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MINIMUM COVERING PARTITION ENERGY OF A GRAPH

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Abstract

The Partition energy of a graph was introduced by E. Sampathkumar et al. (2015). Motivated by this, we introduce the concept of minimum covering partition energy of a graph, $E_p^C(G)$ and compute the minimum covering partition energy $E_p^C(G)$ of few families of graphs. Also, we established the bounds for minimum covering partition energy.

1. Introduction

The graph energy plays a vital role in chemistry to find the total π-electron energy of a molecule. The conjugated hydrocarbons can be represented by a graph called molecular

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graph. We can represent every carbon atom by a vertex and every carbon-carbon bond by an edge and hydrogen atoms are ignored. Recently several matrices like adjacency matrix, Laplacian matrix, distance matrix, maximum degree matrix, minimum degree matrix, matrix of a subset S of V and color energy of a graph are studied in [1, 2, 4, 5, 10, 11 and 16].

2. Minimum Covering Partition Energy of a Graph

Let G be a simple graph of order n with vertex set $V = v_1, v_2, v_3, ..., v_n$ and edge set E. A subset C of V is called a covering set of G if every edge of G is incident to at least one vertex of C (see [4]). Any covering set with minimum cardinality is called a minimum covering set. Let C be a minimum covering set of a graph G. The minimum covering partition matrix is given by

$$a_{ij} = \begin{cases} 2 & \text{if } v_i \text{ and } v_j \text{ are adjacent where } v_i, v_j \in V_r, \\ -1 & \text{if } v_i \text{ and } v_j \text{ are non-adjacent where } v_i, v_j \in V_r, \\ 1 & \text{if } i = j \text{ and } v_i \in C, \\ 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent between the sets} \\ & V_r \text{ and } V_s \text{ for } r \neq s, \text{ where } v_i \in V_r \text{ and } v_j \in V_s, \\ 0 & \text{otherwise.} \end{cases}$$

In this paper, we study minimum covering partition energy of a graph with respect to given partition of a graph. Further, we determine minimum covering partition energy of two types of complements of a partition graph called k-complement and $k(i)$-complement of a graph introduced by E. Sampathkumar in [13].

Definition 2.1 : The complement of a graph G is a graph $\overline{G}$ on the same vertices such that two distinct vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in G.

Definition 2.2 [13] : Let G be a graph and $P_k = \{V_1, V_2, ..., V_k\}$ be a partition of its vertex set V. Then the k-complement of G is obtained as follows: For all V_i and V_j in P_k, $i \neq j$ remove the edges between V_i and V_j and add the edges between the vertices of V_i and V_j which are not in G and is denoted by $\overline{(G)_k}$.

Definition 2.3 [13] : Let G be a graph and $P_k = \{V_1, V_2, ..., V_k\}$ be a partition of its vertex set V. Then the $k(i)$-complement of G is obtained as follows: For each set V_r in P_k, remove the edges of G joining the vertices within V_r and add the edges of $\overline{G}$ (complement of G) joining the vertices of V_r, and is denoted by $\overline{(G)_{k(i)}}$.

3. Some Basic Properties of Minimum Covering Partition Energy of a Graph

Let $G = (V, E)$ be a graph with n vertices and $P_k = \{V_1, V_2, \ldots, V_k\}$ be a partition of V. For $1 \leq i \leq k$, let b_i denote the total number of edges joining the vertices of V_i and c_i be the total number of edges joining the vertices from V_i to V_j for $i \neq j, 1 \leq j \leq k$ and d_i be the number of non-adjacent pairs of vertices within V_i. Let $m_1 = \sum_{i=1}^{k} b_i$, $m_2 = \sum_{i=1}^{k} c_i$ and $m_3 = \sum_{i=1}^{k} d_i$. Let $P_k^C(G)$ be the minimum covering partition matrix. If the characteristic polynomial of $P_k^C(G)$ denoted by $\Phi_k^C(G, \lambda)$ is $a_0\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n$, then the coefficient a_i can be interpreted using the principal minors of $P_k^C(G)$.

The following proposition determines the first three coefficients of the characteristic polynomial of $P_k^C(G)$.

Proposition 3.1 : The first three coefficients of $\phi_k^C(G, \lambda)$ are given as follows:
(i) $a_0 = 1$,
(ii) $a_1 = -|C|$,
(iii) $a_2 = |C|C_2 - [4m_1 + m_2 + m_3]$.

Proof : (i) From the definition $\Phi_k(G, \lambda) = det[\lambda I - P_k^C(G)]$, we get $a_0 = 1$.

(ii) The sum of determinants of all 1×1 principal submatrices of $P_k^C(G)$ is equal to the trace of $P_k^C(G)$.
$\Rightarrow a_1 = (-1)^1$ trace of $[P_k^C(G)] = -|C|$.

