International J. of Math. Sci. & Engg. Appls. (IJMSEA)
ISSN 0973-9424, Vol. 10 No. II (August, 2016), pp. 169-179

# WEIGHTED SHARING OF SMALL FUNCTION OF A MEROMORPHIC FUNCTION OF DIFFERENTIAL POLYNOMIAL WITH THEIR DERIVATIVES

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**Abstract**

In this paper, we pay our attention to the uniqueness of more generalised form of a function namely $f^nP(f)$ and $[f^nP(f)]^{(k)}$ sharing a small function. The Theorems improves the results of C. K. Basu and T. Lowha [1].

## 1. Introduction and Main Results

In this article, a meromorphic function will mean meromorphic in the whole complex plane. We shall use the standard notations in the Nevanlinna value distribution theory of meromorphic functions such as $T(r, f)$, $N(r, f)$, $\overline{N}(r, f)$, $m(r, f)$, (see [5, 8]). The notation $S(r, f)$ is defined to be any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$ outside of a possible exceptional set of finite linear measure. A meromorphic function $a(z)$ is called a small function with respect to $f(z)$, provided that $T(r, a) = S(r, f)$.

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Key Words : *Meromorphic function, Weighted sharing, Differential polynomials, Small function.*

2010 AMS Subject Classification : Primary 30D35.

Let $f$ and $g$ be two non-constant meromorphic functions. Let $a$ be a finite complex number. We denote by $E(a, f)$ the set of zeros of $f - a$ (counting multiplicity), by $\overline{E}(a, f)$ the set of zeros of $f - a$ (ignoring multiplicity). We say $f$ and $g$ share a CM (IM), if $E(a, f) = E(a, g)$ ($\overline{E}(a, f) = \overline{E}(a, g)$). Similarly, we define that $f$ and $g$ share a small function $a(z)$ CM (IM), if $E(a(z), f) = E(a(z), g)$ ($\overline{E}(a(z), f) = \overline{E}(a(z), g)$). Moreover, $GCD(n_1, n_2, ..., n_k)$ denotes the greatest common divisor of positive integers $n_1, n_2, ..., n_k$.

**Definition 1.1 (see[7])** : We denote by $E_{k)}(a, f)$ the set of zeros of $f - a$ with multiplicities at most $k$, where each zero is counted according to its multiplicity. We denote by $\overline{E}_{k)}(a, f)$ the set of zeros of $f - a$ with multiplicities at most $k$, where each zero is counted only once. In addition, we denote by $N_{k)}\left(r, \frac{1}{f-a}\right)$ $\left(\overline{N}_{k)}\left(r, \frac{1}{f-a}\right)\right)$ the counting function with respect to the set $E_{k)}(a, f)$ ($\overline{E}_{k)}(a, f)$). We denote by $N_{(k}\left(r, \frac{1}{f-a}\right)$ the counting function of $a$-points of $f$ (counted with proper multiplicities) whose multiplicities are not less than $k$, we denote by $\overline{N}_{(k}\left(r, \frac{1}{f-a}\right)$ the corresponding reduced counting function (ignoring multiplicity). Set

$$N_k\left(r, \frac{1}{f-a}\right) = \overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}_{(2}\left(r, \frac{1}{f-a}\right) + ... + \overline{N}_{(k}\left(r, \frac{1}{f-a}\right).$$

For any constant $a$, we define

$$\Theta(a, f) = 1 - \limsup_{r\to\infty} \frac{\bar{N}(r, \frac{1}{f-a})}{T(r, f)}, \quad \delta(a, f) = 1 - \limsup_{r\to\infty} \frac{N(r, \frac{1}{f-a})}{T(r, f)}.$$

We also use the notation $\Theta(a; f, g)$ and $\delta(a; f, g)$, where $\Theta(a; f, g) = min\{\Theta(a, f), \Theta(a, g)\}$ and $\delta(a; f, g) = min\{\delta(a, f), \delta(a, g)\}$.

