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ITERATED ENTIRE FUNCTIONS WITH FINITE ITERATED ORDER SHARING THREE VALUES IM

DIBYENDU BANERJEE[1] AND BISWAJIT MANDAL[2]
[1] Department of Mathematics,
Visva-Bharati, Santiniketan-731235, India
[2] Bunia Dangal High School, Bunia,
Labpur-731303, India

Abstract

Considering the relative iterations of entire functions we investigate growth of iterated entire functions of finite iterated order sharing three values IM.

1. Introduction

If $f(z)$ be an entire function, in [7] Kinnunen introduced the notion of iterated order and iterated lower order as follows.

The iterated i order $\rho_i(f)$ of an entire function f is defined by

$$\rho_i(f) = \overline{\lim_{r\to\infty}} \frac{\log^{[i+1]} M(r,f)}{\log r} = \overline{\lim_{r\to\infty}} \frac{\log^{[i]} T(r,f)}{\log r} (i \in \mathbb{N}).$$

Similarly, the iterated i lower order $\mu_i(f)$ of f is defined by

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$$\mu_i(f) = \underline{\lim}_{r\to\infty} \frac{\log^{[i+1]} M(r,f)}{\log r} = \underline{\lim}_{r\to\infty} \frac{\log^{[i]} T(r,f)}{\log r} (i \in \mathbb{N}).$$

Further the finiteness degree of the order of an entire function f is defined by {[2], [7]}

$$i(f) = \begin{cases} 0 & \text{if } f(z) \text{ is a polynomial;} \\ \min\{k \in \{1,2,3,\ldots\}, \rho_k(f) < \infty\} & \text{if } f(z) \text{ is transcendental;} \\ \infty & \text{when } \rho_k(f) = \infty \text{ for all } k. \end{cases}$$

With this definition in 2009, Jin Tu et al., [10] investigated the growth of two composite entire functions of finite iterated order and achieved some great results.

For two entire functions $f(z)$ and $g(z)$, in [8] Lahiri and Banerjee introduced the idea of relative iterations of $f(z)$ with respect to $g(z)$ as follows.

$$\begin{aligned} f_1(z) &= f(z) \\ f_2(z) &= f(g(z)) \\ f_3(z) &= f(g(f(z))) = f_2(f(z)) = f_1(g_2(z)) \\ &\vdots \\ f_n(z) &= f_k(f_{n-k}(z) \text{ or } g_{n-k}(z) \end{aligned}$$

according as k is even or odd) and so are $g_n(z)$.

Recently Banerjee and Mondal [1] introduced a new type of iteration, called generalised iteration which runs as follows.

Let $f(z)$ and $g(z)$ be two entire functions and $\alpha \in (0,1]$ be any number. Then the generalised iteration of $f(z)$ with respect to $g(z)$ is defined as follows.

$$\begin{aligned} f_{1,g}(z) &= (1-\alpha)g_{0,f}(z) + \alpha f(g_{0,f}(z)), \quad \text{where} \quad g_{0,f}(z) = z \\ f_{2,g}(z) &= (1-\alpha)g_{1,f}(z) + \alpha f(g_{1,f}(z)) \\ f_{3,g}(z) &= (1-\alpha)g_{2,f}(z) + \alpha f(g_{2,f}(z)) \\ &\vdots \\ f_{n,g}(z) &= (1-\alpha)g_{n-1,f}(z) + \alpha f(g_{n-1,f}(z)) \end{aligned}$$

and $g_{n,f}(z)$ are defined similarly.

If f and g be two non-constant meromorphic functions and a be a finite value, we say f and g share the value a CM (counting multiplicities) or IM (ignoring multipltcities) provided $f-a$ and $g-a$ have the same zeros with same multiplicities or same zeros ignoring multiplicities and f, g share ∞ CM or IM provided that $\frac{1}{f}$ and $\frac{1}{g}$ share 0 CM or IM.

In 1979, Gundersen [5] proved that if f and g share three values IM then

$$\frac{1}{3}T(r,g)(1+o(1)) \leq T(r,f) \leq 3T(r,g)(1+o(1)) \text{ as } r \to \infty \ (r \notin E).$$

Brosch [3] improved the result by proving that if f and g share three values CM then

$$\frac{3}{8}T(r,g)(1+o(1)) \leq T(r,f) \leq \frac{8}{3}T(r,g)(1+o(1)) \text{ as } r \to \infty \ (r \notin E).$$

It is clear that if f and g share three values (IM or CM) then their orders are same. In this paper we study growth of iterated entire functions of finite iterated orders sharing three values IM.

We use the standard notations and definitions of Nevanlinna's theory available in [6]. Also we mean that $f(z)$ and $g(z)$ are entire functions having finite iterated order if $\rho_p(f) < \infty, \rho_q(g) < \infty$ and positive iterated lower order if $\mu_p(f) > 0, \mu_q(g) > 0$ for some $p, q \in \mathbb{N}$.

