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SOLVE ABEL'S INTEGRAL EQUATION USING POINT INTERPOLATION MESHLESS METHOD

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Abstract

In this paper the numerical method for solving Abel's integral equation this method is based on point interpolation meshless method. Also Radial basis function, zeros of the shifted Legendre polynomial as the collocation points utilized to apply for solving Abel,s integral equation of the first and second kind the result of numerical experiment show that the numerical scheme is very effective and convenient of this method.

1. Introduction

Abel's integral equation an important tool for modeling an numerous phenomena in basic for engineering sciences such as chemistry, physics, biology, mechanics and electronic. Abel's integral equation appearing in two form the first and second kind as follows:

$$f(x) = \int_0^x \frac{y(t)}{\sqrt{x-t}} dt$$

and

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$$y(x) = f(x) + \int_0^x \frac{y(t)}{\sqrt{x-t}}dt.$$

In general

$$y(x) = f(x) + \int_0^x K(x,t,y(t))dt \qquad (1)$$

where $K(x,t,y(t)) = \frac{y(t)}{\sqrt{x-t}}$, $f(x)$ is continuous function, $0 \leq x,t \leq c$, c is constant. Our main aim is offering a new approach for solving these cases.

Able's integral equation with singularity property causes hard and heavy computation, different methods solving this equation but only some of them are efficient [1,2,7].

A point interpolation method (PIM) was proposed to address above two issues [5, 6]. The (PIM) seems attractive several way, first its approximation function passes through each node in an influence domain, second, its shape function are simple compared with any other method, third, is shape function and derivatives are easily developed only if basis function are selected.

This paper proposes a point interpolation meshless method based on radial basis function for the solution of Able's integral equation. This forms a radial PIM, particularly, multiquadric radial basis function are applied in the radial PIM. For convenience the solution we use radial with $\{x_j\}_{j=1}^N$ nodes which are the zeros of the shifted Legendre polynomial $L_N(x), 0 \leq x \leq 1$. The shifted Legendre polynomial $L_i(x)$ are defined on the interval [0,1] and satisfy the following formulae [8].

$$L_0(x) = 1, \qquad L_i(x) = 2x - 1$$

$$L_{i+1}(x) = \frac{2i}{i+1}(2x-1)L_i(x) - \frac{i}{i+1}L_{i-1}(x), \quad i = 1, 2, 3, \cdots$$

Finally, by using the collocation method we obtain the system of linear equation.

The paper it following way: in section 2, we describe the properties of radial(PIM) function. In section 3 we combined point interpolation technique, radial basis function and collocation method for solving Able's integral equation. Some examples are investigated in section 4, the numerical result show the accuracy of the method. The conclusions are discussed in the final section.

2. Radial Basis Functions

2.1 Definition of Radial Basis Function

Let $R^+ = \{x \in R, x \geq 0\}$ be the non-negative half-line and let $B : R^+ \rightarrow R$ be a continuous function with $B(0) \geq 0$. A radial basis functions on R^d is a function of the form $B(\|X - X_i\|)$, where $X, X_i \in R^d$ and $\|\cdot\|$ denotes the Euclidean distance between X and X_i^s. If one chooses N points $\{x\}_{j=1}^N$ in R^d then by custom

$$\theta(X) = \sum_{i=1}^{N} \lambda_i B(\|X - X_i\|); \quad \lambda_i \in R,$$

is called a radial basis functions as well [3].

The multiquadrics function (MQ) define

$$\sqrt{r^2 + c^2}, \text{`} \quad r = \|X - X_i\|, \quad c > 0.$$

2.2 Point Interpolation Based on Radial Basis Function

Consider an approximation function $y(x)$ in an influence domain that has a set of arbitrarily distributed nodes $P_i(x)(i = 1, 2, \cdots, n)$. n is the number of nodes in the influence domain of x. Nodal function value is assumed to be y_i at the node x_i. Radial PIM constructs the approximation function $y(x)$ to pass through all these node points using radial basis function $B_i(x)$ and polynomial basis function $p_j(x)$ [6]

$$y(x) = B(x)a_i + P(x)b_j = B^T(x)a + P^T(x)b \tag{2}$$

where a_i is the coefficient for $B_i(x)$ and b_j the coefficient for $p_i(x)$ (usually, $m < n$). The vectors are defined a

$$\begin{aligned} a^T &= [a_1, a_2, \cdots, a_n] \\ b^T &= [b_1, b_2, \cdots, b_m] \\ B^T &= [B_1(x) \quad B_2(x) \cdots B_n(x)] \\ P^T &= [P_1(x) \quad P_2(x) \cdots P_m(x)]. \end{aligned} \tag{3}$$

A polynomial basis function has the following monomial terms as:

$$P^T(x) = [1, x, x^2, x^3, \cdots]. \tag{4}$$

The coefficients a_i and b_j in Equation (2) are determined by enforcing the interpolation pass through all n scattered nodal points within the influence domain. The interpolation at the k^{th} point has

$$y_k(x) = \sum_{i=1}^{n} a_i B_i(x_k) + \sum_{j=1}^{m} b_j P_j(x_k), \quad k = 1, 2, \cdots, n. \tag{5}$$

The polynomial term is an extra-requirement that guarantees unique approximation [3]. Following constraints are usually imposed: It is expressed in matrix form as follows:

$$\begin{bmatrix} B_0 & P_0 \\ P_0^T & 0 \end{bmatrix} \begin{Bmatrix} a \\ b \end{Bmatrix} == \begin{Bmatrix} y^e \\ 0 \end{Bmatrix} \quad \text{or} \quad G \begin{Bmatrix} a \\ b \end{Bmatrix} = \begin{Bmatrix} y^e \\ 0 \end{Bmatrix} \tag{6}$$

where the vector for function values is defined as

$$y^e = [y_1, y_2, \cdots, y_n]^T. \tag{7}$$

The coefficient matrix B_0 on unknowns a is

$$B_0 = \begin{pmatrix} B_1(x_1) & \cdots & B_n(x_1) \\ \vdots & \ddots & \vdots \\ B_1(x_n) & \cdots & B_n(x_n) \end{pmatrix} \tag{8}$$