Let $a$ be any value in the extended complex plane. Let $k$ be an arbitrary non negative integer. We define

$$\Theta_{k)}(a, f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}_{k)}(r, \frac{1}{f-a})}{T(r, f)}. \tag{1.1}$$

**Definition 1.2 (see[3])** : Suppose $f(z), g(z)$ satisfy $\overline{E}(a, f) = \overline{E}(a, g)$. Let $z_0$ be a common 1-point of $f(z)$ and $g(z)$ with the multiplicity r and q. We denote by $\overline{N}_L(r, \frac{1}{f-1})$ the reduced counting function of those 1-points of $f(z)$ and $g(z)$ where $r > q$; $\overline{N}_E^{(2}(r, \frac{1}{f-1})$ the reduced counting function of those 1-points of $f(z)$ and $g(z)$ where $r = q \geq 2$. In addition, we denote $N_E^{1)}(r, \frac{1}{f-1})(N_E^{1)}(r, \frac{1}{g-1}))$ the counting function of those common simple 1-points of $f(z)$ and $g(z)$.

**Definition 1.3 (see[3])** : Let $f$ and $g$ be two non-constant entire functions. We denote by $N_0(r, \frac{1}{f'})$ the counting function of those zeros of $f'$ which are not the zeros of $f(f-1)$, by $\overline{N}_0(r, \frac{1}{f'})$ the corresponding reduced counting functions. Similarly, we can define $N_0(r, \frac{1}{g'})$ and $\overline{N}_0(r, \frac{1}{g'})$.
**Remark 1.1** : From (1.1) we have

$$0 \leq \Theta(a, f) \leq \Theta_{k)}(a, f) \leq \Theta_{k-1)}(a, f) \leq \Theta_{1)}(a, f) \leq 1.$$

In 1998, Q. C. Zhang [11] proved the following theorem about a meromorphic function and its $k$-th order derivative.
**Theorem A** : Let $f$ be a non-constant meromorphic function and let $k$ be a positive integer. Suppose that $f$ and $f^{(k)}$ share 1CM and $2\overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f'}\right) + N\left(r, \frac{1}{f^{(k)}}\right) \leq (\lambda + o(1))\, T(r, f^{(k)})$ for $r \in I$, where $I$ is a set of infinite linear measure and $\lambda$ satisfies $0 < \lambda < 1$ then $\frac{f^{(k)}-1}{f-1} \equiv c$ for some constant $c \in \mathbb{C} - \{0\}$.
In 2003, Kit-wing [10] discussed the problem of a meromorphic function and its $k$-th derivative sharing one small function and proved the following result.
**Theorem B** : Let $k \geq 1$. Let $f$ be a non-constant non-entire meromorphic function, $a \in s(f)$ and $a(\neq 0, \infty)$ and $f$ do not have any common pole. If $f, f^{(k)}$ share a CM and $4\delta(0, f) + 2(8+k)\Theta(\infty, f) > 19 + 2k$, then $f = f^{(k)}$.
Two years later, in 2005, Q.C.Zhang [12] proved the following theorem.
**Theorem C** : Let $f$ be a non-constant meromorphic function and $k(\geq 1), l(\geq 0)$ be integers. Also let $a \equiv a(z)(\not\equiv 0, \infty)$ be a meromorphic function such that $T(r, a) = S(r, f)$. Suppose thst $f - a$ and $f^{(k)} - a$ share $(0, l)$. If $l \geq 2$ and $(3+k)\Theta(\infty, f) + 2\delta_{2+k}(0, f) > k + 4$ or, If $l = 1$ and $(4+k)\Theta(\infty, f) + 3\delta_{2+k}(0, f) > k + 6$, or, if $l = 0$, i.e., $f - a$ and $f^{(k)} - a$ share the value 0 IM and $(6+k)\Theta(\infty, f) + 5\delta_{2+k}(0, f) > 2k + 10$ then $f \equiv f^{(k)}$.
In 2010, A.Chen, X.Wang and G.Zhang [2] proved the following results.
**Theorem D** : Let $K(\geq 1)$, $n(\geq 1)$ be integers and $f$ be a non-constant meromorphic function. Also let $a(z)(\not\equiv 0, \infty)$ be a small function with respect to $f$. If $f$ and $[f^n]^{(k)}$ share $a(z)$ IM and

$$4\overline{N}(r, f) + 2\overline{N}\left(r, \frac{1}{\left(\frac{f}{a}\right)'}\right) + 2N_2\left(r, \frac{1}{(f^n)^{(k)}}\right) + \overline{N}\left(r, \frac{1}{(f^n)^{(k)}}\right) \leq (\lambda + o(1))T\left(r, (f^n)^{(k)}\right),$$

or, if $f$ and $(f^n)^{(k)}$ share $a(z)$ CM and

$$2\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{\left(\frac{f}{a}\right)'}\right)+N_2\left(r,\frac{1}{(f^n)^{(k)}}\right)\leq(\lambda+o(1))T\left(r,(f^n)^{(k)}\right),\ \text{for } 0<\lambda<1,$$

where $r\in I$ and $I$ is a set of infinite linear measure, then $\dfrac{f-a}{(f^n)^{(k)}-a}=c$, for some constant $c\in\mathbb{C}-\{0\}$.

