MINIMUM COVERING PARTITION ENERGY OF A GRAPH

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Abstract

The Partition energy of a graph was introduced by E. Sampathkumar et al. (2015). Motivated by this, we introduce the concept of minimum covering partition energy of a graph, $E_p^{C}(G)$ and compute the minimum covering partition energy $E_p^{C}(G)$ of few families of graphs. Also, we established the bounds for minimum covering partition energy.

1. Introduction

The graph energy plays a vital role in chemistry to find the total $\pi$-electron energy of a molecule. The conjugated hydrocarbons can be represented by a graph called molecular

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graph. We can represent every carbon atom by a vertex and every carbon-carbon bond by an edge and hydrogen atoms are ignored. Recently several matrices like adjacency matrix, Laplacian matrix, distance matrix, maximum degree matrix, minimum degree matrix, matrix of a subset $S$ of $V$ and color energy of a graph are studied in [1, 2, 4, 5, 10, 11 and 16].

2. Minimum Covering Partition Energy of a Graph

Let $G$ be a simple graph of order $n$ with vertex set $V = v_1, v_2, v_3, ..., v_n$ and edge set $E$. A subset $C$ of $V$ is called a covering set of $G$ if every edge of $G$ is incident to at least one vertex of $C$ (see [4]). Any covering set with minimum cardinality is called a minimum covering set. Let $C$ be a minimum covering set of a graph $G$. The minimum covering partition matrix is given by

$$a_{ij} = \begin{cases} 
2 & \text{if } v_i \text{ and } v_j \text{ are adjacent where } v_i, v_j \in V_r, \\
-1 & \text{if } v_i \text{ and } v_j \text{ are non-adjacent where } v_i, v_j \in V_r, \\
1 & \text{if } i = j \text{ and } v_i \in C, \\
1 & \text{if } v_i \text{ and } v_j \text{ are adjacent between the sets } V_r \text{ and } V_s \text{ for } r \neq s, \text{ where } v_i \in V_r \text{ and } v_j \in V_s, \\
0 & \text{otherwise.}
\end{cases}$$

In this paper, we study minimum covering partition energy of a graph with respect to given partition of a graph. Further, we determine minimum covering partition energy of two types of complements of a partition graph called $k$-complement and $k(i)$-complement of a graph introduced by E. Sampathkumar in [13].

**Definition 2.1**: The complement of a graph $G$ is a graph $\overline{G}$ on the same vertices such that two distinct vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in $G$.

**Definition 2.2** [13]: Let $G$ be a graph and $P_k = \{V_1, V_2, ..., V_k\}$ be a partition of its vertex set $V$. Then the $k$-complement of $G$ is obtained as follows: For all $V_i$ and $V_j$ in $P_k$, $i \neq j$ remove the edges between $V_i$ and $V_j$ and add the edges between the vertices of $V_i$ and $V_j$ which are not in $G$ and is denoted by $(G)_k$.

**Definition 2.3** [13]: Let $G$ be a graph and $P_k = \{V_1, V_2, ..., V_k\}$ be a partition of its vertex set $V$. Then the $k(i)$-complement of $G$ is obtained as follows: For each set $V_r$ in $P_k$, remove the edges of $G$ joining the vertices within $V_r$ and add the edges of $\overline{G}$ (complement of $G$) joining the vertices of $V_r$, and is denoted by $(G)_{k(i)}$. 
3. Some Basic Properties of Minimum Covering Partition Energy of a Graph

Let \( G = (V, E) \) be a graph with \( n \) vertices and \( P_k = \{V_1, V_2, \ldots, V_k\} \) be a partition of \( V \). For \( 1 \leq i \leq k \), let \( b_i \) denote the total number of edges joining the vertices of \( V_i \) and \( c_i \) be the total number of edges joining the vertices from \( V_i \) to \( V_j \) for \( i \neq j, 1 \leq j \leq k \) and \( d_i \) be the number of non-adjacent pairs of vertices within \( V_i \). Let \( m_1 = \sum_{i=1}^{k} b_i \), \( m_2 = \sum_{i=1}^{k} c_i \) and \( m_3 = \sum_{i=1}^{k} d_i \). Let \( P_k^C(G) \) be the minimum covering partition matrix. If the characteristic polynomial of \( P_k^C(G) \) denoted by \( \Phi_k^C(G, \lambda) \) is \( a_0 \lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \cdots + a_n \), then the coefficient \( a_i \) can be interpreted using the principal minors of \( P_k^C(G) \).

The following proposition determines the first three coefficients of the characteristic polynomial of \( P_k^C(G) \).

**Proposition 3.1**: The first three coefficients of \( \phi_k^C(G, \lambda) \) are given as follows:

(i) \( a_0 = 1 \),

(ii) \( a_1 = -|C| \),

(iii) \( a_2 = |C|C_2 - [4m_1 + m_2 + m_3] \).

