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MINIMUM DOMINATING HARARY ENERGY OF A GRAPH

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Abstract

In this paper, we introduce the concept of minimum dominating Harary energy of a graph, $HE_D(G)$ and compute the minimum dominating Harary energy $HE_D(G)$ of few families of graphs. Also, established the bounds for minimum dominating Harary energy.

1. Introduction

Let G be a simple graph of order n with vertex set $V = v_1, v_2, v_3, ..., v_n$ and edge set E. The distance between the vertices v_i and v_j, denoted by $d_{i,j} = d(v_i, v_j)$ is the length of

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shortest path joining them. The Harary matrix of a graph G is an $n \times n$ matrix (a_{ij}), in which

$$h_{ij} = \begin{cases} \frac{1}{d_{ij}} & \text{if } i \neq j, \\ 0 & otherwise \end{cases}$$

A subset $D \subseteq V$ is a dominating set if D is a dominating set and every vertex of $V - D$ is adjacent to at least one vertex in D. Any dominating set with minimum cardinality is called a minimum dominating set. Let D be a minimum dominating set of a graph G. The minimum dominating Harary matrix of G is the $n \times n$ matrix defined by $H_D(G) = (a_{ij})$ where

$$a_{ij} = \begin{cases} 1 & \text{if } i = j \text{ and } v_i \in D, \\ 0 & \text{if } i = j \text{ and } v_i \notin D, \\ \frac{1}{d_{ij}} & \text{otherwise,} \end{cases}$$

The characteristic polynomial of $H_D(G)$ is denoted by $f_n(G, \lambda) = det(\lambda I - H_D(G))$. The minimum dominating Harary eigenvalues of the graph G are the eigenvalues of $H_D(G)$.

Since $H_D(G)$ is real and symmetric, its eigenvalues are real numbers and are labelled in non-increasing order $\lambda_1 \geq \lambda_2 \geq \lambda_3 .... \geq \lambda_n$. The minimum dominating Harary energy of G is defined as

$$HE_D(G) = \sum_{i=1}^{n} |\lambda_i| \tag{1}$$

2. Minimum Dominating Harary Energy of Some Standard Graphs

Theorem 2.1 : If K_n is the complete graph with n vertices has $HE_D(K_n) = (n-2) + \sqrt{n^2 - 2n + 5}$.

Proof : Let K_n be the complete graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$. The minimum dominating set $= D = \{v_1\}$.

$$H_D(K_n) = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 0 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 0 \end{pmatrix}_{n \times n} .$$

characteristic polynomial is

$[\lambda + 1]^{n-2}[\lambda^2 - (n-1)\lambda - 1]$

Minimum dominating Harary eigenvalues are

$$spec_D(K_n) = \begin{pmatrix} -1 & \frac{n-1+\sqrt{(n^2-2n+5)}}{2} & \frac{n-1-\sqrt{(n^2-2n+5)}}{2} \\ n-2 & 1 & 1 \end{pmatrix}.$$

Minimum dominating Harary energy for complete graph is

$$\begin{aligned} HE_D(K_n) &= |-1|(n-2) + |\frac{(n-1)+\sqrt{(n^2-2n+5)}}{2}| \\ &\quad +|\frac{(n-1)-\sqrt{(n^2-2n+5)}}{2}| \\ &= (n-2)+\sqrt{(n^2-2n+5)} \end{aligned}$$

$$HE_D(K_n) = (n-2)+\sqrt{(n^2-2n+5)}$$

□

Theorem 2.2 : If $K_{1,n-1}$ is a star graph of order n, then $HE_D(K_{1,n-1}) = \frac{1}{2}[(n-2)+\sqrt{n^2+8n}]$ for $n \geq 3$..

Proof : Let $K_{1,n-1}$ be a graph with minimum dominating set is $D = \{v_0\}$. Then we have

$$A_D(K_{1,n-1}) \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} \\ 1 & \frac{1}{2} & 0 & \cdots & \frac{1}{2} & \frac{1}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \frac{1}{2} & \frac{1}{2} & \cdots & 0 & \frac{1}{2} \\ 1 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & 0 \end{pmatrix}_{n\times n}.$$

Characteristic equation for $n \geq 3$ is $(2\lambda+1)^{n-2}\left(\lambda^2 - (\frac{n}{2})\lambda - (\frac{n}{2})\right) = 0$

Minimum dominating Harary eigenvalues for $n \geq 3$ are

$$\begin{pmatrix} \frac{-1}{2} & \frac{\frac{n}{2}+\sqrt{\frac{n^2}{4}+2n}}{2} & \frac{\frac{n}{2}-\sqrt{\frac{n^2}{4}+2n}}{2} \\ n-2 & 1 & 1 \end{pmatrix}.$$