The coefficient matrix P_0 on unknowns b is

$$P_0 = \begin{pmatrix} P_1(x_1) & \cdots & P_m(x_1) \\ \vdots & \ddots & \vdots \\ P_1(x_n) & \cdots & P_n(x_n) \end{pmatrix} \tag{9}$$

Because the distance is directionless, there is $B_k(x_i) = B_i(x_k)$, which means that the matrix B_0 is symmetric. Unique solution is obtained if the inverse of matrix B_0 exists,

$$\begin{Bmatrix} a \\ b \end{Bmatrix} = G^{-1} \begin{Bmatrix} y^e \\ 0 \end{Bmatrix}.$$

The interpolation is finally expressed as

$$y(x) = [B^T(x) \;\; P^T(x)] G^{-1} \begin{Bmatrix} y^e \\ 0 \end{Bmatrix} = \Phi(x)\Lambda$$

where the matrix of shape functions $\Phi(x)$ is defined by

$$\Phi(x) = [\Phi_1(x), \Phi_2(x), \cdots, \Phi_n(x)]$$

in which

$$\Phi_k(x) = \sum_{i=1}^{n} B_i(x) G^{-}_{i,k} + \sum_{j=1}^{m} P_j(x) G^{-}_{n+j,k}$$

where $G^{-}_{i,k}$ is the $(i;k)$ element of matrix G^{-1}. After radial basis functions are determined, shape functions depend only upon the position of scattered nodes. Once the inverse of matrix G is obtained.

The results of this section can be summarized in the following algorithm.

Algorithm

The algorithm works in the following manner:

Choose N center point $\{x_j\}_{j=1}^{N}$ from the domain set $[a, b]$.

1. Approximate $y(x)$ as $y_N(x) = \Phi^T(x)\Lambda$.
2. Substitute $y_N(x)$ into the main problem and creat residual function $Res(x)$.
3. Substitute collocation points $\{x_j\}_{j=1}^{N}$ into the $Res(x)$ and create the N equations.
4. Solve the N equations with N unknown coefficients of members of Λ and find the numerical solution.

3. Description of the Method

In the present method, the closed form PIM approximating function Eq. (1) is first obtained from a set of training points, as follows:

$$y(x) = y_N(x) = \sum_{i=1}^{N} \lambda_i \phi_i(x) = \Phi^T(x)\Lambda. \tag{10}$$

Then, from substituting Eq. (11) into Eq. (1), we have

$$\Phi^T(x)\Lambda = f(x) + \int_0^x k(x, t, \Phi^T(t)\Lambda) dt. \tag{11}$$

We now collocate Eq. (11) at points $\{x_i\}_{i=1}^{N}$ as

$$\Phi^T(x_i)\Lambda = f(x_i) + \int_0^{x_i} K(x_i, t, \Phi^T(t)\Lambda) dt \tag{12}$$

$$Res(x_i) = -\Phi^T(x_i)\Lambda + f(x_i) + \int_0^{x_i} K(x_i, t, \Phi^T(t)\Lambda)dt. \tag{13}$$

The set of equations for obtaining the coefficients $\{\lambda_i\}_{i=1}^N$ come from equalizing Eq. (13) to zero at N interpolate nodes. Behavior of the MQ-RBF method, we applied the law root of mean square error (RMSR)

$$RMSR = \sqrt{\frac{\sum_{i=1}^{N}(y(x_i) - y_n(x_i))}{N}}.$$

4. Numerical Results

In order to illustrate the performance of radial point interpolation meshless method in solving Able's integral equation and justify the accuracy and efficiency of our method, we consider the following examples. We use multiquadrics (MQ) RBF.

Example 4.1 : Consider the following Abel's integral equation [9]

$$\int_0^x \frac{y(t)}{\sqrt{x-t}}dt = \frac{2}{105}\sqrt{x}(105 - 56x^2 + 48x^3), \quad x = [0, 1]. \tag{14}$$

With the exact solution $y(x) = x$. Applied present method and solve eq. (14). RMSR value is $1.5483 * 10^{-14}$ by using $m = 5$ and $n = 7$ multiquadrics function.

Example 4.2 : Consider the following Abel's integral equation [9]

$$y(t) = x^2 + \frac{16}{15}x^{\frac{5}{2}} - \int_0^x \frac{y(t)}{\sqrt{x-t}}dt` \quad x = [0, 1]. \tag{15}$$

With the exact solution $y(x) = x^3 - x^2 + 1$. Applied present method and solve eq. (15). RMSR value is $1.4675 * 10^{-14}$ by using m=5 and n=7 multiquadrics function.

Example 4.3 : Consider the following Abel's integral equation [4]

$$y(t) = x - \frac{4}{3}x^{\frac{3}{2}} + \int_0^x \frac{y(t)}{\sqrt{x-t}}dt \quad x = [0, 1]. \tag{16}$$

With the exact solution $y(x) = x$. Applied present method and solve eq. (16). RMSR value is $1.001 * 10^{-15}$ by using $m = 5$ and $n = 7$ multiquadrics function.

5. Conclusion

In this method numerical scheme to solve Ablel's integral equation using collocation points and approximation the solution using the multiqudric (MQ) radial a point interpolation meshless method. We note that this method is easy to computation and through the comparison with exact solution we show that the method is good accuracy and efficiency.

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