2. Known Lemmas

The following lemmas will be needed in the sequel.

Lemma 2.1 [9] : Let $f(z)$ and $g(z)$ be entire functions. If $M(r,g) > \frac{2+\varepsilon}{\varepsilon}|g(0)|$ for any $\varepsilon > 0$, then

$$T(r, f(g)) < (1+\varepsilon)T(M(r,g), f).$$

Lemma 2.2 [4] : Let $f(z)$ and $g(z)$ be entire functions with $g(0) = 0$. Let α satisfy $0 < \alpha < 1$ and let $C(\alpha) = \frac{(1-\alpha)^2}{4\alpha}$. Then for $r > 0$

$$M(r, f(g)) \geq M(C(\alpha)M(\alpha r, g), f).$$

Further if $g(z)$ is any entire function, then with $\alpha = \frac{1}{2}$, for sufficiently large r

$$M(r, f(g)) \geq M(\frac{1}{8}M(\frac{r}{2}, g) - |g(0)|, f).$$

Clearly, $M(r, f(g)) \geq M(\frac{1}{9}M(\frac{r}{2}, g), f)$.

Lemma 2.3 [6] : Let $f(z)$ and $g(z)$ be transcendental entire functions. Then

$$\frac{T(r,f)}{T(r,g(f))} \to 0 \ \text{ as } \ r \to \infty.$$

So it can be easily seen that $\frac{\log M(r,f)}{\log M(r,g(f))} \to 0$ as $r \to \infty$.

Lemma 2.4 [6] : If f be an entire function, then for all large r

$$T(r,f) \leq \log M(r,f) \leq 3T(2r,f).$$

3. Main results

Our main results are the following theorems.

Theorem 3.1 : Let $f(z)$ and $g(z)$ be entire functions of finite iterated order and positive iterated lower order with $i(f) \neq i(g)$. If $f_k(z)$ and $g_k(z)$, $k\ (\geq 2)$ share three values IM, then for any $n\ (\geq k)$

(I) $\rho_{mp}(f_n) = \rho_{mp}(g_n)$, if $n = mk$;

(II) $\rho_{mp+i(g_q)}(f_n) - \rho_{mp+i(f_q)}(g_n) = \rho_{i(g_q)}(g_q) - \rho_{i(f_q)}(f_q)$, if $n = mk + q$ and mk is odd and $\rho_{mp+i(f_q)}(f_n) - \rho_{mp+i(g_q)}(g_n) = \rho_{i(f_q)}(f_q) - \rho_{i(g_q)}(g_q)$, if $n = mk + q$ and mk is even, where $p = i(f_k)$ and m, q $(1 \leq q < n) \in \mathbb{N}$.

Proof : Since $f_k(z)$ and $g_k(z)$ share three values IM, we have

$$\frac{1}{3}(1 + o(1))T(r, g_k) \leq T(r, f_k) \leq 3(1 + o(1))T(r, g_k).$$

So there exists $p \in \mathbb{N}$ such that

$$\rho_p(g_k) \leq \rho_p(f_k) \leq \rho_p(g_k) < \infty$$

i.e.,

$$\rho_p(f_k) = \rho_p(g_k). \tag{3.1}$$

Also we have for given $\varepsilon\ (> 0)$ and for sufficiently large r

$$T(r, f_k) \leq \exp^{[p-1]}\{r^{\rho_p(f_k)+\varepsilon}\}, M(r, f_k) \leq \exp^{[p]}\{r^{\rho_p(f_k)+\varepsilon}\}. \tag{3.2}$$

(I) Suppose $n = mk$. Then we have the following cases.

Case (i) : Let k be even. For sufficiently large r, we have by using Lemma 2.1 and (3.2)

$$\begin{aligned}
T(r,f_n) &\leq 2T(M(r,f_{(m-1)k}),f_k) \\
&\leq 2\exp^{[p]}\{(\rho_p(f_k)+\varepsilon)\log M(r,f_{(m-1)k})\} \\
&\leq \exp^{[p]}[\log\{\exp^{[p]}(M(r,f_{(m-2)k}))^{\rho_p(f_k)+2\varepsilon}\}] \\
&\quad \vdots \\
&\leq \exp^{[(m-1)p]}[(\rho_p(f_k)+2\varepsilon)\log\{M(r,f_k)\}] \\
&\leq \exp^{[mp]}\{\log(r^{\rho_p(f_k)+2\varepsilon})\}. \qquad (3.3)
\end{aligned}$$

Therefore, for all large r

$$\frac{\log^{[mp]} T(r,f_n)}{\log r} \leq \rho_p(f_k)+2\varepsilon. \qquad (3.4)$$