(iii)

$$\begin{aligned}
(-1)^2 a_2 &= \sum_{1 \leq i < j \leq n} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} \\
&= \sum_{1 \leq i < j \leq n} a_{ii}a_{jj} - a_{ji}a_{ij} \\
&= \sum_{1 \leq i < j \leq n} a_{ii}a_{jj} - \sum_{1 \leq i < j \leq n} a_{ji}a_{ij} \\
&= |C|C_2 - [(2)^2 m_1 + (1)^2 m_2 + (-1)^2 m_3] = |C|C_2 - [4m_1 + m_2 + m_3].
\end{aligned}$$

□

Proposition 3.2 : If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are partition eigenvalues of $P_k^C(G)$, then

$$\sum_{i=1}^{n} {\lambda_i}^2 = |C| + 2[4m_1 + m_2 + m_3].$$

We know that

$$\begin{aligned}
\sum_{i=1}^{n} \lambda_i^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}a_{ji} \\
&= 2\sum_{i<j}(a_{ij})^2 + \sum_{i=1}^{n}(a_{ii})^2 \\
&= 2\sum_{i<j}(a_{ij})^2 + |C| \\
&= |C| + 2[4m_1 + m_2 + m_3].
\end{aligned}$$

□

Theorem 3.3 : Let G be a graph with n vertices and P_k be a partition of G. Then

$$E_{P_k}^C(G) \leq \sqrt{n(|C| + 2[4m_1 + m_2 + m_3])}$$

where m_1, m_2, m_3 are as defined above for G.
Proof : Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of $P_k^C(G)$.
Now by Cauchy - Schwartz inequality we have

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} {a_i}^2\right)\left(\sum_{i=1}^{n} {b_i}^2\right).$$

Let $a_i = 1$, $b_i =\mid \lambda_i \mid$. Then
then

$$\left(\sum_{i=1}^{n} |\lambda_i|\right)^2 \leq \left(\sum_{i=1}^{n} 1\right)\left(\sum_{i=1}^{n} |\lambda_i|^2\right)$$

$$[E_{P_k}^C]^2 \leq n(|C| + 2[4m_1 + m_2 + m_3])$$

$$[E_{P_k}^C] \leq \sqrt{n(|C| + 2([4m_1 + m_2 + m_3])}$$

.Which is upper bound. □

Theorem 3.4 : Let G be a partition graph with n vertices. If $R=$ det $P_k^C(G)$, then

$$E_{P_k}^C(G) \geq \sqrt{(|C| + 2[4m_1 + m_2 + m_3]) + n(n-1)R^{\frac{2}{n}}}.$$

Proof : By definition,

$$\begin{aligned}\left(E_{P_k}^C(G)\right)^2 &= \left(\sum_{i=1}^{n} \mid \lambda_i \mid\right)^2 \\ &= \sum_{i=1}^{n} \mid \lambda_i \mid \sum_{j=1}^{n} \mid \lambda_j \mid \\ &= \left(\sum_{i=1}^{n} \mid \lambda_i \mid^2\right) + \sum_{i \neq j} \mid \lambda_i \mid\mid \lambda_j \mid .\end{aligned}$$

Using arithmetic mean and geometric mean inequality, we have

$$\frac{1}{n(n-1)} \sum_{i \neq j} \mid \lambda_i \mid\mid \lambda_j \mid \quad \geq \quad \left(\prod_{i \neq j} \mid \lambda_i \mid\mid \lambda_j \mid\right)^{\frac{1}{n(n-1)}} .$$

Therefore,

$$\begin{aligned}[E_{P_k}^C(G)]^2 &\geq \sum_{i=1}^{n} \mid \lambda_i \mid^2 + n(n-1) \left(\prod_{i \neq j} \mid \lambda_i \mid\mid \lambda_j \mid\right)^{\frac{1}{n(n-1)}} \\ &\geq \sum_{i=1}^{n} \mid \lambda_i \mid^2 + n(n-1) \left(\prod_{i=1}^{n} \mid \lambda_i \mid^{2(n-1)}\right)^{\frac{1}{n(n-1)}} \\ &= \sum_{i=1}^{n} \mid \lambda_i \mid^2 + n(n-1)R^{\frac{2}{n}} \\ &= (|C| + 2[4m_1 + m_2 + m_3]) + n(n-1)R^{\frac{2}{n}}.\end{aligned}$$

Thus,

$$E_{P_k}^C(G) \geq \sqrt{(|C| + 2[4m_1 + m_2 + m_3]) + n(n-1)R^{\frac{2}{n}}}.$$

□

Theorem 3.5 : If the minimum covering partition energy of a graph is a rational number, then it must be a positive even number.

Proof of this theorem is similar to the proof of Theorem 2.12 in [6].