**Theorem E** : Let $K(\geq 1)$, $n(\geq 1)$ be integers and let $f$ be a non-constant meromorphic function . Also
let $a(z)(\not\equiv 0,\infty)$ be a small function with respect to $f$. If $f$ and $(f^n)^{(k)}$ share $a(z)$ IM and

$(2k+6)\Theta(\infty,f)+3\Theta(0,f)+2\delta_{k+2}(0,f)>2k+10$ or , if, $f$ and $(f^n)^{(k)}$ share $a(z)$ $CM$ and

$(k+3)\Theta(\infty,f)+\delta_2(0,f)+\delta_{k+2}(0,f)>k+4$

then $f\equiv(f^n)^{(k)}$.

Recently, in 2014, C. K. Basu And T. Lowha [1] proved the following results.

**Theorem F** : Let $k$, $m$ and $n$ are three positive integers with $m\leq n$ and let $f$ be a non-constant meromorphic function . Also let $a(z)(\not\equiv 0,\infty)$ be a small function with respect to $f$. If $\overline{E}_{l)}(a,f^m(z))=\overline{E}_{l)}\left(a,(f^n)^{(k)}\right)$, where $l$ is a positive integer and

$$\overline{N}(r,f)+2N_2\left(r,\frac{1}{f}\right)+2N_2\left(r,\frac{1}{(f^n)^{(k)}}\right)+\overline{N}\left(r,\frac{1}{(f^n)^{(k)}}\right)\leq(\lambda+O(1))T(r,(f^n)^{(k)}),$$

for $0<\lambda<1$, where $r\in I$ and $I$ is a set of infinite linear measure, then $\dfrac{(f^n)^{(k)}-a}{f^m-a}=c$ for some constant $a\in\mathbb{C}-\{0\}$ where $\mathbb{C}$ is the set of complex numbers.

**Theorem G** : Let $f$ be a non-constant meromorphic function and let $k$ and $n$ be two positive integers. If $\overline{E}_{l)}(a,f)=\overline{E}_{l)}\left(a,(f^n)^{(k)}\right)$, where $l$ is a positive integer and $a(z)(\not\equiv 0,\infty)$ be a small function of $f$ and $(2k+6)\Theta(\infty,f)+\Theta(0,f)+2\delta_2(0,f)+2\delta_{k+2}(0,f)>2k+10$ then $f=(f^n)^{(k)}$.

In this paper, we pay our attention to the uniqueness of more generalized form of a function namely $f^nP(f)$ and $[f^nP(f)]^{(k)}$ sharing a small function.

**Theorem 1** : Let $k$, $m$ and $n$ are three positive integers with $k\leq m$ and $f$ be a non-constant meromorphic function. Also let $a(z)(\not\equiv 0,\infty)$ be a small function with respect to $f$. If $\overline{E}_{l)}\left(a, f^nP(f)\right)$=$\overline{E}_{l)}\left(a, (f^nP(f))^{(k)}\right)$, where $l$ is a positive integer and

$$6\overline{N}\left(r, f\right) + 2N_2\left(r, \frac{1}{f^nP(f)}\right) + 2N_2\left(r, \frac{1}{(f^nP(f))^{(k)}}\right) + \overline{N}\left(r, \frac{1}{(f^nP(f))^{(k)}}\right)$$
$$\leq (\lambda + o(1))T\left(r, (f^nP(f))^{(k)}\right),$$

for $0 < \lambda < 1$, where $r \in I$ and $I$ is a set of infinite linear measure , then $\dfrac{(f^nP(f))^{(k)} - a}{f^nP(f) - a} =$ $c$ for some constant $a \in \mathbb{C} - \{0\}$ where $\mathbb{C}$ is the set of complex numbers.