**Proof**: (i) From the definition \( \Phi_k(G, \lambda) = \det[\lambda I - P_k^C(G)] \), we get \( a_0 = 1 \).

(ii) The sum of determinants of all \( 1 \times 1 \) principal submatrices of \( P_k^C(G) \) is equal to the trace of \( P_k^C(G) \).

\[ \Rightarrow a_1 = (-1)^1 \text{ trace of } [P_k^C(G)] = -|C|. \]

(iii)

\[ (-1)^2 a_2 = \sum_{1 \leq i < j \leq n} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} \]

\[ = \sum_{1 \leq i < j \leq n} a_{ii}a_{jj} - a_{ji}a_{ij} \]

\[ = \sum_{1 \leq i < j \leq n} a_{ii}a_{jj} - \sum_{1 \leq i < j \leq n} a_{ji}a_{ij} \]

\[ = |C|C_2 - [(2)^2 m_1 + (1)^2 m_2 + (-1)^2 m_3] = |C|C_2 - [4m_1 + m_2 + m_3]. \]

\( \square \)
**Proposition 3.2** : If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are partition eigenvalues of $P^C_k(G)$, then

$$\sum_{i=1}^{n} \lambda_i^2 = |C| + 2[4m_1 + m_2 + m_3].$$

We know that

$$\sum_{i=1}^{n} \lambda_i^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}a_{ji}$$

$$= 2\sum_{i<j} (a_{ij})^2 + \sum_{i=1}^{n} (a_{ii})^2$$

$$= 2\sum_{i<j} (a_{ij})^2 + |C|$$

$$= |C| + 2[4m_1 + m_2 + m_3].$$

**Theorem 3.3** : Let $G$ be a graph with $n$ vertices and $P_k$ be a partition of $G$. Then

$$E^C_{P_k}(G) \leq \sqrt{n(|C| + 2[4m_1 + m_2 + m_3])}$$

where $m_1, m_2, m_3$ are as defined above for $G$.

**Proof** : Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of $P^C_k(G)$.

Now by Cauchy - Schwartz inequality we have

$$\left(\sum_{i=1}^{n} a_ib_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right) \left(\sum_{i=1}^{n} b_i^2\right).$$

Let $a_i = 1 , b_i = |\lambda_i|$. Then then

$$\left(\sum_{i=1}^{n} |\lambda_i|\right)^2 \leq \left(\sum_{i=1}^{n} 1\right) \left(\sum_{i=1}^{n} |\lambda_i|^2\right)$$

$$[E^C_{P_k}]^2 \leq n(|C| + 2[4m_1 + m_2 + m_3])$$

$$[E^C_{P_k}] \leq \sqrt{n(|C| + 2[4m_1 + m_2 + m_3])}$$
Which is upper bound.

**Theorem 3.4**: Let $G$ be a partition graph with $n$ vertices. If $R = \det P_k^C(G)$, then

$$E^C_{P_k}(G) \geq \sqrt{(|C| + 2[4m_1 + m_2 + m_3]) + n(n-1)R^2_n}.$$ 

**Proof**: By definition,

$$(E^C_{P_k}(G))^2 = \left(\sum_{i=1}^{n} |\lambda_i| \right)^2$$

$$= \sum_{i=1}^{n} |\lambda_i| \left(\sum_{j=1}^{n} |\lambda_j| \right)$$

$$= \left(\sum_{i=1}^{n} |\lambda_i|^2 \right) + \sum_{i \neq j} |\lambda_i||\lambda_j|.$$ 

Using arithmetic mean and geometric mean inequality, we have

$$\frac{1}{n(n-1)} \sum_{i \neq j} |\lambda_i||\lambda_j| \geq \left(\prod_{i \neq j} |\lambda_i||\lambda_j| \right)^{\frac{1}{n(n-1)}}.$$ 

Therefore,

$$[E^C_{P_k}(G)]^2 \geq \sum_{i=1}^{n} |\lambda_i|^2 + n(n-1) \left(\prod_{i \neq j} |\lambda_i||\lambda_j| \right)^{\frac{1}{n(n-1)}}$$

$$\geq \sum_{i=1}^{n} |\lambda_i|^2 + n(n-1) \left(\prod_{i=1}^{n} |\lambda_i| \right)^{2(n-1)}$$

$$= \sum_{i=1}^{n} |\lambda_i|^2 + n(n-1)R^2_n$$

$$= (|C| + 2[4m_1 + m_2 + m_3]) + n(n-1)R^2_n.$$ 

Thus,

$$E^C_{P_k}(G) \geq \sqrt{(|C| + 2[4m_1 + m_2 + m_3]) + n(n-1)R^2_n}. \quad \square$$ 

**Theorem 3.5**: If the minimum covering partition energy of a graph is a rational number, then it must be a positive even number.
Proof of this theorem is similar to the proof of Theorem 2.12 in [6].