4. Energy of Some Partition Graphs and Their Complements

Theorem 4.1 : The minimum covering 1-partition energy of a complete graph K_n is $E^C_{P_1}(K_n) = (n-2) + \sqrt{4n^2+4n-7}$.

Proof : Let K_n be the complete graph with vertex set $\{v_1, v_2, v_3....V_n\}$. Consider all the vertices is in one partition. The minimum covering set $= C = \{v_1, v_2, v_3....v_{n-1}\}$. The minimum covering 1-partition matrix is

$$P^C_1(K_n) = \begin{bmatrix} 1 & 2 & 2 & \dots & 2 & 2 \\ 2 & 1 & 2 & \dots & 2 & 2 \\ 2 & 2 & 1 & \dots & 2 & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 2 & 2 & \dots & 1 & 2 \\ 2 & 2 & 2 & \dots & 2 & 0 \end{bmatrix}.$$

Characteristic equation is

$$(\lambda+1)^{n-2}(\lambda^2-(2n-3)\lambda-(4n-4)) = 0$$

and the spectrum is $Spec^C_{P_1}(K_n) = \begin{pmatrix} -1 & \frac{(2n-3)+\sqrt{4n^2+4n-7}}{2} & \frac{(2n-3)-\sqrt{4n^2+4n-7}}{2} \\ n-2 & 1 & 1 \end{pmatrix}$.

Therefore, $E^C_{P_1}(K_n) = (n-2) + \sqrt{4n^2+4n-7}$. □

Theorem 4.2 : The minimum covering 1-partition energy of star graph $K_{1,n-1}$ is

$$E^C_{P_1}(K_{1,n-1}) = (n-2) + \sqrt{n^2+14n-15}.$$

Proof : Consider all the vertices is in one partition. The minimum covering set $= C = \{v_1\}$. The minimum covering 1-partition matrix is

$$P^C_1(K_{1,n-1}) = \begin{bmatrix} 1 & 2 & 2 & \dots & 2 & 2 \\ 2 & 0 & -1 & \dots & -1 & -1 \\ 2 & -1 & 0 & \dots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & -1 & -1 & \dots & 0 & -1 \\ 2 & -1 & -1 & \dots & -1 & 0 \end{bmatrix}.$$

Characteristic equation is

$$(\lambda-1)^{n-2}[\lambda^2+(n-3)\lambda-(5n-6)] = 0$$

spectrum is $Spec^C_{P_1}(K_{1,n-1}) = \begin{pmatrix} 1 & \frac{-(n-3)+\sqrt{n^2+14n-15}}{2} & \frac{-(n-3)-\sqrt{n^2+14n-15}}{2} \\ n-2 & 1 & 1 \end{pmatrix}$.

Therefore, $E^C_{P_1}(K_{1,n-1}) = (n-2) + \sqrt{n^2+14n-15}$. □

Definition 4.3 : The Crown graph S^0_n for an integer $n \geq 3$ is the graph with vertex set $\{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$ and edge set $\{u_iv_i : 1 \leq i, j \leq n, i \neq j\}$. S^0_n is therefore equivalent to the complete bipartite graph $K_{n,n}$ with horizontal edges removed.

Theorem 4.4 : The minimum covering 1-partition energy of Crown graph S^0_n is

$$E^C_{P_1}(S^0_n) = \sqrt{37}(n-1) + \sqrt{16n^2 - 48n + 37}$$

Proof : Consider all the vertices is in one partition. Let S^0_n be a crown graph of order $2n$ with vertex set $\{u_1, u_2, \cdots, u_n, v_1, v_2, v_3...v_n\}$ and minimum covering set $= C = \{u_1, u_2, u_3...u_n\}$.The minimum covering 1-partition matrix is

$$P^C_1(S^0_n) = \begin{bmatrix} 1 & -1 & -1 & \dots & -1 & -1 & 2 & \dots & 2 & 2 \\ -1 & 1 & -1 & \dots & -1 & 2 & -1 & \dots & 2 & 2 \\ -1 & -1 & 1 & \dots & -1 & 2 & 2 & \dots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \dots & 1 & 2 & 2 & \dots & 2 & -1 \\ -1 & 2 & 2 & \dots & 2 & 0 & -1 & \dots & -1 & -1 \\ 2 & -1 & 2 & \dots & 2 & -1 & 0 & \dots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 2 & -1 & \dots & 2 & -1 & -1 & \dots & 0 & -1 \\ 2 & 2 & 2 & \dots & -1 & -1 & -1 & \dots & -1 & 0 \end{bmatrix}.$$