**Theorem 2** : Let $f$ be a non-constant meromorphic function and let $k$ and $n$ be two positive integers. If $\overline{E}_{l)}\left(a, f^nP(f)\right)$=$\overline{E}_{l)}\left(a, (f^nP(f))^{(k)}\right)$ , where $l$ is a positive integer and $a(z)(\not\equiv 0, \infty)$ be a small function of $f$ and $(3k+6)\Theta(\infty, f)+(3k+9)\Theta(0, f)+5m\delta(0, f) > 6k + 15 + 4m - n$ then $f^nP(f) = (f^nP(f))^{(k)}$.

## 2. Some Lemmas

Here we mention some existing lemmas of the literature which will be frequently used to prove the aforementioned theorems.

**Lemma 2.1 (see [9])** : Let $f(z)$ be a non-constant meromorphic function and $k$, $p$ be two positive integers. Then

$$N_p\left(r, \frac{1}{f^{(k)}}\right) \leq N_{p+k}\left(r, \frac{1}{f}\right) + k\overline{N}(r, f) + S(r, f).$$

**Lemma 2.2 (see [5])** : Let $f$ be a non-constant meromorphic function and let $n$ be a positive integer. $P(f) = a_nf^n + a_{n-1}f^{n-1} + ... + a_1f$
where $a_i$ is a meromorphic function such that $T(r, a_i)$=$S(r, f)$ ($i$=1,2,...,n). Then
$T\left(r, P(f)\right) = nT(r, f) + S(r, f)$.

## 3. Proof of the Theorems

**Proof of Theorem 1** : Let $F = \frac{f^nP(f)}{a}$ and $G = \frac{(f^nP(f))^{(k)}}{a}$.
Therefore, $F - 1 = \dfrac{f^nP(f) - a}{a}$ and $G - 1 = \dfrac{(f^nP(f))^{(k)} - a}{a}$.
Now, $\overline{E}_{l)}\left(a, f^nP(f)\right)$=$\overline{E}_{l)}\left(a, (f^nP(f))^{(k)}\right)$ except the zeros and poles of $a(z)$. Define,

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right).$$

We now consider two cases:

**Case 1** : Suppose $H \not\equiv 0$. Then $m(r, H) = s(r, f)$. Now if $z_0$ is a common simple zero of $F-1$ and $G-1$ (except the zeros and poles of $a(z)$), then after simple calculation , we get $H(z_0) = 0$. So,

$$N_E^{1)}\left(r, \frac{1}{G-1}\right) \leq N(r, \frac{1}{H}) + S(r, f) \leq T(r, H) + S(r, f) \leq N(r, H) + S(r, f).$$

Again by analysis, we can deduce that,

$$\begin{aligned} N(r, H) \leq \overline{N}(r, f) &+ \overline{N}_{(2}\left(r, \frac{1}{F}\right) + \overline{N}_{(2}\left(r, \frac{1}{G}\right) \\ &+ \overline{N}_L\left(r, \frac{1}{F-1}\right) + \overline{N}_L\left(r, \frac{1}{G-1}\right) + \overline{N}_*\left(r, \frac{1}{F-1}\right) \\ &+ \overline{N}_*\left(r, \frac{1}{G-1}\right) + \overline{N}_0\left(r, \frac{1}{F'}\right) + \overline{N}_0\left(r, \frac{1}{G'}\right) + S(r, f). \end{aligned}$$

Also,

$$\begin{aligned} \overline{N}\left(r, \frac{1}{G-1}\right) = N_E^{1)}\left(r, \frac{1}{G-1}\right) + \overline{N}_E^{(2}\left(r, \frac{1}{G-1}\right) + \overline{N}_L\left(r, \frac{1}{G-1}\right) + \overline{N}_L\left(r, \frac{1}{F-1}\right) \\ + \overline{N}_*\left(r, \frac{1}{G-1}\right) + S(r, f). \end{aligned}$$