4. Energy of Some Partition Graphs and Their Complements

**Theorem 4.1** : The minimum covering 1-partition energy of a complete graph $K_n$ is $E_{C_1}^C(K_n) = (n - 2) + \sqrt{4n^2 + 4n - 7}$.

**Proof** : Let $K_n$ be the complete graph with vertex set $\{v_1, v_2, v_3, \ldots, V_n\}$. Consider all the vertices is in one partition. The minimum covering set $= C = \{v_1, v_2, v_3, \ldots, v_{n-1}\}$.

The minimum covering 1-partition matrix is

$$P_{C_1}^C(K_n) = \begin{bmatrix}
1 & 2 & 2 & \ldots & 2 & 2 \\
2 & 1 & 2 & \ldots & 2 & 2 \\
2 & 2 & 1 & \ldots & 2 & 2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
2 & 2 & 2 & \ldots & 1 & 2 \\
2 & 2 & 2 & \ldots & 2 & 0 \\
\end{bmatrix}.$$ 

Characteristic equation is

$$(\lambda + 1)^{n-2}(\lambda^2 - (2n - 3)\lambda - (4n - 4)) = 0$$

and the spectrum is $Spec_{C_1}^C(K_n) = \left(\begin{array}{cc}
-1 & \frac{2n-3}{n-2} + \sqrt{\frac{4n^2 + 4n - 7}{2}} \\
\frac{2n-3}{n-2} - \sqrt{\frac{4n^2 + 4n - 7}{2}} & 1 \\
\end{array}\right).$

Therefore, $E_{C_1}^C(K_n) = (n - 2) + \sqrt{4n^2 + 4n - 7}$. 

**Theorem 4.2** : The minimum covering 1-partition energy of star graph $K_{1,n-1}$ is $E_{C_1}^C(K_{1,n-1}) = (n - 2) + \sqrt{n^2 + 14n - 15}$.

**Proof** : Consider all the vertices is in one partition. The minimum covering set $= C = \{v_1\}$. The minimum covering 1-partition matrix is

$$P_{C_1}^C(K_{1,n-1}) = \begin{bmatrix}
1 & 2 & 2 & \ldots & 2 & 2 \\
2 & 0 & -1 & \ldots & -1 & -1 \\
2 & -1 & 0 & \ldots & -1 & -1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
2 & -1 & -1 & \ldots & 0 & -1 \\
2 & -1 & -1 & \ldots & -1 & 0 \\
\end{bmatrix}.$$ 

Characteristic equation is

$$(\lambda - 1)^{n-2}[(\lambda^2 + (n - 3)\lambda - (5n - 6))] = 0$$
spectrum is \( \text{Spec}^C_{P_1}(K_{1,n-1}) = \begin{pmatrix} 1 & -\frac{(n-3)+\sqrt{n^2+14n-15}}{2} & -\frac{(n-3)-\sqrt{n^2+14n-15}}{2} \\ n-2 & 1 & 1 \\ \end{pmatrix} \).

Therefore, \( E^C_{P_1}(K_{1,n-1}) = (n - 2) + \sqrt{n^2 + 14n - 15} \).

**Definition 4.3**: The Crown graph \( S^0_n \) for an integer \( n \geq 3 \) is the graph with vertex set \( \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\} \) and edge set \( \{u_i v_i : 1 \leq i, j \leq n, i \neq j\} \). \( S^0_n \) is therefore equivalent to the complete bipartite graph \( K_{n,n} \) with horizontal edges removed.

**Theorem 4.4**: The minimum covering 1-partition energy of Crown graph \( S^0_n \) is

\[
E^C_{P_1}(S^0_n) = \sqrt{37}(n - 1) + \sqrt{16n^2 - 48n + 37}
\]

**Proof**: Consider all the vertices is in one partition. Let \( S^0_n \) be a crown graph of order \( 2n \) with vertex set \( \{u_1, u_2, \ldots, u_n, v_1, v_2, v_3 \ldots v_n\} \) and minimum covering set = \( C = \{u_1, u_2, u_3 \ldots u_n\} \). The minimum covering 1-partition matrix is

\[
P^C_{1}(S^0_n) = \begin{bmatrix}
1 & -1 & -1 & \ldots & -1 & -1 & 2 & \ldots & 2 & 2 \\
-1 & 1 & -1 & \ldots & -1 & 2 & -1 & \ldots & 2 & 2 \\
-1 & -1 & 1 & \ldots & -1 & 2 & 2 & \ldots & -1 & -1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-1 & -1 & -1 & \ldots & 1 & 2 & 2 & \ldots & 2 & -1 \\
-1 & 2 & 2 & \ldots & 2 & 0 & -1 & \ldots & -1 & -1 \\
2 & -1 & 2 & \ldots & 2 & -1 & 0 & \ldots & -1 & -1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
2 & 2 & -1 & \ldots & 2 & -1 & -1 & \ldots & 0 & -1 \\
2 & 2 & 2 & \ldots & -1 & -1 & -1 & \ldots & -1 & 0
\end{bmatrix}
\]