Characteristic equation is

$$[\lambda^2 + (2n-3)\lambda - (3n^2 - 9n + 7)][\lambda^2 - 3\lambda - 7]^{n-1} = 0$$

spectrum is $Spec^C_{P_1}(S^0_n)$

$= \begin{pmatrix} \frac{3+\sqrt{37}}{2} & \frac{3-\sqrt{37}}{2} & \frac{(2n-3)+\sqrt{16n^2-48n+37}}{2} & \frac{(2n-3)-\sqrt{16n^2-48n+37}}{2} \\ (n-1) & (n-1) & 1 & 1 \end{pmatrix}$. Therefore,

$E^C_{P_1}(S^0_n) = \sqrt{37}(n-1) + \sqrt{16n^2 - 48n + 37}$. □

Definition 4.5 : The double star graph $S_{n,m}$ is the graph constructed from $K_{1,n-1}$ and $K_{1,m-1}$ by joining their centers v_0 and u_0. $V(S_{n,m}) = V(K_{1,n-1}) \cup V(K_{1,m-1})$ and $E(S_{n,m}) = \{v_0u_0; v_0v_i; u_0u_j : 1 \leq i \leq n-1, 1\ leqi \leq n-1\}$ Therefore, double star graph is bipartite graph.

Theorem 4.6 : The minimum covering 1-partition energy of Double star graph $S_{n,n}$ is

$$E^C_{P_1}(S_{n,n}) = (2n-4) + \sqrt{4n^2+4n-4} + 2(\sqrt{9n-8}).$$

Proof : Consider all the vertices in the one partition. The minimum covering set $= C = \{u_0, v_0\}$. The minimum covering 1-partition matrix is

$$P^C_1(S_{n,n}) = \begin{bmatrix} 1 & 2 & 2 & \dots & 2 & 2 & -1 & -1 & \dots & -1 \\ 2 & 0 & -1 & \dots & -1 & -1 & -1 & -1 & \dots & -1 \\ 2 & -1 & 0 & \dots & -1 & -1 & -1 & -1 & \dots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & -1 & -1 & \dots & 0 & -1 & -1 & -1 & \dots & -1 \\ 2 & -1 & -1 & \dots & -1 & 1 & 2 & 2 & \dots & 2 \\ -1 & -1 & -1 & \dots & -1 & 2 & 0 & -1 & \dots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \dots & -1 & 2 & -1 & 0 & \dots & -1 \\ -1 & -1 & -1 & \dots & -1 & 2 & -1 & -1 & \dots & 0 \end{bmatrix}.$$

Characteristic equation is

$$(\lambda-1)^{2n-4}[\lambda^2 + (2n-6)\lambda - (7n-10)][\lambda^2 - (9n-8)] = 0$$

Hence, spectrum is

$$Spec^C_{P_1}(S_{n,n}) = \begin{pmatrix} 1 & \sqrt{9n-8} & -\sqrt{9n-8} & \frac{(2n-6)+\sqrt{4n^2+4n-4}}{2} & \frac{(n-6)-\sqrt{4n^2+4n-4}}{2} \\ 2n-4 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Therefore, $E^C_{P_1}(S_{n,n}) = (2n-4) + \sqrt{4n^2+4n-4} + 2(\sqrt{9n-8})$. □

Theorem 4.7 : The minimum covering 1-partition energy of Cocktail party graph $K_{n\times 2}$ is

$$E^C_{P_1}(K_{n\times 2}) = (6n-9) + \sqrt{16n^2+8n-15}.$$

Proof : Consider all the vertices in one partition. The minimum covering set $= C = \{u_1, u_2, ...u_{n-1}, v_1, v_2...v_{n-1}\}$. The minimum covering 1-partition matrix is

$$P^C_1(K_{n\times 2}) = \begin{bmatrix} 1 & -1 & 2 & 2 & \dots & 2 & 2 & 2 & 2 \\ -1 & 1 & 2 & 2 & \dots & 2 & 2 & 2 & 2 \\ 2 & 2 & 1 & -1 & \dots & 2 & 2 & 2 & 2 \\ 2 & 2 & -1 & 1 & \dots & 2 & 2 & 2 & 2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 2 & 2 & 2 & 2 & \dots & 1 & -1 & 2 & 2 \\ 2 & 2 & 2 & 2 & \dots & -1 & 1 & 2 & 2 \\ 2 & 2 & 2 & 2 & \dots & 2 & 2 & 0 & -1 \\ 2 & 2 & 2 & 2 & \dots & 2 & 2 & -1 & 0 \end{bmatrix}.$$

Characteristic equation is

$$(\lambda-1)^1(\lambda-2)^{n-1}(\lambda+4)^{n-2}[\lambda^2-(4n-9)\lambda-(20n-24)]=0$$

Hence, spectrum is $Spec^C_{P_1}(K_{n\times 2})=$
$\begin{pmatrix} 1 & 2 & -4 & \frac{(4n-9)+\sqrt{16n^2+8n-15}}{2} & \frac{(4n-9)-\sqrt{16n^2+8n-15}}{2} \\ 1 & (n-1) & (n-2) & 1 & 1 \end{pmatrix}$. Therefore, $E^C_{P_1}(K_{n\times 2})=$ $(6n-9)+\sqrt{16n^2+8n-15}$. □

