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BOUNDARY n-SIGNED GRAPHS

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Abstract

In this paper, we extended the notion in the graphs called boundary graph to n-signed graphs and then we proved boundary n-signed graph is always identity balanced for given any n-signed graph. Further, we proved several switching equivalence characterizations and structural characterization for boundary n-signed graphs.

1. Introduction

Unless mentioned or defined otherwise, for all terminology and notion in graph theory the reader is refer to [3]. We consider only finite, simple graphs free from self-loops.

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Let $n \geq 1$ be an integer. An *n-tuple* $(a_1, a_2, ..., a_n)$ is *symmetric*, if $a_k = a_{n-k+1}, 1 \leq k \leq n$. Let $H_n = \{(a_1, a_2, ..., a_n) : a_k \in \{+, -\}, a_k = a_{n-k+1}, 1 \leq k \leq n\}$ be the set of all symmetric n-tuples. Note that H_n is a group under coordinate wise multiplication, and the order of H_n is 2^m, where $m = \lceil \frac{n}{2} \rceil$.

A *symmetric n-sigraph (symmetric n-marked graph)* is an ordered pair $\Sigma_n = (\Gamma, \sigma)$ $(\Sigma_n = (\Gamma, \mu))$, where $\Gamma = (V, E)$ is a graph called the *underlying graph* of Σ_n and $\sigma : E \to H_n$ $(\mu : V \to H_n)$ is a function.
In this paper by an *n-tuple/n-signed graph/n-marked graph* we always mean a symmetric n-tuple/symmetric n-sigraph/symmetric n-marked graph.
An n-tuple $(a_1, a_2, ..., a_n)$ is the *identity n-tuple*, if $a_k = +$, for $1 \leq k \leq n$, otherwise it is a *non-identity n-tuple*. In an n-signed graph $\Sigma_n = (\Gamma, \sigma)$ an edge labelled with the identity n-tuple is called an *identity edge*, otherwise it is a *non-identity edge*.
Further, in an n-signed graph $\Sigma_n = (\Gamma, \sigma)$, for any $A \subseteq E(\Gamma)$ the *n-tuple* $\sigma(A)$ is the product of the n-tuples on the edges of A.
In [9], the authors defined two notions of balance in n-signed graph $\Sigma_n = (\Gamma, \sigma)$ as follows (See also R. Rangarajan and P.S.K.Reddy [6] :
Definition : Let $\Sigma_n = (\Gamma, \sigma)$ be an n-signed graph. Then,

(i) Σ_n is *identity balanced* (or *i-balanced*), if product of n-tuples on each cycle of Σ_n is the identity n-tuple, and

(ii) Σ_n is *balanced*, if every cycle in Σ_n contains an even number of non-identity edges.

Note : An i-balanced n-signed graph need not be balanced and conversely.
The following characterization of i-balanced n-signed graphs is obtained in [9].
Theorem 1 : An n-signed graph $\Sigma_n = (\Gamma, \sigma)$ is i-balanced if, and only if, it is possible to assign n-tuples to its vertices such that the n-tuple of each edge uv is equal to the product of the n-tuples of u and v.
Let $\Sigma_n = (\Gamma, \sigma)$ be an n-signed graph. Consider the n-marking μ on vertices of Σ_n defined as follows: each vertex $v \in V$, $\mu(v)$ is the n-tuple which is the product of the n-tuples on the edges incident with v. *Complement* of Σ_n is an n-signed graph $\overline{\Sigma_n} = (\overline{\Gamma}, \sigma^c)$, where for any edge $e = uv \in \overline{\Gamma}$, $\sigma^c(uv) = \mu(u)\mu(v)$. Clearly, $\overline{\Sigma_n}$ as defined here is an i-balanced n-signed graph due to Theorem 1.

In [9], the authors also have defined switching and cycle isomorphism of an n-signed graph $\Sigma_n = (\Gamma, \sigma)$ as follows: (See also ([4], [7], [8]) and ([11]-[17]).

