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FIXED POINT THEOREMS VIA THE CONCEPT SET S_t IN 2-METRIC SPACES

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Abstract

2- metric spaces is an attractive nonlinear generalization of metric spaces which was studied in details by Gahler [2]. In this note some common fixed point results in 2- metric spaces are obtained. In the process, some previously known results in the context of 2-metric spaces are generalized and improved. As an application of the concept of E_α given by Rathore et al. [18]. At the end some open problems are suggested.

1. Introduction and Preliminaries

Kannan [9] introduced a set $S_a = \{z \in X; \rho(z, Tz) \leq a\}$ for any positive number a and a self-mapping T of a metric space X with metric ρ.

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Using the notation of the set S_a, he proved certain properties of the set S_a and also established the well known Banach fixed point theorem [13]. Pal M. and Pal M.C. [17] introduced a set E_α for self mapping in different way than that of Kannan [9]. They obtained some properties of the set E_α and proved some results of fixed points without using iteration method. Rathore M. S., Singh M., Rathore S. and Singh N. [18] defined the set E_α considering two self mappings and established all properties stated in [17]. Recently Lahiri, B.K., Das P. and Dey L. K. [11] proved an analogue of Cantor intersection theorem for complete 2-metric space and defined set S_t. Further they observed that the intersection theorem along with the idea of a set S_t may be conventionally used to prove Banach fixed point theorem in 2-metric space. Later on, some other fixed point theorem have also been obtained by them.

2. Definitions and Preliminaries

Definition 2.1 [6] : A sequence $\{x_n\}$ in 2-metric space (X, σ) is said to converge to $x \in X$ if for any $a \in X, \sigma(x_n, x, a) \to 0$ as $n \to \infty$.

i.e. $x_n \to x$ as $n \to \infty$ or $lim_{n\to\infty} x_n = x$.

Definition 2.2 [6] : A sequence $\{x_n\}$ in 2-metric space (X, σ) is said to be a Cauchy sequence if for any $a \in X, \sigma(x_m, x_n, a) \to 0$ as $m, n \to \infty$.

Definition 2.3 [6, 7] : A 2-metric space (X, σ) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges to a point of X.

Definition 2.4 : A 2-metric space (X, σ) is said to be compact if every sequence $\{x_n\}$ in X has a convergent sub-sequence.

Theorem 2.1 [11] : Suppose that (X, σ) is a complete 2-metric space. If $\{F_n\}$ is any decreasing sequence (i.e, $F_{n+1} \subset F_n \forall n \in N$) of 2-closed sets with $\delta_a(F_n) \to 0$ as $n \to \infty \; \forall \; a \in X$ then $\cap_{n=1}^{\infty} F_n$ is non empty and contains at most one point.

In our subsequent discussion we need a set S_t due to [11] which is defined as follows:

Definition 2.5 : Let (X, ρ) be 2-metric space and $T : X \to X$ be a mapping. For $t > 0$ define

$$S_t = \{x \in X; \sigma(x, Tx, y) \leq t \; \forall \; y \in X\}.$$

3. Main Result

Theorem 3.1 : Let (X, σ) be a 2-metric space and $T : X \to X$ be a continuous mapping such that

$$\sigma(Tx, Ty, a) \leq \alpha\sigma(x, y, a) + \beta\sigma(x, Ty, a) + \gamma\sigma(y, Tx, a) \tag{3.1}$$

where α, β, γ are non-negative real numbers such that $\alpha + \beta + \gamma < 1, \quad \forall x, y \in X$.
Then T has a unique fixed point.

Proof : Let $\{t_n\}$ be a decreasing sequence of positive numbers converging to zero. Clearly $S_{t_{n+1}} \subseteq S_{t_n}$, n=1,2,3...

Clearly S_{t_n} (n = 1,2,...) is closed.