Theorem 4.8 : The minimum covering 1-partition energy of complete bipartite graph $K_{n,n}$ is

$$E^C_{P_1}(K_{n,n})=3(n-1)+\sqrt{1+16n^2}.$$

Proof : Consider all the vertices in the one partition. The minimum covering set $=C=\{u_1,u_2...,u_n\}$. The minimum covering 1-partition matrix is

$$P^C_1(K_{n,n})=\begin{bmatrix} 1 & -1 & -1 & -1 & \dots & 2 & 2 & 2 & 2 \\ -1 & 1 & -1 & -1 & \dots & 2 & 2 & 2 & 2 \\ -1 & -1 & 1 & -1 & \dots & 2 & 2 & 2 & 2 \\ -1 & -1 & -1 & 1 & \dots & 2 & 2 & 2 & 2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 2 & 2 & 2 & 2 & \dots & 0 & -1 & -1 & -1 \\ 2 & 2 & 2 & 2 & \dots & -1 & 0 & -1 & -1 \\ 2 & 2 & 2 & 2 & \dots & -1 & -1 & 0 & -1 \\ 2 & 2 & 2 & 2 & \dots & -1 & -1 & -1 & 0 \end{bmatrix}.$$

Characteristic equation is

$$(\lambda-1)^{n-1}(\lambda-2)^{n-1}[\lambda^2+(2n-3)\lambda-(3n^2+3n-2)]=0$$

Hence, spectrum is $Spec^C_{P_1}(K_{n,n})$
$=\begin{pmatrix} 1 & 2 & \frac{-(2n-3)+\sqrt{16n^2+1}}{2} & \frac{-(2n-3)-\sqrt{16n^2+1}}{2} \\ (n-1) & (n-1) & 1 & 1 \end{pmatrix}$.
Therefore, $E^C_{P_1}(K_{n,n})=3(n-1)+\sqrt{1+16n^2}$. □

Theorem 4.9 : The minimum covering 2-partition energy of star graph $K_{1,n-1}$ in which the vertex of degree $n-1$ is in one partition and vertices of degree 1 are in another partition is $E^C_{P_2}(K_{1,n-1})=(n-2)+\sqrt{n^2+2n-3}$.

Proof : The 2-partition of star graph $K_{1,n-1}$ in which the vertex of degree $n-1$ is in one partition and vertices of degree 1 are in another partition The minimum covering set $= C = \{v_0\}$. The minimum covering 1-partition matrix is

$$P_2^C(K_{1,n-1}) = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 0 & -1 & \dots & -1 & -1 \\ 1 & -1 & 0 & \dots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & -1 & -1 & \dots & 0 & -1 \\ 1 & -1 & -1 & \dots & -1 & 0 \end{bmatrix}.$$

Hence, its characteristic equation is $(\lambda-1)^{n-2}[\lambda^2+(n-3)\lambda-(2n-3)] = 0$ Hence, spectrum is $Spec_{P_2}^C(K_{1,n-1}) = \begin{pmatrix} 1 & \frac{-(n-3)+\sqrt{n^2+2n-3}}{2} & \frac{-(n-3)-\sqrt{n^2+2n-3}}{2} \\ (n-2) & 1 & 1 \end{pmatrix}$. Therefore, $E_{P_2}^C(K_{1,n-1}) = (n-2) + \sqrt{n^2+2n-3}$. □

Theorem 4.10 : The minimum covering 2-partition energy of $2(i)$-complement of star graph $K_{1,n-1}$ in which the vertex of degree $n-1$ is in one partition and vertices of degree 1 are in another partition is $(n-2) + \sqrt{4n^2-8n+5}$.

Proof : Consider $2(i)$-complement of star graph $K_{1,n-1}$, in which the vertex of degree $n-1$ is in one partition and remaining vertices are in other partition.The minimum covering set is$\{v_2, v_3...v_n\}$. Its minimum covering partition matrix is

$$P_2^C(\overline{(K_{1,n-1})_{2(i)}}) = \begin{bmatrix} 0 & 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & 2 & \dots & 2 & 2 \\ 1 & 2 & 1 & \dots & 2 & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 2 & 2 & \dots & 1 & 2 \\ 1 & 2 & 2 & \dots & 2 & 1 \end{bmatrix}.$$

Hence, its characteristic equation is $(\lambda+1)^{n-2}[\lambda^2-(2n-3)\lambda-(n-1)] = 0$ Hence, spectrum is $Spec_{P_2}^C(\overline{(K_{1,n-1})_{2(i)}}) = \begin{pmatrix} -1 & \frac{(2n-3)+\sqrt{4n^2-8n+5}}{2} & \frac{(2n-3)-\sqrt{4n^2-8n+5}}{2} \\ (n-2) & 1 & 1 \end{pmatrix}$.