Let $\Sigma_n = (\Gamma, \sigma)$ and $\Sigma'_n = (\Gamma', \sigma')$, be two n-signed graphs. Then Σ_n and Σ'_n are said to be *isomorphic*, if there exists an isomorphism $\phi : \Gamma \to \Gamma'$ such that if uv is an edge in Σ_n with label $(a_1, a_2, ..., a_n)$ then $\phi(u)\phi(v)$ is an edge in Σ'_n with label $(a_1, a_2, ..., a_n)$.

Let $\Sigma_n = (\Gamma, \sigma)$ be an n-signed graph, switching of n-signed graph with respect to the n-marking μ is the replacing the n-tuple on each edge $e = uv \in E(\Sigma_n)$ as the product of n-tuple of u, $e = uv$ and v (i.e, $\mu(u)\sigma(uv)\mu(v)$). The resulting n-signed graph is denoted by $(\Sigma_n)_\mu$ and is called the switched n-signed graph. Two n-signed graphs $(\Sigma_n)_1 = (\Gamma_2, \sigma)$ and $(\Sigma_n)_2 = (\Gamma_2, \sigma')$ are said to be switching equivalent and is denoted by $(\Sigma_n)_1 \sim (\Sigma_n)_2$, if $((\Sigma_n)_1)_\mu \cong (\Sigma_n)_2$.

Let $(\Sigma_n)_1 = (\Gamma_1, \sigma)$ and $(\Sigma_n)_2 = (\Gamma_2, \sigma')$ be two n-signed graphs with $\Gamma_1 \cong \Gamma_2$. The n-tuple each cycle in $(\Sigma_n)_1$ is same as the n-tuple each cycle in $(\Sigma_n)_1$, then the above two n-signed graphs (with their underlying graphs are isomorphic) are said to cycle isomorphic. We make use of the following known result (see [9]):

Theorem 2 : Given a graph Γ, any two n-signed graphs with Γ as underlying graph are switching equivalent if, and only if, they are cycle isomorphic.

In this paper, we introduced the notion called boundary n-signed graph and we obtained some interesting results in the following sections.

2. Switching Invariant Boundary n-Signed Graphs

Inspired by the concept of boundary vertex introduced by Chartrand et al. ([1], [2]), in [5]), the authors the new notion in the graph theory called boundary graph $\mathcal{B}(\Gamma)$ contingent upon only 1-component graph Γ. Suppose $\Gamma = (V, E)$ be any graph, the boundary graph $\mathcal{B}(\Gamma)$ of Γ with $V(\mathcal{B}(\Gamma)) = V(\Gamma)$ with $p, q \in V(\mathcal{B}(\Gamma))$ and $e = pq \in E(\mathcal{B}(\Gamma))$, if either the distance between p and r is less than or equal to the distance between p and q, for each $r \in N(q)$ or the distance between q and r is less than or equal to the distance between p and q, for each $r \in N(p)$.

In [5], the authors also remarked that the graphs Radius$(\Gamma) = 1 =$ diam(Γ), then Γ is boundary graph.

By using the concept of complement in n-signed graphs, we define the boundary n-signed graph $\mathcal{B}(\Sigma_n) = (\mathcal{B}(\Gamma), \sigma)$ is an n-signed graph whose underlying graph is boundary graph and n-tuple of any edge $pq \in \mathcal{B}(\Sigma_n)$ is the component-wise multiplication of n-tuples of the vertices p and q assigned by using the n-marking. For some n-signed graph $(\Sigma_n)_1$, such that $\mathcal{B}[(\Sigma_n)_1] \cong \Sigma_n$, then Σ_n is called the $\mathcal{B}(\Sigma_n)$.

The n-signed graphs can be classified into two types namely, identity balanced and identity unbalanced signed graphs. Further, the n-signed graph $\Sigma_n = (\Gamma, \sigma)$ is identity balanced and identity unbalanced, we have $\mathcal{B}(\Sigma_n)$ is always identity balanced in either of the cases.

