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A NOTE ON SEMI- CONVERGENT SERIES

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Abstract

In this paper we shall give some properties of a semi-convergent series. We also distinguish a semi- convergent and an absolutely convergent series.

1. Introduction

The study of infinite series includes two important class of series namely, absolute convergent series and conditionally convergent series. A conditionally convergent series is also named as a semi- convergent series. The study of a semi-convergent series plays a vital role in mathematical analysis. Here we shall prove the re-arrangement theorems for a semi-convergent series.

Definition 1 : An **infinite series** is a sum of infinite number of terms $a_1+a_2+a_3+\cdots$. An infinite series is usually denoted by $\sum\limits_{n=1}^{\infty} a_n$.

If the sum of infinite number of terms is a finite number, say S, then we say that the series $\sum\limits_{n=1}^{\infty} a_n$ is **convergent** and S is called the **sum** of the series.

We denote it as $\sum_{n=1}^{\infty} a_n = S$.

If $\sum_{n=1}^{\infty} a_n = \pm\infty$, we say that the series is **divergent**.

Definition 2 : An alternating series is of the form $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ where $a_n > 0$.

Definition 3 : The **sequence of partial sums** of the series $\sum_{n=1}^{\infty} a_n$ is the sum to n terms of the series. It is denoted by S_n. That is $S_n = \sum_{n=1}^{\infty} a_k$.

Theorem : A series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if its sequence of partial sums S_n is convergent.

Definition 4 : A series $\sum_{n=1}^{\infty} a_n$ is called an **absolutely convergent** series, if the series of positive terms $\sum_{n=1}^{\infty} |a_n|$ is convergent.

If a series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then the series itself is also convergent.

Example :

1. The series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^p}$ where $p > 1$
2. The series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n(n+1)}$.

Definition 5 : A series $\sum_{n=1}^{\infty} a_n$ is called a **semi-convergent (conditionally convergent)** series , if the series $\sum_{n=1}^{\infty} a_n$ is convergent, but the series of positive terms $\sum_{n=1}^{\infty} |a_n|$ is not convergent.

Example : The series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^p}$ where $0 \leq p \leq 1$.

Definition 6 : Let $\sum_{n=1}^{\infty} a_n$ be a series of positive and negative terms. Define

$$a_n^+ = \begin{cases} a_n, & \text{if} \quad a_n > 0 \\ 0, & \text{if} \quad a_n \leq 0 \end{cases}$$

$$a_n^- = \begin{cases} -a_n, & \text{if} \quad a_n < 0 \\ 0, & \text{if} \quad a_n \geq 0 \end{cases}$$

Then the series $\sum_{n=1}^{\infty} a_n^+$ is called the **series of positive terms** and series $\sum_{n=1}^{\infty} a_n^-$ is called the **series of negative terms**.

It follows that $a_n = a_n^+ - a_n^-$

$$|a_n| = a_n^+ + a_n^-$$

That is $a_n^+ = \frac{|a_n|+a_n}{2}, \quad a_n^- = \frac{|a_n|-a_n}{2}$.

Note : For the series $\sum_{n=1}^{\infty} a_n$, the series $\sum_{n=1}^{\infty} a_n^+$ is called the series of **positive terms** and the series $\sum_{n=1}^{\infty} a_n^-$ is called the series of **negative terms**.

Theorem : For an absolutely convergent series, the series $\sum_{n=1}^{\infty} a_n^+ +$ and $\sum_{n=1}^{\infty} a_n^-$ are convergent.

Proof : Let $\sum_{n=1}^{\infty} a_n$ be an absolutely convergent series. Then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} |a_n|$ are convergent.

Let $\sum_{n=1}^{\infty} a_n = S$ and $\sum_{n=1}^{\infty} |a_n| = T$.

Let S_n and T_n be the sequence of partial sums of $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} |a_n|$ respectively.

Let P_n and Q_n denote the sequence of partial sums of $\sum_{n=1}^{\infty} a_n^+$ and $\sum_{n=1}^{\infty} a_n^-$ respectively.

Then we have $S_n = P_n - Q_n$ and $T_n = P_n + Q_n$.

That is $P_n = \frac{T_n+S_n}{2}$ and $Q_n = \frac{T_n-S_n}{2}$.