Characteristic equation is

\[
|\lambda^2 + (2n - 3)\lambda - (3n^2 - 9n + 7)|[\lambda^2 - 3\lambda - 7]^{n-1} = 0
\]

spectrum is \( \text{Spec}^C_{P_1}(S^0_n) \)

\[
= \begin{pmatrix}
\frac{3+\sqrt{37}}{2} & \frac{3-\sqrt{37}}{2} & \frac{(2n-3)+\sqrt{16n^2-48n+37}}{2} & \frac{(2n-3)-\sqrt{16n^2-48n+37}}{2} \\
(n-1) & (n-1) & 2 & 1 \\
\end{pmatrix}
\]. Therefore,

\[
E^C_{P_1}(S^0_n) = \sqrt{37}(n - 1) + \sqrt{16n^2 - 48n + 37} \).
\]

**Definition 4.5**: The double star graph \( S_{n,m} \) is the graph constructed from \( K_{1,n-1} \) and \( K_{1,m-1} \) by joining their centers \( v_0 \) and \( u_0 \). \( V(S_{n,m}) = V(K_{1,n-1}) \cup V(K_{1,m-1}) \) and \( E(S_{n,m}) = \{v_0 u_0; v_0 v_i; u_0 u_j : 1 \leq i \leq n - 1, 1 \leq j \leq m - 1\} \). Therefore, double star graph is bipartite graph.
Theorem 4.6 : The minimum covering 1-partition energy of Double star graph $S_{n,n}$ is
\[ E_{PC}^{C}(S_{n,n}) = (2n - 4) + \sqrt{4n^2 + 4n - 4} + 2(\sqrt{9n - 8}). \]

Proof : Consider all the vertices in the one partition. The minimum covering set $C = \{u_0, v_0\}$. The minimum covering 1-partition matrix is
\[
P_{PC}^{C}(S_{n,n}) = \begin{bmatrix}
1 & 2 & 2 & \ldots & 2 & 2 & -1 & -1 & \ldots & -1 \\
2 & 0 & -1 & \ldots & -1 & -1 & -1 & -1 & \ldots & -1 \\
2 & -1 & 0 & \ldots & -1 & -1 & -1 & -1 & \ldots & -1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
2 & -1 & -1 & 0 & -1 & -1 & -1 & -1 & \ldots & -1 \\
2 & -1 & -1 & \ldots & -1 & 1 & 2 & 2 & \ldots & 2 \\
-1 & -1 & -1 & \ldots & -1 & 2 & 0 & -1 & \ldots & -1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-1 & -1 & -1 & \ldots & -1 & 2 & -1 & 0 & \ldots & -1 \\
-1 & -1 & -1 & \ldots & -1 & 2 & -1 & -1 & \ldots & -1
\end{bmatrix}.
\]

Characteristic equation is
\[
(\lambda - 1)^{2n-4}[\lambda^2 + (2n - 6)\lambda - (7n - 10)][\lambda^2 - (9n - 8)] = 0
\]

Hence, spectrum is
\[
Spec_{PC}^{C}(S_{n,n}) = \left( \frac{1}{2n - 4}, \frac{\sqrt{9n - 8}}{1}, \frac{-\sqrt{9n - 8}}{1}, \frac{(2n-6)+\sqrt{4n^2+4n-4}}{2}, \frac{1}{2}, \frac{(n-6)-\sqrt{4n^2+4n-4}}{2} \right)
\]

Therefore, $E_{PC}^{C}(S_{n,n}) = (2n - 4) + \sqrt{4n^2 + 4n - 4} + 2(\sqrt{9n - 8})$. \hfill \Box

Theorem 4.7 : The minimum covering 1-partition energy of Cocktail party graph $K_{n\times2}$ is
\[ E_{PC}^{C}(K_{n\times2}) = (6n - 9) + \sqrt{16n^2 + 8n - 15}. \]

Proof : Consider all the vertices in one partition. The minimum covering set $C = \{u_1, u_2, \ldots u_{n-1}, v_1, v_2, \ldots v_{n-1}\}$. The minimum covering 1-partition matrix is
\[
P_{PC}^{C}(K_{n\times2}) = \begin{bmatrix}
1 & -1 & 2 & 2 & \ldots & 2 & 2 & 2 & 2 \\
-1 & 1 & 2 & 2 & \ldots & 2 & 2 & 2 & 2 \\
2 & 2 & 1 & -1 & \ldots & 2 & 2 & 2 & 2 \\
-1 & 2 & -1 & 1 & \ldots & 2 & 2 & 2 & 2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
2 & 2 & 2 & 2 & \ldots & -1 & 1 & 2 & 2 \\
2 & 2 & 2 & 2 & \ldots & -1 & 1 & 2 & 2 \\
-1 & 2 & 2 & 2 & \ldots & 2 & 2 & 0 & -1 \\
2 & 2 & 2 & 2 & \ldots & 2 & 2 & -1 & 0
\end{bmatrix}.
\]
Characteristic equation is
\[(\lambda - 1)^1(\lambda - 2)^{n-1}(\lambda + 4)^{n-2}[\lambda^2 - (4n - 9)\lambda - (20n - 24)] = 0\]