Minimum dominating Harary energy is

$$\begin{aligned} HE_D(K_{1,n-1}) &= |\frac{-1}{2}|(n-2) + \frac{1}{2}|\frac{n}{2} + \sqrt{\frac{n^2}{4}+2n}| \\ &\quad +\frac{1}{2}|\frac{n}{2} - \sqrt{\frac{n^2}{4}+2n}| \\ &= \frac{1}{2}[(n-2)+\sqrt{n^2+8n}] \end{aligned}$$

$$\therefore HE_D(K_{1,n-1}) = \frac{1}{2}[(n-2) + \sqrt{n^2+8n}].$$

□

Definition 2.3 : The cocktail party graph, denoted by $K_{n\times 2}$, is graph having vertex set $V = \bigcup_{i=1}^{n}\{u_i, v_i\}$ and edge set $E = \{u_iu_j, v_iv_j, u_iv_j, v_iu_j : 1 \le i < j \le n\}$. This graph is also called as complete n-partite graph.

Theorem 2.4 : If $K_{n\times 2}$ is a cocktail party graph of order $2n$, then

$$HE_D(K_{n\times 2}) = (2n-3) + \sqrt{(4n^2-4n+9)}.$$

Proof : Let $K_{n\times 2}$ be a cocktail party graph of order $2n$ with $V(K_{n\times 2}) = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$. The minimum dominating set $= D = \{u_1, v_1\}$. Then

$$H_D(K_{n\times 2}) = \begin{pmatrix} 1 & \frac{1}{2} & 1 & 1 & \cdots & 1 & 1 & 1 & 1 \\ \frac{1}{2} & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & \frac{1}{2} & \cdots & 1 & 1 & 1 & 1 \\ 1 & 1 & \frac{1}{2} & 0 & \cdots & 1 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & 0 & \frac{1}{2} & 1 & 1 \\ 1 & 1 & 1 & 1 & \cdots & \frac{1}{2} & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 0 & \frac{1}{2} \\ 1 & 1 & 1 & 1 & \cdots & 1 & 1 & \frac{1}{2} & 0 \end{pmatrix}_{2n\times 2n}.$$

Characteristic equation is $[\lambda+1.5]^{n-2}[\lambda+0.5]^{n-1}[\lambda-0.5][\lambda^2-(2n-2)\lambda-(n+\frac{5}{4})]$ minimum dominating Harary eigenvalues are

$$= \begin{pmatrix} -1.5 & -0.5 & 0.5 & \frac{2n-2+\sqrt{(4n^2-4n+9)}}{2} & \frac{2n-2-\sqrt{(4n^2-4n+9)}}{2} \\ n-2 & n-1 & 1 & 1 & 1 \end{pmatrix}.$$

minimum dominating Harary energy,

$$\begin{aligned} HE_D(K_{n\times 2}) &= |-1.5|(n-2) + |-0.5|(n-1) + |0.5| + |\frac{2n-2+\sqrt{(4n^2-4n+9)}}{2}| \\ &\quad +|\frac{2n-2-\sqrt{(4n^2-4n+9)}}{2}| \\ &= (2n-3) + \sqrt{(4n^2-4n+9)}. \end{aligned}$$

□

Definition 2.5 : The friendship graph, denoted by $F_3^{(n)}$, is the graph obtained by taking n copies of the cycle graph C_3 with a vertex in common.

Theorem 2.6 : If $F_3^{(n)}$ is a friendship graph, then $HE_D(F_3^{(n)}) = n + \sqrt{(n^2 + 6n + 1)}$.

Proof : Let $F_3^{(n)}$ be a friendship graph with $V(F_3^{(n)}) = \{v_0, v_1, v_2, \ldots, v_n\}$. The minimum dominating set $= D = \{v_3\}$. Then

$$H_D(F_3^{(n)}) = \begin{pmatrix} 0 & 1 & 1 & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 1 & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} \\ 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 & 0 & \cdots & \frac{1}{2} & \frac{1}{2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{2} & \frac{1}{2} & 1 & 1 & \cdots & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 & 1 & \cdots & 1 & 0 \end{pmatrix}_{(2n+1)\times(2n+1)}.$$

