BOUNDARY $n$-SIGNED GRAPHS

P. SIVA KOTA REDDY$^1$, R. RAJENDRA$^2$ AND M. C. GEETHA$^3$

1 Department of Mathematics, Siddaganga Institute of Technology, Tumkur-572 103, India
2 Department of Mathematics, Field Marshal K.M. Cariappa College
   (a constituent college of Mangalore University)
   Madikeri-571 201, India
3 Department of Mathematics, East West Institute of Technology, Bangalore-560 091, India

Abstract
In this paper, we extended the notion in the graphs called boundary graph to $n$-signed graphs and then we proved boundary $n$-signed graph is always identity balanced for given any $n$-signed graph. Further, we proved several switching equivalence characterizations and structural characterization for boundary $n$-signed graphs.

1. Introduction
Unless mentioned or defined otherwise, for all terminology and notion in graph theory the reader is refer to [3]. We consider only finite, simple graphs free from self-loops.

Key Words and Phrases : $n$-Signed graphs, $n$-Marked graphs, Balance, Switching, Balance, Boundary $n$-signed graphs, Complementation.

2000 AMS Subject Classification : 05C 22.

© http: //www.ascent-journals.com

161
Let \( n \geq 1 \) be an integer. An \( n \)-tuple \((a_1, a_2, ..., a_n)\) is symmetric, if \( a_k = a_{n-k+1}, 1 \leq k \leq n \). Let \( H_n = \{(a_1, a_2, ..., a_n) : a_k \in \{+,-\}, a_k = a_{n-k+1}, 1 \leq k \leq n\} \) be the set of all symmetric \( n \)-tuples. Note that \( H_n \) is a group under coordinate wise multiplication, and the order of \( H_n \) is \( 2^m \), where \( m = \lceil \frac{n}{2} \rceil \).

A symmetric \( n \)-sigraph (symmetric \( n \)-marked graph) is an ordered pair \( \Sigma_n = (\Gamma, \sigma) \) (\( \Sigma_n = (\Gamma, \mu) \)), where \( \Gamma = (V, E) \) is a graph called the underlying graph of \( \Sigma_n \) and \( \sigma : E \to H_n \) (\( \mu : V \to H_n \)) is a function.

In this paper by an \( n \)-tuple/\( n \)-signed graph/\( n \)-marked graph we always mean a symmetric \( n \)-tuple/symmetric \( n \)-sigraph/symmetric \( n \)-marked graph.

An \( n \)-tuple \((a_1, a_2, ..., a_n)\) is the identity \( n \)-tuple, if \( a_k = + \), for \( 1 \leq k \leq n \), otherwise it is a non-identity \( n \)-tuple. In an \( n \)-signed graph \( \Sigma_n = (\Gamma, \sigma) \) an edge labelled with the identity \( n \)-tuple is called an identity edge, otherwise it is a non-identity edge.

Further, in an \( n \)-signed graph \( \Sigma_n = (\Gamma, \sigma) \), for any \( A \subseteq E(\Gamma) \) the \( n \)-tuple \( \sigma(A) \) is the product of the \( n \)-tuples on the edges of \( A \).

In [9], the authors defined two notions of balance in \( n \)-signed graph \( \Sigma_n = (\Gamma, \sigma) \) as follows (See also R. Rangarajan and P.S.K.Reddy [6] :

**Definition** : Let \( \Sigma_n = (\Gamma, \sigma) \) be an \( n \)-signed graph. Then,

(i) \( \Sigma_n \) is identity balanced (or \( i \)-balanced), if product of \( n \)-tuples on each cycle of \( \Sigma_n \) is the identity \( n \)-tuple, and

(ii) \( \Sigma_n \) is balanced, if every cycle in \( \Sigma_n \) contains an even number of non-identity edges.

**Note** : An \( i \)-balanced \( n \)-signed graph need not be balanced and conversely.

The following characterization of \( i \)-balanced \( n \)-signed graphs is obtained in [9].

**Theorem 1** : An \( n \)-signed graph \( \Sigma_n = (\Gamma, \sigma) \) is \( i \)-balanced if, and only if, it is possible to assign \( n \)-tuples to its vertices such that the \( n \)-tuple of each edge \( uv \) is equal to the product of the \( n \)-tuples of \( u \) and \( v \).