Now we shall show that $\partial(S_{t_n}) \to 0$ as $n \to 0$. For any $x, y \in S_{t_n},\ \ and\ \ a \in X$, we have

$$\begin{aligned}
\sigma(x, y, a) &\leq \sigma(x, Tx, a) + \sigma(x, y, Tx) + \sigma(Tx, y, a) \\
&\leq 2t_n + \sigma(Tx, y, a) \\
&\leq 2t_n + \sigma(Tx, Ty, a) + \sigma(Tx, y, Ty) + \sigma(Ty, y, a) \\
&\leq 4t_n + \sigma(Tx, Ty, a) \\
&\leq 4t_n + \alpha\sigma(x, y, a) + \beta\sigma(x, Ty, a) + \gamma\sigma(y, Tx, a).
\end{aligned}$$

This implies that

$$(1 - \alpha)\sigma(x, y, a) \leq 4t_n + \beta\sigma(x, Ty, a) + \gamma\sigma(y, Tx, a). \tag{3.2}$$

First step:

From (3.1) and (3.2) we have

$$\begin{aligned}
(1 - \alpha)\sigma(x, y, a) \leq 4t_n &+ \beta\{\sigma(x, Ty, Tx) + \sigma(x, Tx, a) + \sigma(Tx, Ty, a)\} \\
&+\ \gamma\{\sigma(y, Tx, Ty) + \sigma(y, Ty, a) + \sigma(Ty, Tx, a)\}
\end{aligned}$$

or

$$\begin{aligned}
(1 - \alpha)\sigma(x, y, a) &\leq 4t_n + \beta\{2t_n + \sigma(Tx, Ty, a)\} + \gamma\{2t_n + \sigma(Tx, Ty, a)\} \\
&= 4t_n + 2(\beta + \gamma)t_n + (\beta + \gamma)\sigma(Tx, Ty, a) \\
&\leq 4t_n + 2(\beta + \gamma)t_n + (\beta + \gamma)\{\alpha\sigma(x, y, a) + \beta\sigma(x, Ty, a) + \gamma\sigma(y, Tx, a)\} \\
&\leq 4t_n + 2(\beta + \gamma)t_n + (\beta + \gamma)[\alpha\{\sigma(x, y, Tx) + \sigma(x, Tx, a) + \sigma(Tx, y, a)\} \\
&+\ \beta\sigma(x, Ty, a) + \gamma\sigma(y, Tx, a)]
\end{aligned}$$

$$\begin{aligned}(1-\alpha)\sigma(x,y,a) &= 4t_n + 2(\beta+\gamma)t_n + (\beta+\gamma)[\alpha\{2t_n + \sigma(Tx,y,a)\}\\ &\quad + \beta\sigma(x,Ty,a) + \gamma\sigma(y,Tx,a)]\\ &= 4t_n + 2(\beta+\gamma)t_n + 2\alpha(\beta+\gamma)t_n\\ &\quad + (\beta+\gamma)\{\beta\sigma(x,Ty,a) + (\alpha+\gamma)\sigma(y,Tx,a)\}\end{aligned}$$

$$(1-\alpha)\sigma(x,y,a) \leq 4t_n + 2(1+\alpha)(\beta+\gamma)t_n + (\beta+\gamma)\{\beta\sigma(x,Ty,a) + (\alpha+\gamma)\sigma(y,Tx,a)\}. \tag{3.3}$$

Second step:

From (3.1) and (3.3), we have

$$\begin{aligned}&(1-\alpha)\sigma(x,y,a)\\ &\quad\leq 4t_n + 2(1+\alpha)(\beta+\gamma)t_n\\ &\quad + (\beta+\gamma)[\beta\{\sigma(x,Ty,Tx) + \sigma(x,Tx,a) + \sigma(Tx,Ty,a)\}\\ &\quad + (\alpha+\gamma)\{\sigma(y,Tx,Ty) + \sigma(y,Ty,a) + \sigma(Ty,Tx,a)\}]\\ &\quad\leq 4t_n + 2(1+\alpha)(\beta+\gamma)t_n\\ &\quad + (\beta+\gamma)[\beta\{2t_n + \sigma(Tx,Ty,a)\} + (\alpha+\gamma)\{2t_n + \sigma(Tx,Ty,a)\}]\\ &\quad = 4t_n + 2(1+\alpha)(\beta+\gamma)t_n\\ &\quad + (\beta+\gamma)[2(\alpha+\beta+\gamma)t_n + \beta\sigma(Tx,Ty,a) + (\alpha+\gamma)\sigma(Tx,Ty,a)]\end{aligned}$$