Therefore,

$$\begin{aligned} &\overline{N}\left(r, \tfrac{1}{G-1}\right) \leq \overline{N}(r, f) + \overline{N}_{(2}\left(r, \tfrac{1}{F}\right) + \overline{N}_{(2}\left(r, \frac{1}{G}\right) + 2\overline{N}_L\left(r, \tfrac{1}{F-1}\right) + 2\overline{N}_L\left(r, \tfrac{1}{G-1}\right) \\ &+\overline{N}_E^{(2}\left(r, \tfrac{1}{G-1}\right) + 2\overline{N}_*\left(r, \frac{1}{G-1}\right) + \overline{N}_*\left(r, \frac{1}{F-1}\right) + \overline{N}_0\left(r, \tfrac{1}{F'}\right) + \overline{N}_0\left(r, \tfrac{1}{G'}\right) + S(r, f) \end{aligned} \tag{3.1}$$

Since, $\overline{E}_{l)}(1, F) = \overline{E}_{l)}(1, G)$. Therefore,

$$2\overline{N}_L\left(r, \frac{1}{G-1}\right) + 2\overline{N}_*\left(r, \frac{1}{G-1}\right) + \overline{N}_E^{(2}\left(r, \frac{1}{G-1}\right) \leq 2\overline{N}_{(2}\left(r, \frac{1}{G-1}\right).$$

From (3.1), we have,

$$\begin{aligned} &\overline{N}\left(r, \tfrac{1}{G-1}\right) \leq \overline{N}(r, f) + \overline{N}_{(2}\left(r, \tfrac{1}{F}\right) + \overline{N}_{(2}\left(r, \frac{1}{G}\right) + 2\overline{N}_{(2}\left(r, \frac{1}{G-1}\right) + 2\overline{N}_L\left(r, \tfrac{1}{F-1}\right) \\ &+\overline{N}_*\left(r, \frac{1}{F-1}\right) + \overline{N}_0\left(r, \tfrac{1}{F'}\right) + \overline{N}_0\left(r, \tfrac{1}{G'}\right) + S(r, f) \end{aligned} \tag{3.2}$$

We also have ,

$$\overline{N}_2\left(r,\frac{1}{F}\right)+2\overline{N}_L\left(r,\frac{1}{F-1}\right)+\overline{N}_*\left(r,\frac{1}{F-1}\right)+\overline{N}_0\left(r,\frac{1}{F'}\right)\leq 2\overline{N}\left(r,\frac{1}{F'}\right) \quad (3.3)$$

Now by the second fundamental theorem we get,

$$T\left(r,G\right)\leq\overline{N}(r,G)+\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)-\overline{N}_0\left(r,\frac{1}{G'}\right)+S(r,G) \quad (3.4)$$

From (3.4) using (3.2) and (3.3) we get,

$$T(r,G)\leq 2\overline{N}(r,f)+2\overline{N}\left(r,\frac{1}{F'}\right)+2\overline{N}\left(r,\frac{1}{G'}\right)+\overline{N}\left(r,\frac{1}{G}\right)+S(r,f) \quad (3.5)$$

By Lemma (2.1) we have,

$$\begin{aligned}T\left(r,(f^nP(f))^{(k)}\right)\leq 6\overline{N}\left(r,f\right)+2N_2\left(r,\frac{1}{f^nP(f)}\right)+2N_2\left(r,\frac{1}{(f^nP(f))^{(k)}}\right)\\+\overline{N}\left(r,\frac{1}{(f^nP(f))^{(k)}}\right)+S(r,f)\end{aligned}$$

which contradicts the given conditions of the theorem.

**Case II** : Suppose $H(z)\equiv 0$ i.e., $\frac{F''}{F'}-\frac{2F'}{F-1}=\frac{G''}{G'}-\frac{2G'}{G-1}$. Integrating we get,

$\log F^{'}-2\log(F-1)=\log G^{'}-2\log(G-1)+\log A$. Where A is a constant $\neq 0$.

That is, $\log\dfrac{F^{'}}{(F-1)^2}=\log\dfrac{AG^{'}}{(G-1)^2}$.

Again integrating we get,

$$\frac{1}{F-1}=\frac{A}{G-1}+B \quad (3.6)$$

Now if $z_0$ is a pole of $f$ with multiplicity $p$ which is not the poles and the zeros of $a(z)$, then $z_0$ is the pole of $F$ with multiplicity $(n+m)p$ and the pole of $G$ with multiplicity $(n+m)p+k$ $(\neq(n+m)p)$. This contradicts (3.6). This implies $f$ has no pole , that is $f$ is an entire function.