Hence, spectrum is $Spec_{C_{P_1}}(K_{n \times 2}) = \begin{pmatrix} 1 & 2 & -4 (4n-9+\sqrt{16n^2+8n-15}) & (4n-9-\sqrt{16n^2+8n-15}) \\ 1 & (n-1) & (n-2) & \frac{2}{2} \end{pmatrix}$. Therefore, $E_{C_{P_1}}^C(K_{n \times 2}) = (6n - 9) + \sqrt{16n^2 + 8n - 15}$.

**Theorem 4.8**: The minimum covering 1-partition energy of complete bipartite graph $K_{n,n}$ is
\[E_{C_{P_1}}^C(K_{n,n}) = 3(n - 1) + \sqrt{1 + 16n^2}.\]

**Proof**: Consider all the vertices in the one partition. The minimum covering set $C = \{u_1, u_2, ..., u_n\}$. The minimum covering 1-partition matrix is
\[P_{C_{P_1}}^C(K_{n,n}) = \begin{bmatrix} 1 & -1 & -1 & -1 & \ldots & 2 & 2 & 2 & 2 \\ -1 & 1 & -1 & -1 & \ldots & 2 & 2 & 2 & 2 \\ -1 & -1 & 1 & -1 & \ldots & 2 & 2 & 2 & 2 \\ -1 & -1 & -1 & 1 & \ldots & 2 & 2 & 2 & 2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 2 & 2 & 2 & 2 & \ldots & 0 & -1 & -1 & -1 \\ 2 & 2 & 2 & 2 & \ldots & -1 & 0 & -1 & -1 \\ 2 & 2 & 2 & 2 & \ldots & -1 & -1 & 0 & -1 \\ 2 & 2 & 2 & 2 & \ldots & -1 & -1 & -1 & 0 \end{bmatrix} \]

Characteristic equation is
\[(\lambda - 1)^{n-1}(\lambda - 2)^{n-1}[\lambda^2 + (2n - 3)\lambda - (3n^2 + 3n - 2)] = 0\]

Hence, spectrum is $Spec_{C_{P_1}}^C(K_{n,n}) = \begin{pmatrix} 1 & 2 & -(2n-3+\sqrt{16n^2+1}) & -(2n-3-\sqrt{16n^2+1}) \\ (n-1) & (n-1) & \frac{2}{2} & \frac{2}{1} \end{pmatrix}$. Therefore, $E_{C_{P_1}}^C(K_{n,n}) = 3(n - 1) + \sqrt{1 + 16n^2}$.

**Theorem 4.9**: The minimum covering 2-partition energy of star graph $K_{1,n-1}$ in which the vertex of degree $n - 1$ is in one partition and vertices of degree 1 are in another partition is $E_{C_{P_2}}^C(K_{1,n-1}) = (n - 2) + \sqrt{n^2 + 2n - 3}$.

**Proof**: The 2-partition of star graph $K_{1,n-1}$ in which the vertex of degree $n - 1$ is in one partition and vertices of degree 1 are in another partition is...
set = \( C = \{v_0\} \). The minimum covering 1-partition matrix is

\[
P^C_2(K_{1,n-1}) = \begin{bmatrix}
    1 & 1 & 1 & \ldots & 1 & 1 \\
    1 & 0 & -1 & \ldots & -1 & -1 \\
    1 & -1 & 0 & \ldots & -1 & -1 \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    1 & -1 & -1 & \ldots & 0 & -1 \\
    1 & -1 & -1 & \ldots & -1 & 0 \\
\end{bmatrix}.
\]

Hence, its characteristic equation is \((\lambda - 1)^{n-2}[\lambda^2 + (n-3)\lambda - (2n-3)] = 0\) Hence, spectrum is \( \text{Spec}^C_{P_2}(K_{1,n-1}) = \left( \frac{1}{(n-2)}, \frac{1}{2}, \frac{1}{2} \right) \). Therefore, \( E^C_{P_2}(K_{1,n-1}) = (n-2) + \sqrt{n^2 + 2n - 3} \). \( \square \)

**Theorem 4.10**: The minimum covering 2-partition energy of 2(i)-complement of star graph \( K_{1,n-1} \) in which the vertex of degree \( n-1 \) is in one partition and vertices of degree 1 are in another partition is \( (n-2) + \sqrt{4n^2 - 8n + 5} \).