On the other hand, we have

$$\overline{\lim_{r\to\infty}} \frac{\log^{[p+1]} M(r,f_k)}{\log r} = \rho_p(f_k).$$

Since $\rho_p(f_k) > 0$, there exists a sequence $\{r_l\}$ tending to infinity such that for given ε $(0 < \varepsilon < \mu_p(f_k))$ and for sufficiently large r_l, we have

$$M(r_l,f_k) \geq \exp^{[p]}\{r_l^{\rho_p(f_k)-\varepsilon}\}.$$

So for sufficiently large r_l, we have by using Lemma 2.2 and Lemma 2.4

$$\begin{aligned}
T(r_l, f_n) &\geq \frac{1}{3}\log M(\frac{1}{9}M(\frac{r_l}{2^2}, f_{(m-1)k}), f_k) \\
&> \frac{1}{3}\log\{\exp^{[p]}(\frac{1}{9}M(\frac{r_l}{2^2}, f_{(m-1)k}))^{\mu_p(f_k)-\varepsilon}\} \\
&\geq \exp^{[p]}\{(\mu_p(f_k)-2\varepsilon)\log M(\frac{1}{9}M(\frac{r_l}{2^3}, f_{(m-2)k}), f_k)\} \\
&\geq \exp^{[p]}[\log\{\exp^{[p]}(M(\frac{r_l}{2^3}, f_{(m-2)k}))^{\mu_p(f_k)-2\varepsilon}\}] \\
&\geq \exp^{[2p]}\{(\mu_p(f_k)-2\varepsilon)\log M(\frac{r_l}{2^3}, f_{(m-2)k})\} \\
&\vdots \\
&\geq \exp^{[(m-1)p]}\{(\mu_p(f_k)-2\varepsilon)\log M(\frac{r_l}{2^m}, f_k)\} \qquad (3.5) \\
&\geq \exp^{[(m-1)p]}[(\mu_p(f_k)-2\varepsilon)\log\{\exp^{[p]}(\frac{r_l}{2^m})^{\rho_p(f_k)-\varepsilon}\}] \\
&\geq \exp^{[mp]}\{\log(r_l)^{\rho_p(f_k)-2\varepsilon}\}. \qquad (3.6)
\end{aligned}$$

Therefore for a sequence $\{r_l\}$ tending to infinity

$$\frac{\log^{[mp]} T(r_l, f_n)}{\log r_l} \geq \rho_p(f_k) - 2\varepsilon. \qquad (3.7)$$

From (3.4) and (3.7), we get

$$\rho_{mp}(f_n) = \rho_p(f_k). \qquad (3.8)$$

Similarly,

$$\rho_{mp}(g_n) = \rho_p(g_k) \qquad (3.9)$$

From (3.1), (3.8) and (3.9), we have

$$\rho_{mp}(f_n) = \rho_{mp}(g_n).$$

Case (ii) : Let k be odd. For sufficiently large r, we have by using Lemma 2.1 and (3.2)

$$\begin{aligned}
T(r, f_n) &\leq 2\exp^{[p]}\{(\rho_p(f_k)+\varepsilon)\log M(r, g_{(m-1)k})\} \\
&\leq 2\exp^{[p]}\{(\rho_p(f_k)+\varepsilon)\log M(r, g_k(f_{(m-2)k}))\} \\
&\leq \exp^{[2p]}\{(\rho_p(g_k)+2\varepsilon)\log M(r, f_{(m-2)k})\} \\
&\leq \exp^{[3p]}\{(\rho_p(f_k)+2\varepsilon)\log M(r, g_{(m-3)k})\} \\
&\leq \exp^{[4p]}\{(\rho_p(g_k)+2\varepsilon)\log M(r, f_{(m-4)k})\}.
\end{aligned}$$

Now if m be odd, for sufficiently large values of r

$$\begin{aligned} T(r,f_n) &\leq \exp^{[(m-1)p]}\{(\rho_p(g_k)+2\varepsilon)\log M(r,f_k)\} \\ &\leq \exp^{[mp]}\{\log(r^{\rho_p(f_k)+2\varepsilon})\} \end{aligned}$$

and if m be even

$$\begin{aligned} T(r,f_n) &\leq \exp^{[(m-1)p]}\{(\rho_p(f_k)+2\varepsilon)\log M(r,g_k)\} \\ &\leq \exp^{[mp]}\{\log(r^{\rho_p(g_k)+2\varepsilon})\}. \end{aligned} \tag{3.10}$$

Therefore for large r

$$\frac{\log^{[mp]} T(r,f_n)}{\log r} \leq \begin{cases} \rho_p(f_k)+2\varepsilon, & \text{if } m \text{ is odd,} \\ \rho_p(g_k)+2\varepsilon, & \text{if } m \text{ is even.} \end{cases} \tag{3.11}$$