or

$$\begin{aligned}(1-\alpha)\sigma(x,y,a) &\leq 4t_n + 2(1+\alpha)(\beta+\gamma)t_n + 2(\beta+\gamma)(\alpha+\beta+\gamma)t_n\\ &\quad + (\beta+\gamma)(\alpha+\beta+\gamma)\sigma(Tx,Ty,a)\\ &\leq 4t_n + 2(1+\alpha)(\beta+\gamma)t_n + 2(\beta+\gamma)(\alpha+\beta+\gamma)t_n\\ &\quad + (\beta+\gamma)(\alpha+\beta+\gamma)\{\alpha\sigma(x,y,a) + \beta\sigma(x,Ty,a) + \gamma\sigma(y,Tx,a)\}\\ &\leq 4t_n + 2(1+\alpha)(\beta+\gamma)t_n + 2(\beta+\gamma)(\alpha+\beta+\gamma)t_n\\ &\quad + (\beta+\gamma)(\alpha+\beta+\gamma)[\alpha\{\sigma(x,y,Tx) + \sigma(x,Tx,a) + \sigma(Tx,y,a)\}\\ &\quad + \beta\sigma(x,Ty,a) + \gamma\sigma(y,Tx,a)]\\ &\leq 4t_n + 2(1+\alpha)(\beta+\gamma)t_n + 2(\beta+\gamma)(\alpha+\beta+\gamma)t_n\\ &\quad + (\beta+\gamma)(\alpha+\beta+\gamma)[\alpha\{2t_n + \sigma(Tx,y,a)\} + \beta\sigma(x,Ty,a) + \gamma(y,Tx,a)]\end{aligned}$$

$$\begin{aligned}(1-\alpha)\sigma(x,y,a) \quad\leq\quad & 4t_n + 2(1+\alpha)(\beta+\gamma)t_n + 2(1+\alpha)(\beta+\gamma)(\alpha+\beta+\gamma)t_n\\ & +(\beta+\gamma)(\alpha+\beta+\gamma)[\beta\sigma(x,Ty,a) + (\alpha+\gamma)\sigma(y,Tx,a)].\end{aligned} \tag{3.4}$$

Third step:

From (3.1) and (3.4) it follows that

$$\begin{aligned}(1-\alpha)\sigma(x,y,a) \leq & 4t_n + 2(1+\alpha)(\beta+\gamma)t_n + 2(1+\alpha)(\beta+\gamma)(\alpha+\beta+\gamma)t_n \\ & + (\beta+\gamma)(\alpha+\beta+\gamma)[\beta\{\sigma(x,Ty,Tx) + \sigma(x,Tx,a) + \sigma(Tx,Ty,a)\} \\ & + (\alpha+\gamma)\{\sigma(y,Tx,Ty) + \sigma(y,Ty,a) + \sigma(Ty,Tx,a)\}] \\ & \leq 4t_n + 2(1+\alpha)(\beta+\gamma)t_n + 2(1+\alpha)(\beta+\gamma)(\alpha+\beta+\gamma)t_n \\ & + (\beta+\gamma)(\alpha+\beta+\gamma)[\beta\{2t_n + \sigma(Tx,Ty,a)\} \\ & + (\alpha+\gamma)\{2t_n + \sigma(Tx,Ty,a)\}] \\ & = 4t_n + 2(1+\alpha)(\beta+\gamma)t_n + 2(1+\alpha)(\beta+\gamma)(\alpha+\beta+\gamma)t_n \\ & \quad + (\beta+\gamma)(\alpha+\beta+\gamma)[2(\alpha+\beta+\gamma)t_n + (\alpha+\beta+\gamma)\sigma(Tx,Ty,a)] \\ & \leq 4t_n + 2(1+\alpha)(\beta+\gamma)t_n + 2(1+\alpha)(\beta+\gamma)(\alpha+\beta+\gamma)t_n \\ & + 2(\beta+\gamma)(\alpha+\beta+\gamma)^2 t_n \\ & \quad + (\beta+\gamma)(\alpha+\beta+\gamma)^2\{\alpha\sigma(x,y,a) + \beta\sigma(x,Ty,a) + \gamma\sigma(y,Tx,a)\}\end{aligned}$$