**Proof**: Consider 2(i)-complement of star graph \( K_{1,n-1} \), in which the vertex of degree \( n-1 \) is in one partition and remaining vertices are in other partition. The minimum covering set is \( \{v_2, v_3, \ldots, v_n\} \). Its minimum covering partition matrix is

\[
P^C_2((K_{1,n-1})_{2(i)}) = \begin{bmatrix}
    0 & 1 & 1 & \ldots & 1 & 1 \\
    1 & 1 & 2 & \ldots & 2 & 2 \\
    1 & 2 & 1 & \ldots & 2 & 2 \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    1 & 2 & 2 & \ldots & 1 & 2 \\
    1 & 2 & 2 & \ldots & 2 & 1 \\
\end{bmatrix}.
\]

Hence, its characteristic equation is \((\lambda + 1)^{n-2}[\lambda^2 - (2n-3)\lambda - (n-1)] = 0\) Hence, spectrum is \( \text{Spec}^C_{P_2}((K_{1,n-1})_{2(i)}) = \left( \frac{-1}{(n-2)}, \frac{(2n-3)+\sqrt{4n^2-8n+5}}{2}, \frac{(2n-3)-\sqrt{4n^2-8n+5}}{2} \right) \). Therefore, \( E^C_{P_2}((K_{1,n-1})_{2(i)}) = (n-2) + \sqrt{4n^2 - 8n + 5} \). \( \square \)

**Theorem 4.11**: The minimum covering 2-partition energy of Crown graph \( S_n^0 \) is

\[
E^C_{P_2}(S_n^0) = 3(n-1) + \sqrt{4n^2 - 8n + 5}.
\]

**Proof**: Consider the crown graph \( S_n^0 \) whose vertex set is partitioned into \( U_n = \{u_1, u_2, \ldots, u_n\} \), \( V_n = \{v_1, v_2, \ldots, v_n\} \).

The minimum covering set = \( C = \{u_1, u_2, \ldots, u_n\} \). The minimum covering 2-partition
matrix is

$$P^C_2(S_0^n) = \begin{bmatrix}
1 & -1 & -1 & \ldots & -1 & 0 & 1 & \ldots & 1 & 1 \\
-1 & 1 & -1 & \ldots & -1 & 1 & 0 & \ldots & 1 & 1 \\
-1 & -1 & 1 & \ldots & -1 & 1 & 1 & \ldots & 0 & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-1 & -1 & -1 & \ldots & 1 & 1 & 1 & \ldots & 1 & 0 \\
0 & 1 & 1 & \ldots & 0 & 1 & 1 & \ldots & 1 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & 0 & \ldots & 1 & -1 & -1 & \ldots & 0 & -1 \\
1 & 1 & 1 & \ldots & 0 & -1 & -1 & \ldots & -1 & 0 \\
\end{bmatrix}.$$  

Characteristic equation is

$$\lambda^2 - 3\lambda + 1][\lambda^2 + (2n-3)\lambda - (n-1)] = 0$$  

minimum covering 2-partition eigenvalues are

$$\text{spec}_\chi^C(S_0^n) = \frac{3+\sqrt{5}}{2} \frac{3-\sqrt{5}}{2} \left(\frac{-(2n-3)+\sqrt{(4n^2-8n+5)}}{n-1} \frac{-(2n-3)-\sqrt{(4n^2-8n+5)}}{n-1} \right).$$  

$$E^C_{P_1}(S_0^n) = 3(n-1) + \sqrt{4n^2 - 8n + 5}.$$  

\textbf{Theorem 4.12} : The minimum covering 2-partition energy of 2$(i)$-complement of Crown graph $S_0^n$ is

$$E_{P_2}^C((S_0^n)_{2(i)}) = (2n - 4) + \sqrt{n^2 + 2n - 3} + \sqrt{9n^2 + 6n - 11}.$$

\textbf{Proof} : Consider the 2$(i)$-complement of crown graph whose vertex set is partitioned into $U_n = \{u_1, u_2, \ldots, u_n\}, V_n = \{v_1, v_2, \ldots, v_n\}$. The minimum covering set $= C = \{u_1, u_2, \ldots, u_{n-1}, v_1, v_2, \ldots, v_{n-1}\}$. The minimum covering 2-partition matrix is