Now $\lim_{n\to\infty} P_n = \lim_{n\to\infty} \frac{T_n+S_n}{2} = \lim_{n\to\infty} \frac{1}{2}(T_n + S_n) = \frac{T+S}{2}$, a finite number.

Also $\lim_{n\to\infty} Q_n = \lim_{n\to\infty} \frac{T_n-S_n}{2} = \lim_{n\to\infty} \frac{1}{2}(T_n - S_n) = \frac{T-S}{2}$, a finite number.

Hence the sequence of partial sums of $\sum_{n=1}^{\infty} a_n^+$ and $\sum_{n=1}^{\infty} a_n^-$ are convergent.

Thus the two series $\sum_{n=1}^{\infty} a_n^+$ and $\sum_{n=1}^{\infty} a_n^-$ are convergent.

Theorem : For a semi-convergent series $\sum_{n=1}^{\infty} a_n$, the two series $\sum_{n=1}^{\infty} a_n^+$ and $\sum_{n=1}^{\infty} a_n^-$ are divergent.

Proof : Let $\sum_{n=1}^{\infty} a_n$ be a semi-convergent series. Then $\sum_{n=1}^{\infty} a_n$ is convergent, but $\sum_{n=1}^{\infty} |a_n|$ is not convergent.

Let $\sum_{n=1}^{\infty} a_n = S$, $\sum_{n=1}^{\infty} |a_n| = \infty$.

Let S_n and T_n be the sequence of partial sums of $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} |a_n|$ respectively.

Let $\lim_{n\to\infty} S_n = S$. We have $\lim_{n\to\infty} T_n = \infty$.

Let P_n and Q_n denote the sequence of partial sums of $\sum\limits_{n=1}^{\infty} a_n^+$ and $\sum\limits_{n=1}^{\infty} a_n^-$ respectively.
Then we have $S_n = P_n - Q_n$ and $T_n = P_n + Q_n$.
That is $P_n = \frac{T_n+S_n}{2}$ and $Q_n = \frac{T_n-S_n}{2}$.
Now

$$\lim_{n\to\infty} P_n = \lim_{n\to\infty} \frac{T_n + S_n}{2} = \lim_{n\to\infty} \frac{1}{2}(T_n + S_n) = \frac{\infty + S}{2} = \infty.$$

Also

$$\lim_{n\to\infty} Q_n = \lim_{n\to\infty} \frac{T_n - S_n}{2} = \lim_{n\to\infty} \frac{1}{2}(T_n - S_n) = \frac{\infty - S}{2} = \infty.$$

Hence the sequence of partial sums of $\sum\limits_{n=1}^{\infty} a_n^+$ and $\sum\limits_{n=1}^{\infty} a_n^-$ are divergent.

Thus the two series $\sum\limits_{n=1}^{\infty} a_n^+$ and $\sum\limits_{n=1}^{\infty} a_n-$ are divergent.

Lemma : For a semi-convergent series $\sum\limits_{n=1}^{\infty} a_n$, the two series $\sum\limits_{n=1}^{\infty} a_n^+$ and $\sum\limits_{n=1}^{\infty} a_n^-$ have infinitely many positive terms.

Proof : Assume that $\sum\limits_{n=1}^{\infty} a_n^-$ has only a finite number of positive terms. Then

$$\begin{aligned}
\sum_{n=1}^{\infty} |a_n| &= \sum_{n=1}^{\infty} (a_n^+ + a_n^-) \\
&\leq \sum_{n=1}^{\infty} (a_n^+ - a_n^-) + 2\sum_{n=1}^{\infty} a_n^- \\
&= \sum_{n=1}^{\infty} a_n + 2\sum_{n=1}^{\infty} a_n^-
\end{aligned}$$

We have $\sum\limits_{n=1}^{\infty} a_n$ is convergent and by our assumption, $\sum\limits_{n=1}^{\infty} a_n^-$ is convergent. It follows that $|a_n|$ is convergent which is a contradiction. Hence our assumption is wrong.
Thus $\sum\limits_{n=1}^{\infty} a_n^-$ must have infinite number of positive terms.