Theorem 3 : Let $\Sigma_n = (\Gamma, \sigma)$ be any n-signed graph. Then its $\mathcal{B}(\Sigma_n)$ is identity balanced.

Proof : Let p and q be any two vertices in n-signed marked graph. By the definition of $\mathcal{B}(\Sigma_n)$, we observed that $V(\mathcal{B}(\Sigma_n)) = V(\Sigma_n)$. Let pq be any edge in $\mathcal{B}(\Sigma_n)$, then the n-tuple of the edge pq is equal to the component-wise multiplication n-tuples assigned to p and q by n-marking. Hence, by Theorem 1, $\mathcal{B}(\Sigma_n)$ is identity balanced. $\square$

Let $k \in \mathbb{Z}^+$, the k^{th} iterated boundary n-signed graph $\mathcal{B}(\Sigma_n)$ of Σ_n is defined as follows:

$$\mathcal{B}^0(\Sigma_n) = \Sigma_n, \; \mathcal{B}^k(\Sigma_n) = \mathcal{B}(\mathcal{B}^{k-1}(\Sigma_n)).$$

Corollary 4 : Let $\Sigma_n = (\Gamma, \sigma)$ be any n-signed graph and $k \in \mathbb{Z}^+$. Then iterated boundary n-signed graph $\mathcal{B}^k(\Sigma_n)$ is identity balanced.

Theorem 5 : Let $(\Sigma_n)_1 = (\Gamma_1, \sigma_1)$ and $(\Sigma_n)_2 = (\Gamma_2, \sigma_2)$ with $\Gamma_1 \cong \Gamma_2$. Then $\mathcal{B}[(\Sigma_n)_1] \sim \mathcal{B}[(\Sigma_n)_2]$.

Proof : Let $(\Sigma_n)_1$ and $(\Sigma_n)_2$ be any n-signed graphs with their underlying graphs are isomorphic. Then $\mathcal{B}[(\Sigma_n)_1]$ and $\mathcal{B}[(\Sigma_n)_2]$ are identity balanced and they are switching equivalent, by Theorem 2. $\square$

In [5], the authors characterized the graphs for which graph and its boundary graph are isomorphic.

Theorem 6 : Let $\Gamma = (V, E)$ be graph. Then Γ is isomorphic to any complete graph if, and only if, the graph Γ and the boundary graph $\mathcal{B}(\Gamma)$ are isomorphic.

By the motivation of the above work, we obtained the necessary and sufficient conditions for $\mathcal{B}(\Sigma_n) \sim \Sigma_n$.

Theorem 7 : Let $\Sigma_n = (\Gamma, \sigma)$ be any n-signed graph. Then Σ_n is identity balanced and Γ is isomorphic to K_q, for any $q \in \mathbb{Z}^+$ if, and only if, the boundary n-signed graph $\mathcal{B}(\Sigma_n)$ and the n-signed graph Σ_n are switching equivalent.

Proof : Suppose Σ_n is identity balanced and Γ is isomorphic to K_q, for any $q \in \mathbb{Z}^+$. Then, since $\mathcal{B}(\Sigma_n)$ is identity balanced as per Theorem 3. Then $\mathcal{B}(\Sigma_n)$ and Σ_n are identity balanced and hence they are switching equivalent, from the Theorem 2.

Conversely, suppose that $\mathcal{B}(\Sigma_n) \sim \Sigma_n$. Then Γ and $\mathcal{B}(\Gamma)$ are isomorphic. Now Γ is any complete graph. Consider an n-signed graph $\Sigma_n = (\Gamma, \sigma)$ with Γ is isomorphic to any complete graph K_q, for any $q \in \mathbb{Z}^+$, then $\mathcal{B}(\Sigma_n)$ is identity balanced. If Σ_n identity unbalanced (i.e., the component-wise multiplication of all n-tuples of each cycle in Σ_n is non-identity n-tuple), then $\Sigma_n = (\Gamma, \sigma)$ and $\mathcal{B}(\Sigma_n)$ are not switching equivalent, which is a contradiction and hence $\Sigma_n = (\Gamma, \sigma)$ and $\mathcal{B}(\Sigma_n)$ are switching equivalent, by the hypothesis. Therefore, Σ_n is identity n-tuple. □