Let \( \Sigma_n = (\Gamma, \sigma) \) be an \( n \)-signed graph. Consider the \( n \)-marking \( \mu \) on vertices of \( \Sigma_n \) defined as follows: each vertex \( v \in V, \mu(v) \) is the \( n \)-tuple which is the product of the \( n \)-tuples on the edges incident with \( v \). **Complement** of \( \Sigma_n \) is an \( n \)-signed graph \( \Sigma^c_n = (\Gamma, \sigma^c) \), where for any edge \( e = uv \in \Gamma, \sigma^c(uv) = \mu(u)\mu(v) \). Clearly, \( \Sigma^c_n \) as defined here is an \( i \)-balanced \( n \)-signed graph due to Theorem 1.
In [9], the authors also have defined switching and cycle isomorphism of an \( n \)-signed graph \( \Sigma_n = (\Gamma, \sigma) \) as follows: (See also ([4], [7], [8]) and ([11]-[17]).

Let \( \Sigma_n = (\Gamma, \sigma) \) and \( \Sigma'_n = (\Gamma', \sigma') \), be two \( n \)-signed graphs. Then \( \Sigma_n \) and \( \Sigma'_n \) are said to be isomorphic, if there exists an isomorphism \( \phi : \Gamma \rightarrow \Gamma' \) such that if \( uv \) is an edge in \( \Sigma_n \) with label \((a_1, a_2, \ldots, a_n)\) then \( \phi(u)\phi(v) \) is an edge in \( \Sigma'_n \) with label \((a_1, a_2, \ldots, a_n)\).

Let \( \Sigma_n = (\Gamma, \sigma) \) be an \( n \)-signed graph, switching of \( n \)-signed graph with respect to the \( n \)-marking \( \mu \) is the replacing the \( n \)-tuple on each edge \( e = uv \in E(\Sigma_n) \) as the product of \( n \)-tuple of \( u \), \( e = uv \) and \( v \) (i.e, \( \mu(u)\sigma(uv)\mu(v) \)). The resulting \( n \)-signed graph is denoted by \( (\Sigma_n)_\mu \) and is called the switched \( n \)-signed graph. Two \( n \)-signed graphs \( (\Sigma_n)_1 = (\Gamma_2, \sigma) \) and \( (\Sigma_n)_2 = (\Gamma_2, \sigma') \) are said to be switching equivalent and is denoted by \( (\Sigma_n)_1 \sim (\Sigma_n)_2 \), if \( ((\Sigma_n)_1)_\mu \cong (\Sigma_n)_2 \).

Let \( (\Sigma_n)_1 = (\Gamma_1, \sigma) \) and \( (\Sigma_n)_2 = (\Gamma_2, \sigma') \) be two \( n \)-signed graphs with \( \Gamma_1 \cong \Gamma_2 \). The \( n \)-tuple each cycle in \( (\Sigma_n)_1 \) is same as the \( n \)-tuple each cycle in \( (\Sigma_n)_1 \), then the above two \( n \)-signed graphs (with their underlying graphs are isomorphic) are said to cycle isomorphic. We make use of the following known result (see [9]):

**Theorem 2**: Given a graph \( \Gamma \), any two \( n \)-signed graphs with \( \Gamma \) as underlying graph are switching equivalent if, and only if, they are cycle isomorphic.

In this paper, we introduced the notion called boundary \( n \)-signed graph and we obtained some interesting results in the following sections.

### 2. Switching Invariant Boundary \( n \)-Signed Graphs

Inspired by the concept of boundary vertex introduced by Chartrand et al. ([1], [2]), in [5]), the authors the new notion in the graph theory called boundary graph \( B(\Gamma) \) contingent upon only 1-component graph \( \Gamma \). Suppose \( \Gamma = (V, E) \) be any graph, the boundary graph \( B(\Gamma) \) of \( \Gamma \) with \( V(B(\Gamma)) = V(\Gamma) \) with \( p, q \in V(B(\Gamma)) \) and \( e = pq \in E(B(\Gamma)) \), if either the distance between \( p \) and \( r \) is less than or equal to the distance between \( p \) and \( q \), for each \( r \in N(q) \) or the distance between \( q \) and \( r \) is less than or equal to the distance between \( p \) and \( q \), for each \( r \in N(p) \).

In [5], the authors also remarked that the graphs \( \text{Radius}(\Gamma) = 1 = \text{diam}(\Gamma) \), then \( \Gamma \) is boundary graph.