or

$$\begin{aligned}(1-\alpha)\sigma(x,y,a) \leq & 4t_n + 2(1+\alpha)(\beta+\gamma)t_n + 2(1+\alpha)(\beta+\gamma)(\alpha+\beta+\gamma)t_n \\ & + 2(\beta+\gamma)(\alpha+\beta+\gamma)^2 t_n \\ & + (\beta+\gamma)(\alpha+\beta+\gamma)^2[\alpha\{\sigma(x,y,Tx) + \sigma(x,Tx,a) + \sigma(Tx,y,a)\} \\ & + \beta\sigma(x,Ty,a) + \gamma\sigma(y,Tx,a)] \\ & \leq 4t_n + 2(1+\alpha)(\beta+\gamma)t_n + 2(1+\alpha)(\beta+\gamma)(\alpha+\beta+\gamma)t_n \\ & + 2(\beta+\gamma)(\alpha+\beta+\gamma)^2 t_n \\ & + (\beta+\gamma)(\alpha+\beta+\gamma)^2[2\alpha t_n + \alpha\sigma(Tx,y,a) + \beta\sigma(x,Ty,a) + \gamma\sigma(y,Tx,a)]\end{aligned}$$

$$\begin{aligned}(1-\alpha)\sigma(x,y,a) \quad \leq \quad & 4t_n + 2(1+\alpha)(\beta+\gamma)t_n + 2(1+\alpha)(\beta+\gamma)(\alpha+\beta+\gamma)t_n \\ & +2(1+\alpha)(\beta+\gamma)(\alpha+\beta+\gamma)^2 t_n \\ & +(\beta+\gamma)(\alpha+\beta+\gamma)^2[\{\beta\sigma(x,Ty,a) + (\alpha+\gamma)\sigma(y,Tx,a)\}].\end{aligned} \tag{3.5}$$

Fourth step:

From (3.1) and (3.5), we have

$$\begin{aligned}(1-\alpha)\sigma(x,y,a) \;&\leq 4t_n + 2(1+\alpha)(\beta+\gamma)t_n + 2(1+\alpha)(\beta+\gamma)(\alpha+\beta+\gamma)t_n\\ &+2(1+\alpha)(\beta+\gamma)(\alpha+\beta+\gamma)^2 t_n + (\beta+\gamma)(\alpha+\beta+\gamma)^2[\beta\{\alpha(x,Ty,Tx)\\ &+\sigma(x,Tx,a)+\sigma(Tx,Ty,a)\}\\ &+(\alpha+\gamma)\{\sigma(y,Tx,Ty)+\sigma(y,Ty,a)+\sigma(Ty,Tx,a)\}]\\ &\leq 4t_n + 2(1+\alpha)(\beta+\gamma)t_n + 2(1+\alpha)(\beta+\gamma)(\alpha+\beta+\gamma)t_n\\ &+2(1+\alpha)(\beta+\gamma)(\alpha+\beta+\gamma)^2 t_n\\ &+(\beta+\gamma)(\alpha+\beta+\gamma)^2[\beta\{2t_n+\sigma(Tx,Ty,a)\}\\ &+(\alpha+\gamma)\{2t_n+\sigma(Tx,Ty,a)\}]\end{aligned}$$