Let $\Gamma = (V, E)$ be any graph and a vertex $p \in V(\Gamma)$ is said to be complete vertex, if $\langle N(p) \rangle$ is complete. In [5], the authors characterized the graphs for which $\mathcal{B}(\Gamma)$ and $\overline{\Gamma}$ are isomorphic. The neighborhood of the $p \in \Gamma$ is denoted by $N_k(p)$ and is defined as the set of all vertices $r \in N(p)$ such that the distance between the vertices p and r is k.

Theorem 8 : Let $\Gamma = (V, E)$ be graph. Then $\overline{\Gamma}$ is isomorphic to $\mathcal{B}(\Gamma)$ if, and only if, the graph Γ satisfies the following conditions:

i . for each $p \in V(\Gamma)$, $\langle N(p) \rangle$ is not complete

ii . if $e = pq \in E(\Gamma)$, then neither $N(p) - \{q\} \subseteq N(q) - \{p\}$ nor $N(q) - \{p\} \subseteq N(p) - \{q\}$

iii . for each pair of non-adjacent vertices $p, q \in V(\Gamma)$, the sets $N_k(p)$ and $N_k(q)$ are empty sets, where $k = d(p, q) + 1$.

By the motivation of the above work, we characterized n-signed graphs, the boundary n-signed graph and complement of Σ_n are switching equivalent.

Theorem 9 : Let $\Sigma_n = (\Gamma, \sigma)$ be any n-signed graph. Then the complement of n-signed graph $\overline{\Sigma_n}$ and boundary n-signed graph $\mathcal{B}(\Sigma_n)$ are switching equivalent if, and only if, Γ satisfies the conditions of Theorem 8.

Proof : Suppose Γ satisfies the conditions of Theorem 8. Now, we have $\overline{\Gamma} \cong \mathcal{B}(\Gamma)$ from the above result. Consider an n-signed graph $\Sigma_n = (\Gamma, \sigma)$ with Γ satisfies the conditions of Theorem 8. Then $\mathcal{B}(\Sigma_n)$ and $\overline{\Sigma_n}$ are identity balanced and hence they are switching equivalent.

Conversely, suppose that $\overline{\Sigma_n} \sim \mathcal{SA}(\Sigma_n)$. Then $\overline{\Gamma} \cong \mathcal{B}(\Gamma)$ and the underlying graph Γ satisfies conditions of Theorem 8. □

The c-complement of n-tuple $b = (b_1, b_2, \cdots, b_n)$, where c is any element of H_n is an n-tuple such that $b^c = bc$. Let M be a subset of H_n and c is any element of H_n, the c-complement of the subset M of H_n is $M^c = \{b^c : b \in M\}$.

Suppose $\Sigma_n = (\Gamma, \sigma)$ be any n-signed graph and c is any n-tuple of the set of all n-tuples H_n, the c-complement of Σ_n is an n-signed graph $(\Sigma_n)^c = (\Gamma, \sigma)$ and each n-tuple $b = (b_1, b_2, \cdots, b_n)$ in Σ_n changed as b^c.
In view of the above concept, we have the following switching equivalent characterizations in the flavor of Theorems 7 & 9.
Corollary 10 : Let $\Sigma_n = (\Gamma, \sigma)$ be an n-signed graph. Then

i . Σ_n is identity unbalanced and Γ is isomorphic to K_q, for any $q \in \mathbb{Z}^+$ if, and only if, the boundary n-signed graph $\mathcal{B}(\Sigma_n)$ and the c-complement of n-signed graph $[\Sigma_n]^c$ are switching equivalent.

ii . Σ_n is identity balanced and Γ is isomorphic to K_q, for any $q \in \mathbb{Z}^+$ if, and only if, $\mathcal{B}[(\Sigma_n)^c]$ and the n-signed graph Σ_n are switching equivalent.