For the reverse inequality to obtain similar relation as in (3.6), in this case also there exists a sequence $\{r_l\}$ tending to infinity for which

$$\begin{aligned} T(r_l,f_n) &> \frac{1}{3}\log\{\exp^{[p]}(\frac{1}{9}M(\frac{r_l}{2^2},g_{(m-1)k}))^{\mu_p(f_k)-\varepsilon}\} \\ &\geq \exp^{[p]}\{(\mu_p(f_k)-2\varepsilon)\log M(\frac{r_l}{2^2},g_{(m-1)k})\} \\ &\geq \exp^{[2p]}\{(\mu_p(g_k)-2\varepsilon)\log M(\frac{r_l}{2^3},f_{(m-2)k})\} \\ &\geq \exp^{[3p]}\{(\mu_p(f_k)-2\varepsilon)\log M(\frac{r_l}{2^4},g_{(m-3)k})\}. \end{aligned}$$

Now if m be odd, for a sequence $\{r_l\}$ tending to infinity

$$\begin{aligned} T(r_l,f_n) &\geq \exp^{[(m-1)p]}\{(\mu_p(g_k)-2\varepsilon)\log M(\frac{r_l}{2^m},f_k)\} \\ &\geq \exp^{[mp]}\{\log(r_l)^{\rho_p(f_k)-2\varepsilon}\} \end{aligned}$$

and if m be even

$$\begin{aligned} T(r_l,f_n) &\geq \exp^{[(m-1)p]}\{(\mu_p(f_k)-2\varepsilon)\log M(\tfrac{r_l}{2^m},g_k)\} \\ &\geq \exp^{[mp]}\{\log(r_l)^{\rho_p(g_k)-2\varepsilon}\}. \end{aligned} \tag{3.12}$$

Therefore for a sequence $\{r_l\}$ tending to infinity, we have

$$\frac{\log^{[mp]} T(r_l,f_n)}{\log r_l} \geq \begin{cases} \rho_p(f_k)-2\varepsilon, & \text{if } m \text{ is odd,} \\ \rho_p(g_k)-2\varepsilon, & \text{if } m \text{ is even.} \end{cases} \tag{3.13}$$

From (3.11) and (3.13), we get

$$\rho_{mp}(f_n) = \begin{cases} \rho_p(f_k), & \text{if } m \text{ is odd,} \\ \rho_p(g_k), & \text{if } m \text{ is even.} \end{cases} \tag{3.14}$$

By similar reasoning as above, we have

$$\rho_{mp}(g_n) = \begin{cases} \rho_p(g_k), & \text{if } m \text{ is odd,} \\ \rho_p(f_k), & \text{if } m \text{ is even.} \end{cases} \tag{3.15}$$

From (3.1), (3.14) and (3.15), we have for any m

$$\rho_{mp}(f_n) = \rho_{mp}(g_n).$$

(II) Suppose $n = mk + q$. Then also the following cases arise.

Case (i) : Let mk be odd. Then m and k both are odd. Proceeding similarly as in case (ii) of (I), for sufficiently large r, we have

$$\begin{aligned} T(r, f_n) &\leq \exp^{[4p]}\{(\rho_p(g_k) + 2\varepsilon)\log M(r, f_{(m-4)k+q})\} \\ &\quad \cdots \\ &\leq \exp^{[(m-1)p]}\{(\rho_p(g_k) + 2\varepsilon)\log(M(r, f_{k+q}))\} \\ &\leq \exp^{[(m-1)p]}\{(\rho_p(g_k) + 2\varepsilon)\log(M(M(r, g_q), f_k))\} \\ &\leq \exp^{[mp]}\{(\rho_p(f_k) + 2\varepsilon)\log M(r, g_q)\} \\ &\leq \exp^{[mp]}[(\rho_p(f_k) + 2\varepsilon)\log\{\exp^{[i(g_q)]}(r^{\rho_{i(g_q)}(g_q)+\varepsilon})\}] \\ &\leq \exp^{[mp+i(g_q)]}\{\log(r^{\rho_{i(g_q)}(g_q)+2\varepsilon})\}. \end{aligned}$$

Therefore for large r

$$\frac{\log^{[mp+i(g_q)]} T(r, f_n)}{\log r} \leq \rho_{i(g_q)}(g_q) + 2\varepsilon \tag{3.16}$$