Characteristic equation is $\lambda^{n-1}[\lambda + 1]^n[\lambda^2 - (n + 1)\lambda - n)]$

minimum dominating Harary eigenvalues are

$$= \begin{pmatrix} -1 & 0 & \frac{(n+1)+\sqrt{(n^2+6n+1)}}{2} & \frac{(n+1)-\sqrt{(n^2+6n+1)}}{2} \\ n & n-1 & 1 & 1 \end{pmatrix}$$

minimum dominating Harary energy,

$$\begin{aligned} HE_D(F_3^{(n)}) &= |-1|n + 0(n-1) + |\frac{(n+1) + \sqrt{(n^2+6n+1)}}{2}| \\ &\quad +|\frac{(n+1) - \sqrt{(n^2+6n+1)}}{2}| \\ &= n + \sqrt{(n^2+6n+1)}. \end{aligned}$$

□

3. Properties of Minimum Dominating Harary Energy of a Graph

Theorem 3.1 : Let $|\lambda I - H_D| = a_0\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + ... + a_n$be the charecterastic polynomial of H_D. Then

(i) $a_0 = 1$,

(ii) $a_1 = -|D|$,

(iii) $a_2 = (|D|_2) - \sum_{i<j}^{n} \frac{1}{d(v_i, v_j)^2}$,

Proof :

(i) It follows from the definition, $P_D(G,\lambda) = det(\lambda I - A_D(G))$, that $a_0 = 1$.

(ii) The sum of determinants of all 1×1 principal submatrices of H_D is equal to the trace of H_D.
$\Rightarrow a_1 = (-1)^1$ trace of $[H_D(G)] = -|D|$.

(iii) The sum of determinants of all the 2×2 principal submatrices of $[H_D(G)]$ is

$$\begin{aligned} a_2 &= (-1)^2 \sum_{1\le i<j\le n} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} = \sum_{1\le i<j\le n} (a_{ii}a_{jj} - a_{ij}a_{ji}) \\ &= \sum_{1\le i<j\le n} a_{ii}a_{jj} - \sum_{1\le i<j\le n} a_{ji}a_{ij} \\ &= (D|2) - \sum_{i<j}^{n} \frac{1}{d(v_i,v_j)^2} \end{aligned}$$

□

Theorem 3.2 : If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of $H_D(G)$,
then

$$\sum_{i=1}^{n} \lambda_i = |D| \quad \text{and} \quad \sum_{i=1}^{n} \lambda_i^2 = |D| + 2\sum_{i<j}^{n} \frac{1}{d(v_i,v_j)^2}.$$

Proof : We know that the sum of the eigenvalues of $A_D(G)$ is the trace of $A_D(G)$

$$= \sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{ii} = |D|$$

Consider

$$\begin{aligned} \sum_{i=1}^{n} \lambda_i^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}a_{ji} = \sum_{i=1}^{n} (a_{ii})^2 + \sum_{i\neq j} a_{ij}a_{ji} \\ &= \sum_{i=1}^{n} (a_{ii})^2 + 2\sum_{i<j} (a_{ij})^2 \\ &= \sum_{i=1}^{n} \lambda_i^2 = |D| + 2\sum_{i<j}^{n} \frac{1}{d(v_i,v_j)^2}. \end{aligned}$$

□

Theorem 3.3 : Let G be graph with n vertices m edges and minimum dominating set D . Then

$$\sqrt{\left(|D|+2\left(\sum_{i<j}^{n}\frac{1}{d(v_i,v_j)^2}\right)\right)+n(n-1)|detA_D(G)|^{\frac{2}{n}}}$$
$$\leq HE_D(G) \leq \sqrt{n\left(|D|+2\sum_{i<j}^{n}\frac{1}{d(v_i,v_j)^2}\right)}.$$

Theorem 3.4 : If $\lambda_1(G)$ is the largest minimum dominating Harary eigen value of $H_D(G)$, then $\lambda_1 \geq \frac{2\sum_{i<j}^{n}\frac{1}{d(v_i,v_j)}+|D|}{n}$, where $|D|$ is the cardinality of minimum dominating set.

Proof : Let X be any nonzero vector. Then by [2], We have

$$\lambda_1(H_D) = \max_{X\neq 0}\frac{X'H_DX}{X'X}$$
$$= \lambda_1(H_D) \geq \frac{J'H_DJ}{J'J}$$
$$= \frac{2\sum_{i<j}^{n}\frac{1}{d(v_i,v_j)}+|D|}{n},$$

where J is a unit matrix. □

Theorem 3.5 : Let G be a graph with a minimum dominating set D. If the minimum dominating Harary energy $HE_D(G)$ is a rational number, then $HE_D(G) \equiv |D|(mod2)$.

Proof : Proof is similar to Theorem 3.7 of [1]. □

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References

[1] Adiga C., Bayad A., Gutman I., Srinivas S. A., The minimum covering energy of a graph, Kragujevac J. Sci., 34 (2012), 39-56.

[2] Bapat R. S., Graphs and Matrices, Hindustan Book Agency, (2011).