Therefore, $E_{P_2}^C(\overline{(K_{1,n-1})_{2(i)}}) = (n-2) + \sqrt{4n^2-8n+5}$.. □

Theorem 4.11 : The minimum covering 2-partition energy of Crown graph S_n^0 is

$$E_{P_2}^C(S_n^0) = 3(n-1) + \sqrt{4n^2-8n+5}.$$

Proof : Consider the crown graph S_n^0 whose vertex set is partitioned into $U_n = \{u_1, u_2, \dots, u_n\}$, $V_n = \{v_1, v_2, \dots, v_n\}$.

The minimum covering set $= C = \{u_1, u_2, \dots, u_n\}$.The minimum covering 2-partition matrix is

$$P_2^C(S_n^0) = \begin{bmatrix} 1 & -1 & -1 & \dots & -1 & 0 & 1 & \dots & 1 & 1 \\ -1 & 1 & -1 & \dots & -1 & 1 & 0 & \dots & 1 & 1 \\ -1 & -1 & 1 & \dots & -1 & 1 & 1 & \dots & 0 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \dots & 1 & 1 & 1 & \dots & 1 & 0 \\ 0 & 1 & 1 & \dots & 1 & 0 & -1 & \dots & -1 & -1 \\ 1 & 0 & 1 & \dots & 1 & -1 & 0 & \dots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 0 & \dots & 1 & -1 & -1 & \dots & 0 & -1 \\ 1 & 1 & 1 & \dots & 0 & -1 & -1 & \dots & -1 & 0 \end{bmatrix}.$$

Characteristic equation is

$[\lambda^2 - 3\lambda + 1][\lambda^2 + (2n-3)\lambda - (n-1)] = 0$

minimum covering 2-partition eigenvalues are

$spec_\chi^C(S_n^0) =$

$\begin{pmatrix} \frac{3+\sqrt{5}}{2} & \frac{3-\sqrt{5}}{2} & \frac{-(2n-3)+\sqrt{(4n^2-8n+5)}}{2} & \frac{-(2n-3)-\sqrt{(4n^2-8n+5)}}{2} \\ n-1 & n-1 & 1 & 1 \end{pmatrix}.$

$E_{P_1}^C(S_n^0) = 3(n-1) + \sqrt{4n^2 - 8n + 5}.$ □

Theorem 4.12 : The minimum covering 2-partition energy of $2(i)$-complement of Crown graph S_n^0 is

$$E_{P_2}^C(\overline{(S_n^0)_{2(i)}}) = (2n-4) + \sqrt{n^2 + 2n - 3} + \sqrt{9n^2 + 6n - 11}.$$

Proof : Consider the $2(i)$-complement of crown graph whose vertex set is partitioned into $U_n = \{u_1, u_2, \dots, u_n\}$, $V_n = \{v_1, v_2, \dots, v_n\}$. The minimum covering set $= C = \{u_1, u_2, \dots, u_{n-1}, v_1, v_2, \dots, v_{n-1}\}$.The minimum covering 2-partition matrix is

$$P_2^C(\overline{(S_n^0)_{2(i)}}) = \begin{bmatrix} 1 & 2 & 2 & \dots & 2 & 0 & 1 & \dots & 1 & 1 \\ 2 & 1 & 2 & \dots & 2 & 1 & 0 & \dots & 1 & 1 \\ 2 & 2 & 1 & \dots & 2 & 1 & 1 & \dots & 0 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 2 & 2 & \dots & 0 & 1 & 1 & \dots & 1 & 0 \\ 0 & 1 & 1 & \dots & 1 & 1 & 2 & \dots & 2 & 2 \\ 1 & 0 & 1 & \dots & 1 & 2 & 1 & \dots & 2 & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 0 & \dots & 1 & 2 & 2 & \dots & 1 & 2 \\ 1 & 1 & 1 & \dots & 0 & 2 & 2 & \dots & 2 & 0 \end{bmatrix}.$$

Characteristic polynomial is
$[\lambda]^{n-2}[\lambda+2]^{n-2}[\lambda^2-(n-1)\lambda-(n-1)][\lambda^2-(3n-5)\lambda-(9n-9)]=0$
minimum covering 2-partition spectra is
$Spec^D_{P_2}(\overline{(S^0_n)_2}) =$
$$\begin{pmatrix} -2 & 0 & \frac{(n-1)+\sqrt{(n^2+2n-3)}}{2} & \frac{(n-1)-\sqrt{(n^2+2n-3)}}{2} \\ n-2 & n-2 & 1 & 1 \\ -2 & 0 & \frac{(3n-5)+\sqrt{(9n^2+6n-11)}}{2} & \frac{(3n-5)-\sqrt{(9n^2+6n-11)}}{2} \\ n-2 & n-2 & 1 & 1 \end{pmatrix}.$$
$E^C_{P_2}(\overline{(S^0_n)_{2(i)}}) = (2n-4)+\sqrt{n^2+2n-3}+\sqrt{9n^2+6n-11}.$ □