For the reverse inequality as in case (ii) of (I), since $\mu_p(f_k) > 0$ and $\mu_p(g_k) > 0$, in this case also there exists a sequence $\{r_l\}$ tending to infinity for which

$$\begin{aligned} T(r_l, f_n) &\geq \exp^{[3p]}\{(\mu_p(f_k) - 2\varepsilon)\log M(\frac{r_l}{2^4}, g_{(m-3)k+q})\} \\ &\quad \vdots \\ &\geq \exp^{[(m-1)p]}[(\mu_p(g_k) - 2\varepsilon)\log M(\frac{r_l}{2^m}, f_{k+q}) \\ &\geq \exp^{[mp]}\{(\mu_p(f_k) - 2\varepsilon)\log M(\frac{r_l}{2^{m+1}}, g_q)\} \\ &\geq \exp^{[mp]}[(\mu_p(f_k) - 2\varepsilon)\log\{\exp^{[i(g_q)]}(\frac{r_l}{2^{m+1}})^{\rho_{i(g_q)}(g_q)-\varepsilon}\}] \\ &\geq \exp^{[mp+i(g_q)]}\{\log(r_l)^{\rho_{i(g_q)}(g_q)-2\varepsilon}\}. \end{aligned}$$

Therefore, for a sequence $\{r_l\}$ tending to infinity, we have

$$\frac{\log^{[mp+i(g_q)]} T(r_l, f_n)}{\log r_l} \geq \rho_{i(g_q)}(g_q) - 2\varepsilon \tag{3.17}$$

From (3.16) and (3.17), we get

$$\rho_{mp+i(g_q)}(f_n) = \rho_{i(g_q)}(g_q) \tag{3.18}$$

By similar reasoning as above, we have

$$\rho_{mp+i(f_q)}(g_n) = \rho_{i(f_q)}(f_q). \tag{3.19}$$

From (3.18) and (3.19), we have

$$\rho_{mp+i(g_q)}(f_n) - \rho_{mp+i(f_q)}(g_n) = \rho_{i(g_q)}(g_q) - \rho_{i(f_q)}(f_q).$$

Case (ii) : Let mk be even. Then two subcases may arise.
Subcase (a). Let k be even. Using similar reasoning as in case (i) of (I), for sufficiently large r, we have instead of (3.3)

$$\begin{aligned} T(r, f_n) &\leq \exp^{[(m-1)p]}\{(\rho_p(f_k) + 2\varepsilon) \log M(r, f_{k+q})\} \\ &\leq \exp^{[mp]}\{(\rho_p(f_k) + 2\varepsilon) \log M(r, f_q)\} \\ &\leq \exp^{[mp]}[(\rho_p(f_k) + 2\varepsilon) \log\{\exp^{[i(f_q)]}(r^{\rho_{i(f_q)}(f_q)+\varepsilon})\}] \\ &\leq \exp^{[mp+i(f_q)]}\{\log(r^{\rho_{i(f_q)}(f_q)+2\varepsilon})\}. \end{aligned}$$

Therefore for large r

$$\frac{\log^{[mp+i(f_q)]} T(r, f_n)}{\log r} \leq \rho_{i(f_q)}(f_q) + 2\varepsilon. \tag{3.20}$$

For the reverse inequality, in this case also there exists a sequence $\{r_l\}$ tending to infinity for which instead of (3.5) we have

$$\begin{aligned} T(r_l, f_n) &\geq \exp^{[(m-1)p]}\{(\mu_p(f_k) - 2\varepsilon) \log M(\frac{r_l}{2^m}, f_{k+q})\} \\ &\geq \exp^{[mp]}\{(\mu_p(f_k) - 2\varepsilon) \log M(\frac{r_l}{2^{m+1}}, f_q)\} \\ &\geq \exp^{[mp]}[(\mu_p(f_k) - 2\varepsilon) \log\{\exp^{[i(f_q)]}(\frac{r_l}{2^{m+1}})^{\rho_{i(f_q)}(f_q)-\varepsilon}\}] \\ &\geq \exp^{[mp+i(f_q)]}\{\log(r_l)^{\rho_{i(f_q)}(f_q)-2\varepsilon}\}. \end{aligned}$$

Therefore, for a sequence $\{r_l\}$ tending to infinity, we have

$$\frac{\log^{[mp+i(f_q)]} T(r_l, f_n)}{\log r_l} \geq \rho_{i(f_q)}(f_q) - 2\varepsilon. \tag{3.21}$$

From (3.20) and (3.21), we get

$$\rho_{mp+i(f_q)}(f_n) = \rho_{i(f_q)}(f_q). \tag{3.22}$$

By similar reasoning as above, we have

$$\rho_{mp+i(g_q)}(g_n) = \rho_{i(g_q)}(g_q). \tag{3.23}$$

From (3.22) and (3.23), we have

$$\rho_{mp+i(f_q)}(f_n) - \rho_{mp+i(g_q)}(g_n) = \rho_{i(f_q)}(f_q) - \rho_{i(g_q)}(g_q).$$