Similarly assume that $\sum\limits_{n=1}^{\infty} a_n^+$ has only finite number of terms. Since

$$\begin{aligned}
\sum_{n=1}^{\infty} |a_n| &= \sum_{n=1}^{\infty} (a_n^+ + a_n^-) \\
&\leq \sum_{n=1}^{\infty} (a_n^- - a_n^+) + 2\sum_{n=1}^{\infty} a_n^+ \\
&= (-1)\sum_{n=1}^{\infty} a_n + 2\sum_{n=1}^{\infty} a_n^+.
\end{aligned}$$

Since $\sum\limits_{n=1}^{\infty} a_n$ is convergent, we have $(-1)\sum\limits_{n=1}^{\infty} a_n$ is also convergent.
By our assumption $\sum\limits_{n=1}^{\infty} a_n^+$ is convergent.
It follows that $\sum\limits_{n=1}^{\infty} |a_n|$ is convergent, which is a contradiction.
Hence $\sum\limits_{n=1}^{\infty} a_n^+$ must have infinite number of positive terms.

Lemma : The series $\sum\limits_{n=1}^{\infty} a_n^+$ and $\sum\limits_{n=1}^{\infty} a_n^-$ are unbounded below.

Proof : We have $\sum\limits_{n=1}^{\infty} a_n^-$ is a series of non-positive terms and so it is monotone decreasing.
If the series $\sum\limits_{n=1}^{\infty} a_n^-$ is bounded below, then by monotone convergence theorem, the series will converge which is a contradiction.
Thus the series $\sum\limits_{n=1}^{\infty} a_n^-$ is unbounded below.
Similarly we can prove that $\sum\limits_{n=1}^{\infty} a_n^+ +$ is unbounded below.

Theorem : A semi-convergent series $\sum\limits_{n=1}^{\infty} a_n$, the terms can be rearranged so that the re-arranged series diverges to ∞.

Proof : In this proof we shall denote $u_n = a_n^+$ and $v_n = a_n^-$. Then clearly u_n and v_n are non-negative real numbers.
Now we make a re-arrangement of $\sum\limits_{n=1}^{\infty} a_n$ as follows.

$$(u_1 + u_2 + \cdots + u_{m1})v_1 + (u_{m1+1} + u_{m1+2} + \cdots + u_{m2})$$
$$v_2 + (u_{m2+1} + u_{m2+2} + \cdots + u_{m3})v_3 + \cdots$$

where a group of positive terms is followed by a single negative term.
We denote the re-arranged series as $\sum\limits_{n=1}^{\infty} b_n$ and let S_n denote the sequence of partial sums of $\sum\limits_{n=1}^{\infty} b_n$.
Since $\sum\limits_{n=1}^{\infty} u_n$ is divergent, its sequence of partial sums is unbounded.
Let us choose m_1 so large that $u_1 + u_2 + \cdots + u_{m1} > 1 + v_1$.
Now choose $m_2 > m_1$ such that

$$u_1 + u_2 + \cdots + u_{m1} + u_{m1+1} + u_{m1+2} + \cdots + u_{m2} > 2 + v_1 + v_2.$$

In general, choose $m_n > m_{n-1}$ such that

$$u_1+u_2+\cdots+u_{m1}+u_{m1+1}+u_{m1+2}+\cdots+u_{m2}+\cdots+u_{mn} > n+v_1+v_2+\cdots+v_n \text{ for } n \in N.$$

Since each of the partial sums $S_{m1+1}, S_{m2+2}, \cdots$ of the rearranged series, whose last term is a negative term $-v_n$, is greater than n, these partial sums are unbounded above and hence the series $\sum\limits_{n=1}^{\infty} b_n$ diverges to ∞.

Theorem : The terms of a semi-convergent series $\sum\limits_{n=1}^{\infty} a_n$ can be rearranged so that the rearranged series diverges to $-\infty$.

Proof : The non negative real numbers u_n and v_n are defined as in the proof of previous theorem.

Here we make another re-arrangement of as follows.

$$(-v_1 - v2 - \cdots - v_{m1}) + u_1 + (v_{m1+1} - v_{m1+2} - \cdots - v_{m2})$$
$$+u_2 + (-vm_{2+1} - v_{m2+2} - \cdots - v_{m3}) + u_3 + \cdots$$

where a group of negative terms is followed by a single positive term.