Definition 4.13 : The cocktail party graph, denoted by $K_{n\times 2}$, is a graph having vertex set $V=\bigcup_{i=1}^{n}\{u_i,v_i\}$ and edge set $E=\{u_iu_j, v_iv_j, u_iv_j, v_iu_j : 1\le i<j\le n\}$. This graph is also called as complete n-partite graph.

Theorem 4.14 : The minimum covering 2-partition energy of 2-complement of Cocktail party graph $K_{n\times 2}$ is

$$E^C_{P_2}(\overline{(K_{n\times 2})_{(2)}}) = 2[(n-2)+\sqrt{4n^2+4n-7}].$$

Proof : Consider the 2-complement of Cocktail party graph $\overline{(K_{n\times 2})_{(2)}}$ whose vertex set is partitioned into $U_n=\{u_1,u_2,\ldots,u_n\}$, $V_n=\{v_1,v_2,\ldots,v_n\}$. The minimum covering set $= C = \{u_1,v_1\}$.The minimum covering 2-partition matrix is

$$P^C_2(\overline{(K_{n\times 2})_{(2)}}) = \begin{bmatrix} 1 & 2 & 2 & \ldots & 2 & 1 & 0 & \ldots & 0 & 0 \\ 2 & 0 & 2 & \ldots & 2 & 0 & 1 & \ldots & 0 & 0 \\ 2 & 2 & 0 & \ldots & 2 & 0 & 1 & \ldots & 0 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 2 & 2 & \ldots & 0 & 0 & 0 & \ldots & 1 & 0 \\ 1 & 0 & 0 & \ldots & 0 & 1 & 2 & \ldots & 2 & 2 \\ 0 & 1 & 0 & \ldots & 0 & 2 & 0 & \ldots & 2 & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 1 & \ldots & 0 & 2 & 2 & \ldots & 0 & 2 \\ 0 & 0 & 0 & \ldots & 1 & 2 & 2 & \ldots & 2 & 0 \end{bmatrix}.$$

Characteristic polynomial is
$[\lambda+3]^{n-2}[\lambda+1]^{n-2}[\lambda^2-(2n-5)\lambda-(4n-4)][\lambda^2-(2n-1)\lambda-2]=0$
minimum covering 2-partition spectra is
$Spec^C_{P_2}(\overline{(K_{n\times 2})_{(2)}}) =$

$$\begin{pmatrix} -3 & -1 & \frac{(2n-5)+\sqrt{(4n^2-4n+9)}}{2} & \frac{(2n-5)-\sqrt{(4n^2-4n+9)}}{2} \\ n-2 & n-2 & 1 & 1 \\ -3 & -1 & \frac{(2n-1)+\sqrt{(4n^2-4n+9)}}{2} & \frac{(2n-1)-\sqrt{(4n^2-4n+9)}}{2} \\ n-2 & n-2 & 1 & 1 \end{pmatrix}.$$

$E^C_{P_2}(\overline{(K_{n\times 2})_{(2)}}) = 4(n-2) + 2\sqrt{4n^2-4n+9}$ □

Theorem 4.15 : The minimum covering 2-partition energy of complete bipartite graph $K_{n,n}$ is

$$E^C_{P_2}(K_{n,n}) = 3(n-1) + \sqrt{4n^2+1}.$$

Proof : Consider the complete bipartite graph $K_{n,n}$ whose vertex set is partitioned into $U_n = \{u_1, u_2, \ldots, u_n\}$, $V_n = \{v_1, v_2, \ldots, v_n\}$. The minimum covering set $= C = \{u_1, u_2, ..u_n\}$.The minimum covering 2-partition matrix is

$$P^C_2(K_{n,n}) = \begin{bmatrix} 1 & -1 & -1 & -1 & \ldots & 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & -1 & \ldots & 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & -1 & \ldots & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 & \ldots & 1 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \ldots & 0 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & \ldots & -1 & 0 & -1 & -1 \\ 1 & 1 & 1 & 1 & \ldots & -1 & -1 & 0 & -1 \\ 1 & 1 & 1 & 1 & \ldots & -1 & -1 & -1 & 0 \end{bmatrix}.$$

Characteristic equation is

$$(\lambda-1)^{n-1}(\lambda-2)^{n-1}[\lambda^2 + (2n-3)\lambda - (3n-2)] = 0$$

Hence, spectrum is $Spec^C_{P_2}(K_{n,n}) = \begin{pmatrix} 1 & 2 & \frac{-(2n-3)+\sqrt{4n^2+1}}{2} & \frac{-(2n-3)-\sqrt{4n^2+1}}{2} \\ (n-1) & n-1 & 1 & 1 \end{pmatrix}$.