Subcase (b). Let k be odd. Using similar reasoning as in case (ii) of (I), for sufficiently large r instead of (3.10), we have

$$\begin{aligned} T(r, f_n) &\leq \exp^{[(m-1)p]}\{(\rho_p(f_k) + 2\varepsilon)\log M(r, g_{k+q})\} \\ &\leq \exp^{[mp]}\{(\rho_p(g_k) + 2\varepsilon)\log M(r, f_q)\} \\ &\leq \exp^{[mp]}[(\rho_p(g_k) + 2\varepsilon)\log\{\exp^{[i(f_q)]}(r^{\rho_{i(f_q)}(f_q)+\varepsilon})\}] \\ &\leq \exp^{[mp+i(f_q)]}\{\log(r^{\rho_{i(f_q)}(f_q)+2\varepsilon})\}. \end{aligned}$$

Therefore for large r

$$\frac{\log^{[mp+i(f_q)]} T(r, f_n)}{\log r} \leq \rho_{i(f_q)}(f_q) + 2\varepsilon. \tag{3.24}$$

For the reverse inequality using similar reasoning as in case (ii) of (I), there exists a sequence $\{r_l\}$ tending to infinity for which instead of (3.12), we have

$$\begin{aligned} T(r_l, f_n) &\geq \exp^{[(m-1)p]}\{(\mu_p(f_k) - 2\varepsilon)\log M(\frac{r_l}{2^m}, g_{k+q})\} \\ &\geq \exp^{[mp]}\{(\mu_p(g_k) - 2\varepsilon)\log M(\frac{r_l}{2^{m+1}}, f_q)\} \\ &\geq \exp^{[mp+i(f_q)]}\{\log(r_l)^{\rho_{i(f_q)}(f_q)-2\varepsilon}\}. \end{aligned}$$

Therefore, for a sequence $\{r_l\}$ tending to infinity, we have

$$\frac{\log^{[mp+i(f_q)]} T(r_l, f_n)}{\log r_l} \geq \rho_{i(f_q)}(f_q) - 2\varepsilon. \tag{3.25}$$

From (3.24) and (3.25), we have

$$\rho_{mp+i(f_q)}(f_n) = \rho_{i(f_q)}(f_q). \tag{3.26}$$

By similar reasoning as above, we have

$$\rho_{mp+i(g_q)}(g_n) = \rho_{i(g_q)}(g_q). \tag{3.27}$$

From (3.26) and (3.27), we have

$$\rho_{mp+i(f_q)}(f_n) - \rho_{mp+i(g_q)}(g_n) = \rho_{i(f_q)}(f_q) - \rho_{i(g_q)}(g_q).$$

Example 3.1 : Let $f(z) = e^z$ and $g(z) = z^2$. Then $i(f) = 1 \neq 0 = i(g)$ and $i(f_3) = 2$. Here $f_3(z)$ and $g_3(z)$ do not share three values and $\rho_{2i(f_3)}(f_6) \neq \rho_{2i(g_3)}(g_6)$.

Theorem 3.2 : Let $f(z)$ and $g(z)$ be entire functions of finite iterated order and positive iterated lower order with $i(f) \neq i(g)$. If $f_{k,g}(z)$ and $g_{k,f}(z)$, k (≥ 2) share three values IM, then for any n $(\geq k)$

(I) $\rho_{\frac{(n+1)-k}{2}p+\frac{(n-1)-k}{2}q+s}(f_{n,g}) = \rho_{\frac{(n+1)-k}{2}q+\frac{(n-1)-k}{2}p+s}(g_{n,f})$, if $n-k$ is odd;

and (II) $\rho_{\frac{n-k}{2}(p+q)+s}(f_{n,g}) = \rho_{\frac{n-k}{2}(p+q)+s}(g_{n,f})$, if $n-k$ is even,

where $p = i(f)$, $q = i(g)$ and $s = i(f_{k,g})$.

Proof : Since $f_{k,g}(z)$ and $g_{k,f}(z)$ share three values IM, as in (3.1), we have

$$\rho_s(f_{k,g}) = \rho_s(g_{k,f}), \tag{3.28}$$

for some $s \in \mathbb{N}$.

Also we have for given ε (> 0) and for sufficiently large r

$$T(r, f_{k,g}) \leq \exp^{[s-1]}\{r^{\rho_s(f_{k,g})+\varepsilon}\}, M(r, f_{k,g}) \leq \exp^{[s]}\{r^{\rho_s(f_{k,g})+\varepsilon}\}. \tag{3.29}$$