or

$$\begin{aligned}(1-\alpha)\sigma(x,y,a) \;&\leq 4t_n + 2(1+\alpha)(\beta+\gamma)t_n + 2(1+\alpha)(\beta+\gamma)(\alpha+\beta+\gamma)t_n\\ &+2(1+\alpha)(\beta+\gamma)(\alpha+\beta+\gamma)^2 t_n\\ &+(\beta+\gamma)(\alpha+\beta+\gamma)^2[2(\alpha+\beta+\gamma)t_n+(\alpha+\beta+\gamma)\sigma(Tx,Ty,a)]\end{aligned}$$

or

$$\begin{aligned}(1-\alpha)\sigma(x,y,a) \;&\leq 4t_n + 2(1+\alpha)(\beta+\gamma)t_n + 2(1+\alpha)(\beta+\gamma)(\alpha+\beta+\gamma)t_n\\ &+2(1+\alpha)(\beta+\gamma)(\alpha+\beta+\gamma)^2 t_n + 2(\beta+\gamma)(\alpha+\beta+\gamma)^3 t_n\\ &+(\beta+\gamma)(\alpha+\beta+\gamma)^3\sigma(Tx,Ty,a)\end{aligned}$$

or

$$\begin{aligned}(1-\alpha)\sigma(x,y,a) \;&\leq 4t_n + 2(1+\alpha)(\beta+\gamma)t_n + 2(1+\alpha)(\beta+\gamma)(\alpha+\beta+\gamma)t_n\\ &+2(1+\alpha)(\beta+\gamma)(\alpha+\beta+\gamma)^2 t_n + 2(\beta+\gamma)(\alpha+\beta+\gamma)^3 t_n\\ &+(\beta+\gamma)(\alpha+\beta+\gamma)^3\{\alpha\sigma(x,y,a)+\beta\sigma(x,Ty,a)+\gamma\sigma(y,Tx,a)\}\\ &\leq 4t_n + 2(1+\alpha)(\beta+\gamma)t_n + 2(1+\alpha)(\beta+\gamma)(\alpha+\beta+\gamma)t_n\\ &+2(1+\alpha)(\beta+\gamma)(\alpha+\beta+\gamma)^2 t_n + 2(\beta+\gamma)(\alpha+\beta+\gamma)^3 t_n\\ &+(\beta+\gamma)(\alpha+\beta+\gamma)^3[\alpha\{\sigma(x,y,Tx)+\sigma(x,Tx,a)+\sigma(Tx,y,a)\}\\ &+\beta\sigma(x,Ty,a)+\gamma\sigma(y,Tx,a)]\end{aligned}$$

$$\begin{aligned}
&\leq 4t_n + 2(1+\alpha)(\beta+\gamma)t_n + 2(1+\alpha)(\beta+\gamma)(\alpha+\beta+\gamma)t_n \\
&+2(1+\alpha)(\beta+\gamma)(\alpha+\beta+\gamma)^2 t_n + 2(\beta+\gamma)(\alpha+\beta+\gamma)^3 t_n \\
&+(\beta+\gamma)(\alpha+\beta+\gamma)^3[\alpha\{2t_n + \sigma(Tx,y,a)\} + \beta\sigma(x,Ty,a) + \gamma\sigma(y,Tx,a)] \\
&\quad \leq 4t_n + 2(1+\alpha)(\beta+\gamma)t_n + 2(1+\alpha)(\beta+\gamma)(\alpha+\beta+\gamma)t_n \\
&+2(1+\alpha)(\beta+\gamma)(\alpha+\beta+\gamma)^2 t_n + 2(\beta+\gamma)(\alpha+\beta+\gamma)^3 t_n \\
&+2\alpha(\beta+\gamma)(\alpha+\beta+\gamma)^3 \\
&+(\beta+\gamma)(\alpha+\beta+\gamma)^3\{\beta\sigma(x,Ty,a) + (\alpha+\gamma)\sigma(y,Tx,a)\}
\end{aligned}$$