So, $\overline{N}(r,F)=S(r,f)$ and $\overline{N}(r,G)=S(r,f)$. Now we prove that $B=0$.

We first assume that $B\neq 0$, then $\dfrac{1}{F-1}=\dfrac{B\left(G-1+\frac{A}{B}\right)}{G-1}$.

Therefore, $\overline{N}\left(r,\dfrac{1}{G-1+\frac{A}{B}}\right)=\overline{N}(r,F)=S(r,f)$

Now we assume $\frac{A}{B} \neq 1$.
By the Second fundamental theorem,

$$\begin{aligned}
T(r, G) &\leq \overline{N}(r, G) + \overline{N}\left(r, \frac{1}{G}\right) + \overline{N}\left(r, \frac{1}{G - 1 + \frac{A}{B}}\right) + S(r, G) \\
&\leq \overline{N}\left(r, \frac{1}{G}\right) + S(r, f) \\
&\leq T(r, G) + S(r, f)
\end{aligned}$$

Hence , $T(r,G) = \overline{N}\left(r, \frac{1}{G}\right) + S(r, f)$ i.e., $T\left(r, (f^nP(f))^{(k)}\right) = \overline{N}\left(r, \frac{1}{(f^nP(f))^{(k)}}\right) + S(r, f)$

This contradicts the given condition of the theorem.
Next, we assume $\frac{A}{B} = 1$. Then, $(AF - A - 1)G = -1$.
So, $\frac{a^2}{f^nP(f)\left(Af^nP(f) - Aa - A\right)} = -\frac{(f^nP(f))^{(k)}}{f^nP(f)}$

Now by lemma (2.1) and (2.2), we get,

$$\begin{aligned}
2(n+m)T(r, f) &= T\left(r, \frac{(f^nP(f))^{(k)}}{f^nP(f)}\right) + S(r, f) \\
&\leq N\left(r, \frac{(f^nP(f))^{(k)}}{f^nP(f)}\right) + S(r, f) \\
&\leq N_k\left(r, \frac{1}{f^nP(f)}\right) + k\overline{N}(r, f) + S(r, f) \\
&\leq k\overline{N}\left(r, \frac{1}{f}\right) + mN\left(r, \frac{1}{f}\right) + S(r, f)
\end{aligned}$$

i.e., $(2n + m - k)T(r, f) = S(r, f)$. Which is impossible since $m \geq k$.
Hence our assumption is not true and therefore $B = 0$. So, $\frac{G-1}{F-1} = A$.
This proves the theorem.

**Proof of Theorem 2** : Let $F = \frac{f^nP(f)}{a(z)}$ and $G = \frac{(f^nP(f))^{(k)}}{a(z)}$. So, $\overline{E}_{l)}\left(a, f^nP(f)\right) = \overline{E}_{l)}\left(a, (f^nP(f))^{(k)}\right)$ implies, $\overline{E}_{l)}(1, F) = \overline{E}_{l)}(1, G)$, except the zeros and poles of $a(z)$.
We define,

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right).$$

Now we consider two cases:

**Case I** : Suppose $H \not\equiv 0$,

Then (3.5) of the proof in theorem 1 still holds. Writing (3.5) for the function $F$, we get,

$$T(r,F) \leq 2\overline{N}(r,f) + 2\overline{N}\left(r,\frac{1}{G'}\right) + 2\overline{N}\left(r,\frac{1}{F'}\right) + \overline{N}\left(r,\frac{1}{G}\right) + S(r,f)$$

$$\begin{aligned}\text{i.e.,}\quad (n+m)T(r,f) &\leq 2\overline{N}(r,f) + 2N_2\left(r,\frac{1}{(f^nP(f))^{(k)}}\right) + 2\overline{N}(r,f) \\ &\quad + 2N_2\left(r,\frac{1}{(f^nP(f))}\right) + 2\overline{N}(r,f) + \overline{N}\left(r,\frac{1}{(f^nP(f))^{(k)}}\right) + S(r,f) \\ &\leq (3k+6)\overline{N}(r,f) + (3k+9)\overline{N}\left(r,\frac{1}{f}\right) + 5mN\left(r,\frac{1}{f}\right) + S(r,f).\end{aligned}$$

i.e., $(3k+6)\Theta(\infty,f)$ + (3k+9)$\Theta(0,f)$ +5$m\delta(0,f) \leq 6k+4m+15-n$.