Using Lemma (2.1), Lemma (2.3) and (3.29) for sufficiently large r, we have

$$\begin{aligned} T(r,f_{n,g}) &\leq T(r,g_{n-1,f})+T(r,f(g_{n-1,f}))+O(1) \\ &= T(r,f(g_{n-1,f}))[1+\frac{T(r,g_{n-1,f})}{T(r,f(g_{n-1,f}))}+\frac{O(1)}{T(r,f(g_{n-1,f}))}] \\ &\leq (1+o(1))T(M(r,g_{n-1,f}),f) \\ &= \exp^{[p]}\{(\rho_p(f)+2\varepsilon)\log M(r,g_{n-1,f})\} \\ &\leq \exp^{[p]}[(\rho_p(f)+2\varepsilon)\{\log M(r,f_{n-2,g})+\log M(r,g(f_{n-2,g}))+O(1)\}] \\ &\leq \exp^{[p]}\{(\rho_p(f)+2\varepsilon)(1+o(1))\log M(M(r,f_{n-2,g}),g)\} \\ &\leq \exp^{[p+q]}\{(\rho_q(g)+2\varepsilon)\log M(r,f_{n-2,g})\} \\ &\leq \exp^{[2p+q]}\{(\rho_p(f)+2\varepsilon)\log M(r,g_{n-3,f})\} \\ &\leq \exp^{[2p+2q]}\{(\rho_q(g)+2\varepsilon)\log M(r,f_{n-4,g})\}. \end{aligned} \tag{3.30}$$

Now two cases may arise.

Case (i) : When $(n-k)$ is odd. Using (3.29), for sufficiently large r, we have from (3.30)

$$\begin{aligned} T(r,f_{n,g}) &\leq \exp^{[\frac{(n+1)-k}{2}p+\frac{(n-1)-k}{2}q]}\{(\rho_p(f)+2\varepsilon)\log M(r,g_{k,f})\} \\ &\leq \exp^{[\frac{(n+1)-k}{2}p+\frac{(n-1)-k}{2}q]}[\log\{\exp^{[s]}(r^{\rho_s(g_{k,f})+2\varepsilon})\}] \\ &\leq \exp^{[\frac{(n+1)-k}{2}p+\frac{(n-1)-k}{2}q+s]}\{\log(r^{\rho_s(g_{k,f})+2\varepsilon})\}. \end{aligned}$$

Therefore for large r

$$\frac{\log^{[\frac{(n+1)-k}{2}p+\frac{(n-1)-k}{2}q+s]}T(r,f_{n,g})}{\log r}\leq\rho_s(g_{k,f})+2\varepsilon. \tag{3.31}$$

On the otherhand, since $i(g_{k,f})=s$, we have

$$\limsup_{r\to\infty}\frac{\log^{[s+1]}M(r,g_{k,f})}{\log r}=\rho_s(g_{k,f}).$$

Since $\rho_s(g_{k,f})>0$, there exists a sequence $\{r_l\}$ tending to infinity such that for given ε $(0<\varepsilon<\mu_s(g_{k,f}))$ we have

$$M(r_l,g_{k,f})\geq\exp^{[s]}\{r_l^{\rho_s(g_{k,f})-\varepsilon}\}.$$

So using Lemma 2.2, Lemma 2.3 and Lemma 2.4, we have for sufficiently large r_l

$$\begin{aligned}
T(r_l, f_{n,g}) &\geq T(r_l, f(g_{n-1,f})) - T(r_l, g_{n-1,f}) + O(1) \\
&= (1+o(1))T(r_l, f(g_{n-1,f})) \\
&\geq \frac{1}{3}(1+o(1))\log M(\frac{r_l}{2}, f(g_{n-1,f})) \\
&\geq \frac{1}{3}(1+o(1))\log M(\frac{1}{9}M(\frac{r_l}{2^2}, g_{n-1,f}), f) \\
&\geq \exp^{[p]}[\log\{M(\frac{r_l}{2^2}, g_{n-1,f})\}^{\mu_p(f)-2\varepsilon}] \\
&= \exp^{[p]}\{(\mu_p(f)-2\varepsilon)(1+o(1))\log M(\frac{r_l}{2^2}, g(f_{n-2,g}))\} \\
&\geq \exp^{[p]}[(\mu_p(f)-2\varepsilon)(1+o(1))\log\{\exp^{[q]}(\frac{1}{9}M(\frac{r_l}{2^3}, f_{n-2,g}))^{\mu_q(g)-\varepsilon}\}] \\
&\geq \exp^{[p+q]}\{(\mu_q(g)-2\varepsilon)\log M(\frac{r_l}{2^3}, f_{n-2,g})\} \\
&\geq \exp^{[2p+q]}\{(\mu_p(f)-2\varepsilon)\log M(\frac{r_l}{2^4}, g_{n-3,f})\} \\
&\geq \exp^{[2p+2q]}\{(\mu_q(g)-2\varepsilon)\log M(\frac{r_l}{2^5}, f_{n-4,g})\} \qquad (3.32) \\
&\vdots \\
&\geq \exp^{[\frac{(n+1)-k}{2}p+\frac{(n-1)-k}{2}q]}\{(\mu_p(f)-2\varepsilon)\log M(\frac{r_l}{2^{n-k+1}}, g_{k,f})\} \\
&\geq \exp^{[\frac{(n+1)-k}{2}p+\frac{(n-1)-k}{2}q]}[(\mu_p(f)-2\varepsilon)\log\{\exp^{[s]}(\frac{r_l}{2^{n-k+1}})^{\rho_s(g_{k,f})-\varepsilon}\}] \\
&\geq \exp^{[\frac{(n+1)-k}{2}p+\frac{(n-1)-k}{2}q+s]}\{\log(r_l)^{\rho_s(g_{k,f})-2\varepsilon}\}.
\end{aligned}$$