By using the concept of complement in \( n \)-signed graphs, we define the boundary \( n \)-signed graph \( B(\Sigma_n) = (B(\Gamma), \sigma) \) is an \( n \)-signed graph whose underlying graph is boundary graph
and \( n \)-tuple of any edge \( pq \in B(\Sigma_n) \) is the component-wise multiplication of \( n \)-tuples of the vertices \( p \) and \( q \) assigned by using the \( n \)-marking. For some \( n \)-signed graph \( (\Sigma_n)_1 \), such that \( B[(\Sigma_n)_1] \cong \Sigma_n \), then \( \Sigma_n \) is called the \( B(\Sigma_n) \).

The \( n \)-signed graphs can be classified into two types namely, identity balanced and identity unbalanced signed graphs. Further, the \( n \)-signed graph \( \Sigma_n = (\Gamma, \sigma) \) is identity balanced and identity unbalanced, we have \( B(\Sigma_n) \) is always identity balanced in either of the cases.

**Theorem 3**: Let \( \Sigma_n = (\Gamma, \sigma) \) be any \( n \)-signed graph. Then its \( B(\Sigma_n) \) is identity balanced.

**Proof**: Let \( p \) and \( q \) be any two vertices in \( n \)-signed marked graph. By the definition of \( B(\Sigma_n) \), we observed that \( V(B(\Sigma_n)) = V(\Sigma_n) \). Let \( pq \) be any edge in \( B(\Sigma_n) \), then the \( n \)-tuple of the edge \( pq \) is equal to the component-wise multiplication \( n \)-tuples assigned to \( p \) and \( q \) by \( n \)-marking. Hence, by Theorem 1, \( B(\Sigma_n) \) is identity balanced. \( \qed \)

Let \( k \in \mathbb{Z}^+ \), the \( k \)th iterated boundary \( n \)-signed graph \( B(\Sigma_n) \) of \( \Sigma_n \) is defined as follows:

\[
B^0(\Sigma_n) = \Sigma_n, \quad B^k(\Sigma_n) = B(B^{k-1}(\Sigma_n)).
\]

**Corollary 4**: Let \( \Sigma_n = (\Gamma, \sigma) \) be any \( n \)-signed graph and \( k \in \mathbb{Z}^+ \). Then iterated boundary \( n \)-signed graph \( B^k(\Sigma_n) \) is identity balanced.

**Theorem 5**: Let \( (\Sigma_n)_1 = (\Gamma_1, \sigma_1) \) and \( (\Sigma_n)_2 = (\Gamma_2, \sigma_2) \) with \( \Gamma_1 \cong \Gamma_2 \). Then \( B[(\Sigma_n)_1] \sim B[(\Sigma_n)_2] \).

**Proof**: Let \( (\Sigma_n)_1 \) and \( (\Sigma_n)_2 \) be any \( n \)-signed graphs with their underlying graphs are isomorphic. Then \( B[(\Sigma_n)_1] \) and \( B[(\Sigma_n)_2] \) are identity balanced and they are switching equivalent, by Theorem 2. \( \qed \)

In [5], the authors characterized the graphs for which graph and its boundary graph are isomorphic.

**Theorem 6**: Let \( \Gamma = (V, E) \) be graph. Then \( \Gamma \) is isomorphic to any complete graph if, and only if, the graph \( \Gamma \) and the boundary graph \( B(\Gamma) \) are isomorphic.

By the motivation of the above work, we obtained the necessary and sufficient conditions for \( B(\Sigma_n) \sim \Sigma_n \).

**Theorem 7**: Let \( \Sigma_n = (\Gamma, \sigma) \) be any \( n \)-signed graph. Then \( \Sigma_n \) is identity balanced and \( \Gamma \) is isomorphic to \( K_q \), for any \( q \in \mathbb{Z}^+ \) if, and only if, the boundary \( n \)-signed graph \( B(\Sigma_n) \) and the \( n \)-signed graph \( \Sigma_n \) are switching equivalent.
**Proof**: Suppose \( \Sigma_n \) is identity balanced and \( \Gamma \) is isomorphic to \( K_q \), for any \( q \in \mathbb{Z}^+ \). Then, since \( B(\Sigma_n) \) is identity balanced as per Theorem 3. Then \( B(\Sigma_n) \) and \( \Sigma_n \) are identity balanced and hence they are switching equivalent, from the Theorem 2.