$$\begin{aligned}
(1-\alpha)\sigma(x,y,a) \quad \leq \quad & 4t_n + 2(1+\alpha)(\beta+\gamma)t_n + 2(1+\alpha)(\beta+\gamma)(\alpha+\beta+\gamma)t_n \\
& +2(1+\alpha)(\beta+\gamma)(\alpha+\beta+\gamma)^2 t_n + 2(1+\alpha)(\beta+\gamma)(\alpha+\beta+\gamma)^3 t_n \\
& +(\beta+\gamma)(\alpha+\beta+\gamma)^3[\beta\sigma(x,Ty,a) + (\alpha+\gamma)\sigma(y,Tx,a)].
\end{aligned} \tag{3.6}$$

Continuing in this way in n^{th} step, we get

$$\begin{aligned}
(1-\alpha)\sigma(x,y,a) \leq & 4t_n + 2(1+\alpha)(\beta+\gamma)t_n + 2(1+\alpha)(\beta+\gamma)(\alpha+\beta+\gamma)t_n \\
& +2(1+\alpha)(\beta+\gamma)(\alpha+\beta+\gamma)^2 t_n + 2(1+\alpha)(\beta+\gamma)(\alpha+\beta+\gamma)^3 t_n + ... \\
& +(1+\alpha)(\beta+\gamma)(\alpha+\beta+\gamma)^{n-1} t_n \\
& +(\beta+\gamma)(\alpha+\beta+\gamma)^{n-1}\{\beta\sigma(x,Ty,a) + (\alpha+\gamma)\sigma(y,Tx,a)\}
\end{aligned}$$

or

$$\begin{aligned}
(1-\alpha)\sigma(x,y,a) \leq & 4t_n + 2(1+\alpha)(\beta+\gamma)t_n + 2(1+\alpha)(\beta+\gamma)(\alpha+\beta+\gamma)t_n. \\
& [1 + (\alpha+\beta+\gamma) + (\alpha+\beta+\gamma)^2 + ...(\alpha+\beta+\gamma)^{n-2}] \\
& + (\beta+\gamma)(\alpha+\beta+\gamma)^{n-1}\{\beta\sigma(x,Ty,a) + (\alpha+\gamma)\sigma(y,Tx,a)\}
\end{aligned}$$

i.e.

$$\begin{aligned}
(1-\alpha)\sigma(x,y,a) \leq & 4t_n + 2(1+\alpha)(\beta+\gamma)t_n + 2(1+\alpha)(\beta+\gamma)(\alpha+\beta+\gamma)t_n \\
& \{1 + (\alpha+\beta+\gamma) + (\alpha+\beta+\gamma)^2 + ...\} \\
& +(\beta+\gamma)(\alpha+\beta+\gamma)^{n-1}\{\beta\sigma(x,Ty,a) + (\alpha+\gamma)\sigma(y,Tx,a)\}
\end{aligned}$$

or

$$\begin{aligned}
\sigma(x,y,a) \leq & \frac{4t_n}{1-\alpha} + \frac{2(1+\alpha)(\beta+\gamma)t_n}{1-\alpha} + \frac{2(1+\alpha)(\beta+\gamma)(\alpha+\beta+\gamma)t_n}{(1-\alpha)\{1-(\alpha+\beta+\gamma)\}} \\
& + \frac{(\beta+\gamma)(\alpha+\beta+\gamma)^{n-1}}{1-\alpha}\{\beta\sigma(x,Ty,a) + (\alpha+\gamma)\sigma(y,Tx,a)\} \to 0 \ as \ n \to \infty.
\end{aligned}$$

Hence $\delta(S_{t_n}) \to 0$ as $n \to \infty$. So $\{S_{t_n}\}$ is a sequence of sets such that

(i) S_{t_n} is closed

(ii) $S_{t_{n+1}} \subseteq S_{t_n} \ \forall \ n = 1, 2, 3...$

(iii) $\delta(S_{t_n}) \to 0$ as $n \to \infty$

Hence S_{t_n} contains exactly one point. Let $x_0 \in \bigcap_{n=1}^{\infty} S_{t_n}$.