iii . Σ_n is identity unbalanced and Γ is isomorphic to K_q, for any $q \in \mathbb{Z}^+$ if, and only if, $\mathcal{B}[(\Sigma_n)^c]$ and the c-complement of n-signed graph $[\Sigma_n]^c$ are switching equivalent.

iv . the complement of n-signed graph $\overline{\Sigma_n}$ and $\mathcal{B}[(\Sigma_n)^c]$ are switching equivalent if, and only if, Γ satisfies the conditions of Theorem 8.

v . $\overline{[\Sigma_n]^c}$ and boundary $s-n$-signed graph $\mathcal{B}(\Sigma_n)$ are switching equivalent if, and only if, Γ satisfies the conditions of Theorem 8.

vi . $\overline{[\Sigma_n]^c}$ and $\mathcal{B}[(\Sigma_n)^c]$ are switching equivalent if, and only if, Γ satisfies the conditions of Theorem 8.

Remark 11 : Let $(\Sigma_n)_1 = (\Gamma_1, \sigma_1)$ and $(\Sigma_n)_2 = (\Gamma_2, \sigma_2)$ with $\Gamma_1 \cong \Gamma_2$. Then $\mathcal{B}[((\Sigma_n)_1)^c] \sim \mathcal{B}[((\Sigma_n)_2)^c]$.
Given any n-signed graph, we have proved that $\mathcal{B}(\Sigma_n)$ is identity balanced. Using the c-complement, we have the following result with respect to the notion $\mathcal{B}(\Sigma_n)$.
Theorem 12 : Suppose the boundary graph $\mathcal{B}(\Gamma)$ is bipartite. Then the c-complement of boundary n-signed graph $\mathcal{B}[(\Sigma_n)^c]$ is identity balanced.

Proof : In consideration of Theorem 3, the boundary n-signed graph $\mathcal{B}(\Sigma_n)$ is identity balanced. Then n-tuple of each cycle in $\mathcal{B}(\Sigma_n)$ is identity n-tuple. By the hypothesis, $\mathcal{B}(\Gamma)$ is bipartite. Then n-tuple of each cycle in $\mathcal{B}(\Sigma_n)$ having identity n-tuple. Therefore, the c-complement of boundary n-signed graph $\mathcal{B}[(\Sigma_n)^c]$ is identity balanced. □

3. Structural Characterization of $\mathcal{B}(\Sigma_n)$

In this section, we present the structural characterization of $\mathcal{B}(\Sigma_n)$.

Theorem 13 : Let $\Sigma_n = (\Gamma, \sigma)$ be any n-signed graph. Then Σ_n is identity balanced and Γ is a boundary graph if, and only if, $\Sigma_n = (\Gamma, \sigma)$ is $\mathcal{B}(\Sigma_n)$.

Proof : Let us assume that Σ_n is $\mathcal{B}(\Sigma_n)$. Then $\Sigma_n \cong \mathcal{B}[(\Sigma_n)_1]$, where $(\Sigma_n)_1$ for some n-signed graph. Therefore, the n-signed graph Σ_n identity balanced, because $\Sigma_n = \mathcal{B}(\Sigma_n)$. Conversely, suppose that $\Sigma_n = (\Gamma, \sigma)$ is identity balanced and Γ is a $\mathcal{B}(\Gamma)$ (i.e, $\Gamma \cong \mathcal{B}(\Gamma_1)$, for some graph Γ_1). By the hypothesis, Σ_n is identity balanced, then construct the marked n-signed graph. According to the Theorem 1, each edge pq in $(\Sigma_n)_\mu$ satisfies $\sigma(pq) = \mu(p)\mu(q)$. Consider the n-signed graph $(\Sigma_n)_1 = (\Gamma_1, \sigma_1)$ in which each edge $e = (pq)$ in Γ_1, $\sigma_1(e) = \mu(p)\mu(q)$. Hence $\Sigma_n \cong \mathcal{B}[(\Sigma_n)_1]$. Therefore, Σ_n is a $\mathcal{B}(\Sigma_n)$. □

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