Conversely, suppose that \( B(\Sigma_n) \sim \Sigma_n \). Then \( \Gamma \) and \( B(\Gamma) \) are isomorphic. Now \( \Gamma \) is any complete graph. Consider an \( n \)-signed graph \( \Sigma_n = (\Gamma, \sigma) \) with \( \Gamma \) is isomorphic to any complete graph \( K_q \), for any \( q \in \mathbb{Z}^+ \), then \( B(\Sigma_n) \) is identity balanced. If \( \Sigma_n \) identity unbalanced (i.e., the component-wise multiplication of all \( n \)-tuples of each cycle in \( \Sigma_n \) is non-identity \( n \)-tuple), then \( \Sigma_n = (\Gamma, \sigma) \) and \( B(\Sigma_n) \) are not switching equivalent, which is a contradiction and hence \( \Sigma_n = (\Gamma, \sigma) \) and \( B(\Sigma_n) \) are switching equivalent, by the hypothesis. Therefore, \( \Sigma_n \) is identity \( n \)-tuple. \( \Box \)

Let \( \Gamma = (V, E) \) be any graph and a vertex \( p \in V(\Gamma) \) is said to be complete vertex, if \( \langle N(p) \rangle \) is complete. In [5], the authors characterized the graphs for which \( B(\Gamma) \) and \( \overline{\Gamma} \) are isomorphic. The neighborhood of the \( p \in \Gamma \) is denoted by \( N_k(p) \) and is defined as the set of all vertices \( r \in N(p) \) such that the distance between the vertices \( p \) and \( r \) is \( k \).

**Theorem 8**: Let \( \Gamma = (V, E) \) be graph. Then \( \Gamma \) is isomorphic to \( B(\Gamma) \) if, and only if, the graph \( \Gamma \) satisfies the following conditions:

**i**. for each \( p \in V(\Gamma) \), \( \langle N(p) \rangle \) is not complete

**ii**. if \( e = pq \in E(\Gamma) \), then neither \( N(p) - \{q\} \subseteq N(q) - \{p\} \) nor \( N(q) - \{p\} \subseteq N(p) - \{q\} \)

**iii**. for each pair of non-adjacent vertices \( p, q \in V(\Gamma) \), the sets \( N_k(p) \) and \( N_k(q) \) are empty sets, where \( k = d(p, q) + 1 \).

By the motivation of the above work, we characterized \( n \)-signed graphs, the boundary \( n \)-signed graph and complement of \( \Sigma_n \) are switching equivalent.

**Theorem 9**: Let \( \Sigma_n = (\Gamma, \sigma) \) be any \( n \)-signed graph. Then the complement of \( n \)-signed graph \( \overline{\Sigma_n} \) and boundary \( n \)-signed graph \( B(\Sigma_n) \) are switching equivalent if, and only if, \( \Gamma \) satisfies the conditions of Theorem 8.

**Proof**: Suppose \( \Gamma \) satisfies the conditions of Theorem 8. Now, we have \( \Gamma \cong B(\Gamma) \) from the above result. Consider an \( n \)-signed graph \( \Sigma_n = (\Gamma, \sigma) \) with \( \Gamma \) satisfies the conditions of Theorem 8. Then \( B(\Sigma_n) \) and \( \overline{\Sigma_n} \) are identity balanced and hence they are switching equivalent.

Conversely, suppose that \( \overline{\Sigma_n} \sim SA(\Sigma_n) \). Then \( \Gamma \cong B(\Gamma) \) and the underlying graph \( \Gamma \) satisfies conditions of Theorem 8. \( \Box \)
The $c$-complement of $n$-tuple $b = (b_1, b_2, \ldots, b_n)$, where $c$ is any element of $H_n$ is an $n$-tuple such that $b^c = bc$. Let $M$ be a subset of $H_n$ and $c$ is any element of $H_n$, the $c$-complement of the subset $M$ of $H_n$ is $M^c = \{ b^c : b \in M \}$.

Suppose $\Sigma_n = (\Gamma, \sigma)$ be any $n$-signed graph and $c$ is any $n$-tuple of the set of all $n$-tuples $H_n$, the $c$-complement of $\Sigma_n$ is an $n$-signed graph $(\Sigma_n)^c = (\Gamma, \sigma)$ and each $n$-tuple $b = (b_1, b_2, \ldots, b_n)$ in $\Sigma_n$ changed as $b^c$.

In view of the above concept, we have the following switching equivalent characterizations in the flavor of Theorems 7 & 9.