Therefore for a sequence $\{r_l\}$ tending to infinity

$$\frac{\log^{[\frac{(n+1)-k}{2}p+\frac{(n-1)-k}{2}q+s]} T(r_l, f_{n,g})}{\log r_l} \geq \rho_s(g_{k,f}) - 2\varepsilon. \qquad (3.33)$$

From (3.31) and (3.33), we get

$$\rho_{\frac{(n+1)-k}{2}p+\frac{(n-1)-k}{2}q+s}(f_{n,g}) = \rho_s(g_{k,f}). \qquad (3.34)$$

By similar reasoning as above, we have

$$\rho_{\frac{(n+1)-k}{2}q+\frac{(n-1)-k}{2}p+s}(g_{n,f}) = \rho_s(f_{k,g}). \qquad (3.35)$$

From (3.28), (3.34) and (3.35), we have

$$\rho_{\frac{(n+1)-k}{2}p+\frac{(n-1)-k}{2}q+s}(f_{n,g}) = \rho_{\frac{(n+1)-k}{2}q+\frac{(n-1)-k}{2}p+s}(g_{n,f}).$$

Case (ii) : When $n-k$ is even. For sufficiently large r, we have from (3.30)

$$\begin{aligned} T(r, f_{n,g}) &\leq \exp^{[\frac{n-k}{2}(p+q)]}\{(\rho_q(g)+2\varepsilon)\log M(r, f_{k,g})\} \\ &\leq \exp^{[\frac{n-k}{2}(p+q)]}[\log\{\exp^{[s]}(r^{\rho_s(f_{k,g})+2\varepsilon})\}] \\ &\leq \exp^{[\frac{n-k}{2}(p+q)+s]}\{\log(r^{\rho_s(f_{k,g})+2\varepsilon})\}. \end{aligned}$$

Therefore, for large r

$$\frac{\log^{[\frac{n-k}{2}(p+q)+s]} T(r, f_{n,g})}{\log r} \leq \rho_s(f_{k,g}) + 2\varepsilon. \tag{3.36}$$

For reverse inequality, to obtain similar relation as in (3.33), there also exists a sequence $\{r_l\}$ tending to infinity such that from (3.32)

$$\begin{aligned} T(r_l, f_{n,g}) &\geq \exp^{[\frac{n-k}{2}(p+q)]}\{(\mu_q(g)-2\varepsilon)\log M(\frac{r_l}{2^{n-k+1}}, f_{k,g})\} \\ &\geq \exp^{[\frac{n-k}{2}(p+q)]}[(\mu_q(g)-2\varepsilon)\log\{\exp^{[s]}(\frac{r_l}{2^{n-k+1}})^{\rho_s(f_{k,g})-\varepsilon}\}] \\ &\geq \exp^{[\frac{n-k}{2}(p+q)+s]}\{\log(r_l)^{\rho_s(f_{k,g})-2\varepsilon}\}. \end{aligned}$$

Therefore for a sequence $\{r_l\}$ tending to infinity

$$\frac{\log^{[\frac{n-k}{2}(p+q)+s]} T(r, f_{n,g})}{\log r_l} \geq \rho_s(f_{k,g}) - 2\varepsilon. \tag{3.37}$$

From (3.36) and (3.37), we get

$$\rho_{\frac{n-k}{2}(p+q)+s}(f_{n,g}) = \rho_s(f_{k,g}). \tag{3.38}$$

By similar reasoning as above, we have

$$\rho_{\frac{n-k}{2}(p+q)+s}(g_{n,f}) = \rho_s(g_{k,f}). \tag{3.39}$$

From (3.28), (3.38) and (3.39), we have

$$\rho_{\frac{n-k}{2}(p+q)+s}(f_{n,g}) = \rho_{\frac{n-k}{2}(p+q)+s}(g_{n,f}).$$

Note 3.1 : It is clear from Theorem 3.2 that when $(n-k)$ is odd, $i(f_{n,g}) \sim i(g_{n,f}) = i(f) \sim i(g)$; where as when $(n-k)$ is even, $i(f_{n,g}) = i(g_{n,f})$.

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