This contradicts the given condition of the theorem.

**Case II** : Suppose $H \equiv 0$.

So, $\dfrac{1}{F-1} = \dfrac{A}{G-1} + B$, where $A \neq 0$, $B$ are constants. By the same argument of the proof of the

theorem 1, we get,

$\overline{N}(r,F) = S(r,f)$ and $\overline{N}(r,G) = S(r,f)$.

So, $\Theta(\infty,f) = 1$.

Assume that, $B \neq 0$, then $\dfrac{B\left(F-1-\frac{1}{B}\right)}{F-1} = -\dfrac{A}{G-1}$

So, $\overline{N}\left(r,\dfrac{1}{F-1+\frac{1}{B}}\right) = \overline{N}(r,G) = S(r,f)$.

If $B \neq -1$, then by the second fundamental theorem for $F$, we have

$$\begin{aligned}T\left(r,F\right) &\leq \overline{N}(r,F) + \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}\left(r,\frac{1}{F-1+\frac{A}{B}}\right) + S(r,f) \\ &\leq \overline{N}\left(r,\frac{1}{F}\right) + S(r,f) \\ &\leq T(r,F) + S(r,f)\end{aligned}$$

So, $T(r,F) \leq \overline{N}\left(r,\frac{1}{F}\right) + S(r,f)$ i.e.,$(n+m)T(r,f) \leq \overline{N}\left(r,\frac{1}{f}\right) + mN\left(r,\frac{1}{f}\right)$.
Hence, $\Theta(0,f) + m\delta(0,f) \leq 1-n$.
Putting $\Theta(\infty,f)=1$; $\Theta(0,f)+m\delta(0,f) \leq 1-n$ in the given condition of the theorem we have, $\Theta(0,f) > 1$, which is not true . Hence $B=-1$.
Therefore, $\dfrac{F}{F-1} = \dfrac{A}{G-1}$.
i.e., $F(G-1-A) = -A$ that is $F = \dfrac{A}{-G+(1+A)}$.
So, $f^nP(f) = \dfrac{A}{-(f^nP(f))^{(k)}+(1+A)}$. Therefore, $\overline{N}\left(r,\dfrac{1}{(f^nP(f))^{(k)}+(1+A)}\right) = \overline{N}(r,f) = S(r,f)$.
Hence, $T(r,f) = T\left(r,(f^nP(f))^{(k)}\right) = S(r,f)$. Which is not true. Thus $B=0$.
So, $\dfrac{1}{F-1} = \dfrac{A}{G-1}$, i.e., $G-1 = A(F-1)$.

If $A \neq 1$ then $G = A\left(F-1+\dfrac{1}{A}\right)$. So, $N\left(r,\dfrac{1}{G}\right) = N\left(r,\dfrac{1}{F-1+\frac{1}{A}}\right)$.
By the second fundamental theorem, we have,

$$T(r,F) \leq \overline{N}(r,F) + \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}\left(r,\frac{1}{F-1+\frac{1}{A}}\right) + S(r,f)$$

$$\begin{aligned}
T(r,f^nP(f)) &\leq \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}\left(r,\frac{1}{G}\right) + S(r,f) \\
&\leq \overline{N}\left(r,\frac{1}{f^nP(f)}\right) + \overline{N}\left(r,\frac{1}{(f^nP(f))^{(k)}}\right) + S(r,f) \\
&\leq \overline{N}\left(r,\frac{1}{f}\right) + mN\left(r,\frac{1}{f}\right) + N_{k+1}\left(r,\frac{1}{f^nP(f)}\right) + k\overline{N}(r,f) + S(r,f) \\
&= \overline{N}\left(r,\frac{1}{f}\right) + mN\left(r,\frac{1}{f}\right) + N_{k+1}\left(r,\frac{1}{f^nP(f)}\right) + S(r,f)
\end{aligned}$$

So,

$$(k+2)\Theta(0,f) + 2m\delta(0,f) \leq k+2+m-n \tag{3.7}$$

Now by the given condition of the theorem and by (3.7) we have, $\Theta(0,f) > 1$. This is not possible.
So, $A=1$ and hence $F=G$ i.e., $f^nP(f) = (f^nP(f))^{(k)}$.
This proves the theorem.

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