Therefore, $E^C_{P_2}(K_{n,n}) = 3(n-1) + \sqrt{4n^2+1}$. □

Theorem 4.16 : The minimum covering 2-partition energy of Double star graph $S_{n,n}$ is

$$E^C_{P_2}(S_{n,n}) = (2n-2) + 2\sqrt{n+1} + \sqrt{4n^2+12n+4}.$$

Proof : In double star graph, the centers $\{u_0, v_0\}$ are taken in one partition and remaining vertices are taken in other partition. The minimum covering set $= C =$

$\{u_0, v_0\}$. The minimum covering 2-partition matrix is

$$P_2^C(S_{n,n}) = \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 & 2 & 0 & 0 & \ldots & 0 \\ 1 & 0 & -1 & \ldots & -1 & 0 & -1 & -1 & \ldots & -1 \\ 1 & -1 & 0 & \ldots & -1 & 0 & -1 & -1 & \ldots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & -1 & -1 & \ldots & 0 & 0 & -1 & -1 & \ldots & -1 \\ 2 & 0 & 0 & \ldots & 0 & 1 & 1 & 1 & \ldots & 1 \\ 0 & -1 & -1 & \ldots & -1 & 1 & 0 & -1 & \ldots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & -1 & -1 & \ldots & -1 & 1 & -1 & 0 & \ldots & -1 \\ 0 & -1 & -1 & \ldots & -1 & 1 & -1 & -1 & \ldots & 0 \end{bmatrix}.$$

Characteristic equation is

$$(\lambda - 1)^{2n-4}[\lambda^2 - n][\lambda^2 + (2n-6)\lambda - (7n-10)] = 0$$

Hence, spectrum is

$$Spec_{P_2}^C(S_{n,n}) = \begin{pmatrix} 1 & \sqrt{n} & -\sqrt{n} & \frac{-(2n-6)+\sqrt{4n^2+4n-4}}{2} & \frac{-(2n-6)-\sqrt{4n^2+4n-4}}{2} \\ 2n-4 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Therefore, $E_{P_2}^C(S_{n,n}) = (2n-4) + 2\sqrt{n} + \sqrt{4n^2+4n-4}$. □

Theorem 4.17 : The minimum covering 2-partition energy of 2(i)-complement of Double star graph $S_{n,n}$ is

$$E_{P_2}^C(\overline{(S_{n,n})_{2(i)}}) = (2n-4) + 2\sqrt{n} + \sqrt{16n^2 - 28n + 12}.$$

Proof : In 2(i)-complement of double star graph, the centers $\{u_0, v_0\}$ are taken in one partition and remaining vertices are taken in other partition. . The minimum covering set $= C = \{u_1, u_2 \ldots u_n, v_1, v_2 \ldots v_n\}$. The minimum covering 2-partition matrix is

$$P_1^C(\overline{(S_{n,n})_{2(i)}}) = \begin{bmatrix} 0 & 1 & 1 & \ldots & 1 & -1 & 0 & 0 & \ldots & 0 \\ 1 & 1 & 2 & \ldots & 2 & 0 & 2 & 2 & \ldots & 2 \\ 1 & 2 & 1 & \ldots & 2 & 0 & 2 & 2 & \ldots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 2 & 2 & \ldots & 1 & 0 & 2 & 2 & \ldots & 2 \\ -1 & 0 & 0 & \ldots & 0 & 0 & 1 & 1 & \ldots & 1 \\ 0 & 2 & 2 & \ldots & 2 & 1 & 1 & 2 & \ldots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 2 & 2 & \ldots & 2 & 1 & 2 & 1 & \ldots & 2 \\ 0 & 2 & 2 & \ldots & 2 & 1 & 2 & 2 & \ldots & 1 \end{bmatrix}.$$

Characteristic equation is

$$(\lambda+1)^{2n-4}[\lambda^2-n][\lambda^2-(4n-6)\lambda-(5n-6)]=0$$

Hence, spectrum is

$Spec_{P_2}^{C}(\overline{(S_{n,n})_{2(i)}}) = \begin{pmatrix} -1 & \sqrt{n} & -\sqrt{n} & \frac{(4n-6)+\sqrt{16n^2-28n+12}}{2} & \frac{(4n-6)-\sqrt{16n^2-28n+12}}{2} \\ 2n-4 & 1 & 1 & 1 & 1 \end{pmatrix}.$

Therefore, $E_{P_2}^{C}(\overline{(S_{n,n})_{2(i)}}) = (2n-4)+2\sqrt{n}+\sqrt{16n^2-28n+12}.$ □

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