s[C_4 \times C_m] \leq |S_1| + |S_2| + |S_s|. \]

**Case(ii)** If $m$ be an even number.

Let 
\[ S_3 = \left\{ \{(4,2k),(4,2k+1)\} : k = 1, 2, \cdots \left\lfloor \frac{m}{2} \right\rfloor \right\} \cup \{(3,m),(4,m)\}. \]

Then $F_1 = S_1 \cup S_3 \cup S_s$ is an edge dominating set and the induced subgraph $\langle E - F_1 \rangle$ of $C_4 \times C_m$ is disconnected. Thus $F_1$ is a split edge dominating set of $C_4 \times C_m$.

\[ \gamma'_s[C_4 \times C_m] \leq |F_1|, \]
\[ \gamma'_s[C_4 \times C_m] \leq |S_1| + |S_3| + |S_s|. \]

Hence the proof.

**Theorem 6.6** : Let $P_3$ be a path with 3 vertices and $C_m$ be any cycle with $m \geq 3$. Then split edge domination number of the cartesian product, 
\[ \gamma'_s[P_3 \times C_m] = m + 2. \]
Proof: Consider an edge set
\[ S_s = \left\{ \{(1,1), (2,1)\}, \{(1,2), (2,2)\}, \{(1,1), (1,3)\}, \{(1,2), (3,3)\} \right\}. \]

Let us discuss the following cases:

Case (i) If \( m \equiv 0 \pmod{3} \).

Then \( F = S_1 \cup S_s \) is an edge dominating set and the induced subgraph \( \langle E - F \rangle \) of \( P_3 \times C_m \) is disconnected. Thus \( F \) is a split edge dominating set of \( P_3 \times C_m \) with minimum cardinality. Therefore \( \gamma_s'[P_3 \times C_m] = |S_1| + |S_s|. \)

Case (ii) If \( m \equiv 1 \pmod{3} \).

Let \( F_1 = S_1 \cup S_s \cup \{(2, m), (3, m)\} \) is an edge dominating set and the induced subgraph \( \langle E - F_1 \rangle \) is disconnected. Thus \( F_1 \) is a split edge dominating set of \( P_3 \times C_m \) with minimum cardinality. Therefore \( \gamma_s'[P_3 \times C_m] = |S_1| + |S_s| + 1. \)

Case (iii) If \( m \equiv 2 \pmod{3} \).

Let \( F_2 = S_1 \cup S_s \cup \{(2, m - 2), (2, m - 1)\}, \{(3, m - 1), (3, m)\} \) is an edge dominating set and the induced subgraph \( \langle E - F_2 \rangle \) is disconnected. Thus \( F_2 \) is a split edge dominating set of \( P_3 \times C_m \) with minimum cardinality. Therefore \( \gamma_s'[P_3 \times C_m] = |S_1| + |S_s| + 2. \)

Hence the proof.

Theorem 6.7: Let \( P_n \) be a path of length 2 and \( K_m \) be any complete graph with \( m \geq 3 \). Then split edge domination number of the cartesian product,

\[ \gamma_s'[P_3 \times K_m] \leq \begin{cases} 3m - 3 & \text{if } m \text{ is an even number,} \\ 3m - 4 & \text{if } m \text{ is an odd number.} \end{cases} \]

Proof: Consider an edge set
\[ S_s = \left\{ \{(1,1), (2,1)\}, \{(1,2), (2,2)\} \right\} \cup \left\{ \{(1,1), (i,1)\}, \{(1,2), (i,1)\} : i = 3, 4, \cdots m. \right\}. \]

Let us discuss the following cases:
Case (i) If $m$ is an even number.

Let $S_1 = \left\{ \left\{ (2, 2k + 3), (2, 2k + 4) \right\} : k = 0, 1, \ldots, \left\lfloor \frac{m}{2} \right\rfloor - 2 \right\} \cup \left\{ \left\{ (3, 2k + 1), (3, 2k + 2) \right\} : k = 0, 1, \ldots, \left\lfloor \frac{m}{2} \right\rfloor - 1 \right\}$. Then $S_1 \cup S_s$ is an edge dominating set and the induced subgraph $\langle E - S_1 \cup S_s \rangle$ of $P_3 \times K_m$ is disconnected. Thus $S_1 \cup S_s$ is a split edge dominating set of $P_3 \times K_m$ with minimum cardinality. Therefore $\gamma'_s[P_3 \times K_m] = |S_1| + |S_s|$.

Case (ii) If $m$ is an odd number.

Let $S_2 = \left\{ \left\{ (2, 2k + 4), (2, 2k + 5) \right\} : k = 0, 1, \ldots, \left\lfloor \frac{m}{2} \right\rfloor - 2 \right\} \cup \left\{ \left\{ (3, 2k + 1), (3, 2k + 2) \right\} : k = 0, 1, \ldots, \left\lfloor \frac{m}{2} \right\rfloor - 1 \right\}$. Then $S_2 \cup S_s$ is an edge dominating set and the induced subgraph $\langle E - (S_2 \cup S_s) \rangle$ of $P_3 \times K_m$ is disconnected. Thus $S_2 \cup S_s$ is a split edge dominating set of $P_3 \times K_m$ with minimum cardinality. Therefore $\gamma'_s[P_3 \times K_m] = |S_2| + |S_s|$.

Hence the proof. □

References