We denote the re-arranged series as $\sum\limits_{n=1}^{\infty} b_n$ and let S_n denote the sequence of partial sum of $\sum\limits_{n=1}^{\infty} b_n$.

Since $\sum\limits_{n=1}^{\infty} v_n$ is divergent, its sequence of partial sums is unbounded.

Let us choose m_1 so large that $-v_1 - v_2 - \cdots - v_{m1} < 1 - u_1$.

Now choose $m_2 > m_1$ such that

$$-v_1 - v_2 \cdots - v_{m1} - v_{m1+1} - v_{m1+2} - \cdots - v_{m2} < 2 - u_1 - u_2.$$

In general, choose $m_n > m_{n-1}$ such that

$$-v_1-v_2-\cdots-v_{m1}-v_{m1+1}-v_{m1+2}-\cdots-v_{m2}-\cdots-v_{mn} < n-u_1-u_2-\cdots-u_n \text{ for } n \in N.$$

Since each of the partial sums $S_{m1+1}, S_{m2+2}, \cdots$ of the rearranged series $\sum\limits_{n=1}^{\infty} b_n$, whose last term is a positive term u_n is less than n, these partial sums are unbounded below. Hence the re-arranged series $\sum\limits_{n=1}^{\infty} b_n$ diverges to ∞.

Theorem (Riemanns re-arrangement theorem) : Let $\sum\limits_{n=1}^{\infty} a_n$ be a semi-convergent series and let $r \in R$. Then there exists a rearrangement of the terms of $\sum\limits_{n=1}^{\infty} a_n$ that converges to r.

Proof : The non negative real numbers u_n and v_n are defined as in the proof of previous theorem.

Here we make another re-arrangement of $\sum\limits_{n=1}^{\infty} a_n$ as follows.

Choose first m_1 positive terms such that their sum exceeds r. i.e. $(u_1+u_2+\cdots+u_{m1}) > r$.

Then add first m_2 negative terms such that the sum is less than r.

i.e. $(u_1 + u_2 + \cdots + u_{m1}) + (-v_{m1+1} - v_{m1+2} - \cdots - v_{m2}) < r.$.

Again add next m_3 positive terms so that the sum exceeds r.

i.e. $(u_1+u_2+\cdots+u_{m1})+(-v_{m1+1}-v_{m1+2}-\cdots-v_{m2})+(u_{m2+1}+u_{m2+2}+\cdots+u_{m3}) > r.$

Next we add m_4 negative terms so that the sum is less than r. i.e.

$$
\begin{aligned}
&(u_1 + u_2 + \cdots + u_{m1}) + (-v_{m1+1} - v_{m1+2} - \cdots - v_{m2}) \\
&+(u_{m2+1} + u_{m2+2} + \cdots + u_{m3}) + (-v_{m1+1} - v_{m1+2} - \cdots - v_{m4}) < r.
\end{aligned}
$$

We continue the above process, where a group of positive terms is followed by another group of negative terms.

We denote the re-arranged series as $\sum\limits_{n=1}^{\infty} b_n$ and let S_n denote the sequence of partial sum of $\sum\limits_{n=1}^{\infty} b_n$.

We can see that the partial sums $S_{m1+m2+m3}+\cdots$ of the rearranged seris $\sum\limits_{n=1}^{\infty} b_n$ brackets the real number r.

We have $u_n \to 0$ and $v_n \to 0$ as $n \to \infty$.

Hence for given $\epsilon > 0$, there exists a natural number N such that $|S_n - r| < \epsilon$ for $n \geq N$.

Thus$S_n \to r$ as $n \to \infty$.

Hence the rearranged series $\sum\limits_{n=1}^{\infty} b_n$ converges to the real number r.

References

[1] Knopp Konra, Theoryand Application of Infinite Series, Blackie and Son Limited (London and Glasgow).

[2] Hardy G. H., A Course of Pure Mathematics, Cambridge at the University Press, (Tenth edition), (1963).

[3] Knopp K., Infinite Sequences and Series, Dover, (1956).

[4] Apostol Tom M., Mathematical Analysis, Narosa Publishing House, (Second Edition), ISBN 81-85015-66.

[5] Galanor Stewart, Reimann's rearrangement theorem, mathematics teacher, 80(8), (Novermber 1987), 675-681.