**Corollary 10**: Let $\Sigma_n = (\Gamma, \sigma)$ be an $n$-signed graph. Then

i. $\Sigma_n$ is identity unbalanced and $\Gamma$ is isomorphic to $K_q$, for any $q \in \mathbb{Z}^+$ if, and only if, the boundary $n$-signed graph $B(\Sigma_n)$ and the $c$-complement of $n$-signed graph $[\Sigma_n]^c$ are switching equivalent.

ii. $\Sigma_n$ is identity balanced and $\Gamma$ is isomorphic to $K_q$, for any $q \in \mathbb{Z}^+$ if, and only if, $B[(\Sigma_n)^c]$ and the $n$-signed graph $\Sigma_n$ are switching equivalent.

iii. $\Sigma_n$ is identity unbalanced and $\Gamma$ is isomorphic to $K_q$, for any $q \in \mathbb{Z}^+$ if, and only if, $B[(\Sigma_n)^c]$ and the $c$-complement of $n$-signed graph $[\Sigma_n]^c$ are switching equivalent.

iv. the complement of $n$-signed graph $\Sigma_n$ and $B[(\Sigma_n)^c]$ are switching equivalent if, and only if, $\Gamma$ satisfies the conditions of Theorem 8.

v. $[\Sigma_n]^c$ and boundary $s - n$-signed graph $B(\Sigma_n)$ are switching equivalent if, and only if, $\Gamma$ satisfies the conditions of Theorem 8.

vi. $[\Sigma_n]^c$ and $B[(\Sigma_n)^c]$ are switching equivalent if, and only if, $\Gamma$ satisfies the conditions of Theorem 8.

**Remark 11**: Let $(\Sigma_n)_1 = (\Gamma_1, \sigma_1)$ and $(\Sigma_n)_2 = (\Gamma_2, \sigma_2)$ with $\Gamma_1 \cong \Gamma_2$. Then $B[(\Sigma_n)_1]^c \sim B[(\Sigma_n)_2]^c]$.

Given any $n$-signed graph, we have proved that $B(\Sigma_n)$ is identity balanced. Using the $c$-complement, we have the following result with respect to the notion $B(\Sigma_n)$.

**Theorem 12**: Suppose the boundary graph $B(\Gamma)$ is bipartite. Then the $c$-complement of boundary $n$-signed graph $B[(\Sigma_n)^c]$ is identity balanced.
Proof: In consideration of Theorem 3, the boundary $n$-signed graph $B(\Sigma_n)$ is identity balanced. Then $n$-tuple of each cycle in $B(\Sigma_n)$ is identity $n$-tuple. By the hypothesis, $B(\Gamma)$ is bipartite. Then $n$-tuple of each cycle in $B(\Sigma_n)$ having identity $n$-tuple. Therefore, the $c$-complement of boundary $n$-signed graph $B[(\Sigma_n)^c]$ is identity balanced. 

\section{3. Structural Characterization of $B(\Sigma_n)$}

In this section, we present the structural characterization of $B(\Sigma_n)$.

**Theorem 13**: Let $\Sigma_n = (\Gamma, \sigma)$ be any $n$-signed graph. Then $\Sigma_n$ is identity balanced and $\Gamma$ is a boundary graph if, and only if, $\Sigma_n = (\Gamma, \sigma)$ is $B(\Sigma_n)$.

**Proof**: Let us assume that $\Sigma_n$ is $B(\Sigma_n)$. Then $\Sigma_n \cong B[(\Sigma_n)_1]$, where $(\Sigma_n)_1$ for some $n$-signed graph. Therefore, the $n$-signed graph $\Sigma_n$ identity balanced, because $\Sigma_n = B(\Sigma_n)$. Conversely, suppose that $\Sigma_n = (\Gamma, \sigma)$ is identity balanced and $\Gamma$ is a $B(\Gamma)$ (i.e, $\Gamma \cong B(\Gamma_1)$, for some graph $\Gamma_1$). By the hypothesis, $\Sigma_n$ is identity balanced, then construct the marked $n$-signed graph. According to the Theorem 1, each edge $pq$ in $(\Sigma_n)_\mu$ satisfies $\sigma(pq) = \mu(p)\mu(q)$. Consider the $n$-signed graph $(\Sigma_n)_1 = (\Gamma_1, \sigma_1)$ in which each edge $e = (pq)$ in $\Gamma_1$, $\sigma_1(e) = \mu(p)\mu(q)$. Hence $\Sigma_n \cong B[(\Sigma_n)_1]$. Therefore, $\Sigma_n$ is a $B(\Sigma_n)$. \hfill \Box

\section*{Acknowledgement}

The authors are thankful to the anonymous referee for valuable suggestions and comments for the improvement of the paper. Also, the first author is grateful to Dr. M. N. Channabasappa, Director and Dr. Shivakumaraiah, Principal, Siddaganga Institute of Technology, Tumkur, for their constant support and encouragement.

\section*{References}