$$P^C_2((S_0^n)_{2(i)}) = \begin{bmatrix}
1 & 2 & 2 & \ldots & 2 & 0 & 1 & \ldots & 1 & 1 \\
2 & 1 & 2 & \ldots & 2 & 1 & 0 & \ldots & 1 & 1 \\
2 & 2 & 1 & \ldots & 2 & 1 & 1 & \ldots & 0 & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
2 & 2 & 2 & \ldots & 0 & 1 & 1 & \ldots & 1 & 0 \\
0 & 1 & 1 & \ldots & 1 & 1 & 2 & \ldots & 2 & 2 \\
1 & 0 & 1 & \ldots & 1 & 2 & 1 & \ldots & 2 & 2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & 0 & \ldots & 1 & 2 & 2 & \ldots & 1 & 2 \\
1 & 1 & 1 & \ldots & 0 & 2 & 2 & \ldots & 2 & 0 \\
\end{bmatrix}.$$
Characteristic polynomial is
\[ \lambda^{n-2} \lambda^{n-2} [\lambda^2 - (n - 1) \lambda - (n - 1)] [\lambda^2 - (3n - 5) \lambda - (9n - 9)] = 0 \]
minimum covering 2-partition spectra is
\[ \text{Spec}^D_{P_2}(\mathcal{S}_n^0) = \begin{pmatrix} -2 & 0 & \frac{(n-1)+\sqrt{n^2+2n-3}}{2} & \frac{(n-1)-\sqrt{n^2+2n-3}}{2} \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{(n-1)+\sqrt{n^2+2n-3}}{2} & \frac{(n-1)-\sqrt{n^2+2n-3}}{2} & -2 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & -2 \end{pmatrix}. \]
\[ E^C_{P_2}(\mathcal{S}_n^0) = (2n - 4) + \sqrt{n^2 + 2n - 3} + \sqrt{9n^2 + 6n - 11}. \square \]

**Definition 4.13**: The cocktail party graph, denoted by \( K_n \times 2 \), is a graph having vertex set \( V = \bigcup_{i=1}^n \{u_i, v_i\} \) and edge set \( E = \{u_iu_j, v_iv_j, u_iv_j, v_iu_j : 1 \leq i < j \leq n\} \). This graph is also called as complete \( n \)-partite graph.

**Theorem 4.14**: The minimum covering 2-partition energy of 2-complement of Cocktail party graph \( K_n \times 2 \) is
\[ E^C_{P_2}(K_n \times 2) = 2[(n - 2) + \sqrt{4n^2 + 4n - 7}]. \]

**Proof**: Consider the 2-complement of Cocktail party graph \( \overline{(K_n \times 2)} \) whose vertex set is partitioned into \( U_n = \{u_1, u_2, \ldots, u_n\}, V_n = \{v_1, v_2, \ldots, v_n\} \). The minimum covering set is \( C = \{u_1, v_1\} \). The minimum covering 2-partition matrix is
\[ P^C_{P_2}(\overline{(K_n \times 2)}) = \]
\[ \begin{bmatrix} 1 & 2 & 2 & \ldots & 2 & 0 & 1 & \ldots & 0 & 0 \\ 2 & 0 & 2 & \ldots & 2 & 0 & 1 & \ldots & 0 & 0 \\ 2 & 2 & 0 & \ldots & 2 & 0 & 1 & \ldots & 0 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \ldots & 0 & 1 & 2 & \ldots & 2 & 2 \\ 0 & 1 & 0 & \ldots & 0 & 2 & 0 & \ldots & 2 & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 1 & \ldots & 0 & 2 & 2 & \ldots & 0 & 2 \\ 0 & 0 & 0 & \ldots & 1 & 2 & 2 & \ldots & 2 & 0 \end{bmatrix}. \]

Characteristic polynomial is
\[ \lambda^{n-2} [\lambda + 1]^{n-2} [\lambda^2 - (2n - 5) \lambda - (4n - 4)] [\lambda^2 - (2n - 1) \lambda - 2] = 0 \]
minimum covering 2-partition spectra is
\[ \text{Spec}^C_{P_2}(\overline{(K_n \times 2)}) = \]
Theorem 4.15: The minimum covering 2-partition energy of complete bipartite graph $K_{n,n}$ is

$$E_{CP}^2(K_{n,n}) = 3(n - 1) + \sqrt{4n^2 + 1}. $$

Proof: Consider the complete bipartite graph $K_{n,n}$ whose vertex set is partitioned into $U_n = \{u_1, u_2, \ldots, u_n\}$, $V_n = \{v_1, v_2, \ldots, v_n\}$. The minimum covering set $C = \{u_1, u_2, \ldots, u_n\}$, and the minimum covering 2-partition matrix is

$$P_C^2(K_{n,n}) = \begin{pmatrix}
1 & -1 & -1 & -1 & \ldots & 1 & 1 & 1 & 1 \\
-1 & 1 & -1 & -1 & \ldots & 1 & 1 & 1 & 1 \\
-1 & -1 & 1 & -1 & \ldots & 1 & 1 & 1 & 1 \\
-1 & -1 & -1 & 1 & \ldots & 1 & 1 & 1 & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 1 & 1 & 1 & \ldots & 0 & -1 & -1 & -1 \\
1 & 1 & 1 & 1 & \ldots & -1 & 0 & -1 & -1 \\
1 & 1 & 1 & 1 & \ldots & -1 & -1 & 0 & -1 \\
1 & 1 & 1 & 1 & \ldots & -1 & -1 & -1 & 0
\end{pmatrix}.
$$

Characteristic equation is

$$(\lambda - 1)^{n-1}(\lambda - 2)^{n-1}[\lambda^2 + (2n - 3)\lambda - (3n - 2)] = 0$$

Hence, spectrum is $Spec_{CP}^2(K_{n,n}) = \begin{pmatrix} 1 & 2 & -2(n-3)+\sqrt{4n^2+1} & -2(n-3)-\sqrt{4n^2+1} \\
(n-1) & n-1 & 2 & 1
\end{pmatrix}.$

Therefore, $E_{CP}^2(K_{n,n}) = 3(n - 1) + \sqrt{4n^2 + 1}.$

Theorem 4.16: The minimum covering 2-partition energy of Double star graph $S_{n,n}$ is

$$E_{CP}^2(S_{n,n}) = (2n - 2) + 2\sqrt{n+1} + \sqrt{4n^2 + 12n + 4}.$$ 

Proof: In double star graph, the centers $\{u_0, v_0\}$ are taken in one partition and remaining vertices are taken in other partition. The minimum covering set $C =$
The minimum covering 2-partition matrix is

\[ P_{2C}(S_{n,n}) = \begin{bmatrix}
1 & 1 & 1 & \ldots & 1 & 2 & 0 & 0 & \ldots & 0 \\
1 & 0 & -1 & \ldots & -1 & 0 & -1 & -1 & \ldots & -1 \\
1 & -1 & 0 & \ldots & -1 & 0 & -1 & -1 & \ldots & -1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & -1 & -1 & \ldots & 0 & 0 & -1 & -1 & \ldots & -1 \\
2 & 0 & 0 & \ldots & 0 & 1 & 1 & 1 & \ldots & 1 \\
0 & -1 & -1 & \ldots & -1 & 1 & 0 & -1 & \ldots & -1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & -1 & -1 & \ldots & -1 & 1 & -1 & -1 & \ldots & 0 \\
0 & -1 & -1 & \ldots & -1 & 1 & -1 & -1 & \ldots & 0
\end{bmatrix}. \]

Characteristic equation is

\[
(\lambda - 1)^{2n-4}[\lambda^2 - n][\lambda^2 + (2n - 6)\lambda - (7n - 10)] = 0
\]

Hence, spectrum is

\[
Spec_{P_{2C}}(S_{n,n}) = \begin{pmatrix}
1 & \sqrt{n} & -\sqrt{n} & -(2n-6) + \sqrt{4n^2+4n-4} & -(2n-6) - \sqrt{4n^2+4n-4}
\end{pmatrix}.
\]

Therefore, \( E_{P_{2C}}(S_{n,n}) = (2n - 4) + 2\sqrt{n} + \sqrt{4n^2 + 4n - 4}. \)

**Theorem 4.17:** The minimum covering 2-partition energy of 2\((i)\)-complement of Double star graph \( S_{n,n} \) is

\[
E_{P_{2C}}((S_{n,n})_{2(i)}) = (2n - 4) + 2\sqrt{n} + \sqrt{16n^2 - 28n + 12}.
\]

**Proof:** In 2\((i)\)-complement of double star graph, the centers \( \{u_0, v_0\} \) are taken in one partition and remaining vertices are taken in other partition. The minimum covering set = \( C = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\} \). The minimum covering 2-partition matrix is

\[ P_{1C}((S_{n,n})_{2(i)}) = \begin{bmatrix}
0 & 1 & 1 & \ldots & 1 & -1 & 0 & 0 & \ldots & 0 \\
1 & 2 & 1 & \ldots & 2 & 0 & 2 & 2 & \ldots & 2 \\
1 & 2 & 1 & \ldots & 2 & 0 & 2 & 2 & \ldots & 2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 2 & 2 & \ldots & 1 & 0 & 2 & 2 & \ldots & 2 \\
-1 & 0 & 0 & \ldots & 0 & 0 & 1 & 1 & \ldots & 1 \\
0 & 2 & 2 & \ldots & 2 & 1 & 1 & 2 & \ldots & 2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 2 & 2 & \ldots & 2 & 1 & 2 & 1 & \ldots & 2 \\
0 & 2 & 2 & \ldots & 2 & 1 & 2 & 2 & \ldots & 1
\end{bmatrix}. \]
Characteristic equation is

\[(\lambda + 1)^{2n-4}[\lambda^2 - n][\lambda^2 - (4n - 6)\lambda - (5n - 6)] = 0\]

Hence, spectrum is

\[
\text{Spec}_{C_2}(\{(S_{n,n})^2_1\}) = \begin{pmatrix}
-1 & \sqrt{n} & -\sqrt{n} & (4n-6)+\sqrt{16n^2-28n+12} \\
2n-4 & 1 & 1 & 2 \sqrt{4n-6} \sqrt{16n^2-28n+12} \\
\end{pmatrix}.
\]

Therefore, \(E_{C_2}(\{(S_{n,n})^2_1\}) = (2n-4) + 2\sqrt{n} + \sqrt{16n^2-28n+12}.\)

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