Then $\sigma(x_0, Tx_0, a) \leq t_n \ \forall \ n = 1, 2, 3... \ and \ \forall a \in X$

$\Longrightarrow \sigma(x_0, Tx_0, a) = 0 \ \forall \ a \in X$

$\Longrightarrow Tx_0 = x_0$.

Hence x_0 is a fixed point of T.

To prove uniqueness, let u and v be two distinct fixed point of T,then for a point $a \in X$, $a \neq u \ or \ v$,

$$\begin{aligned}\sigma(u, v, a) &= \sigma(Tu, Tv, a)\\ &\leq \alpha\sigma(u, v, a) + \beta\sigma(u, Tv, a) + \gamma\sigma(v, Tu, a)\\ &= \alpha\sigma(u, v, a) + \beta\sigma(u, v, a) + \gamma\sigma(v, u, a)\\ &= (\alpha + \beta + \gamma)\sigma(u, v, a)\end{aligned}$$

or

$\{1 - (\alpha + \beta + \gamma)\}\sigma(u, v, a) \leq 0$.

which implies that $u = v$.

Hence T has a unique fixed point.

Remark 3.1 : If $\beta = \gamma = 0$, then we get result of B. K. Lahiri, Prafulananda Das and Lakshmi Kant Day [11].

If we put $\alpha = 0$ in Theorem 3.1 then we get the following corollary.

Corollary 3.1 : Let (X, σ) be a 2-metric space and $T : X \to X$ be a continuous mapping such that

$$\sigma(Tx, Ty, a) \leq \beta\sigma(x, Ty, a) + \gamma\sigma(y, Tx, a) \tag{3.7}$$

where β, γ are non-negative real numbers such that $\beta + \gamma < 1 \ \ \forall x, y \in X$. If we put $\beta = 0$ in Theorem 3.1 then we get the following corollary

Corollary 3.2 : Let (X, σ) be a 2-metric space and $T : X \to X$ be a continuous mapping such that

$$\sigma(Tx, Ty, a) \leq \alpha\sigma(x, y, a) + \gamma\sigma(y, Tx, a) \tag{3.8}$$

where α, γ are non-negative real numbers such that $\alpha + \gamma < 1 \quad \forall x, y \in X$.

If we put $\gamma = 0$ in Theorem 3.1 then we get the following corollary.

Corllary 3.3 : Let (X, σ) be a 2-metric space and $T : X \to X$ be a continuous mapping such that

$$\sigma(Tx, Ty, a) \leq \alpha\sigma(x, y, a) + \beta\sigma(x, Ty, a) \tag{3.9}$$

where α, β are non-negative real numbers such that $\alpha + \beta < 1 \quad \forall x, y \in X$.

Theorem 3.2 : Let (X, σ) be a 2-metric space and $T : X \to X$ be a continuous mapping such that

$$\rho(Tx, Ty, a) \leq \alpha[\sigma(x, Ty, a) + \sigma(y, Tx, a)] + \beta[\sigma(x, Tx, a) + \sigma(y, Ty, a)] \tag{3.10}$$

where α, β, are non-negative reals such that $\alpha + \beta < 1 \quad \forall x, y \in X$.

Then T has a unique fixed point.

Proof : Proof follows on similar lines of Theorem 3.1.

If we put $\alpha = 0$ in Theorem 3.2 then we get following corollary.

Corollary 3.4 : Let (X, σ) be a 2-metric space and $T : X \to X$ be a continuous mapping such that

$$\sigma(Tx, Ty, a) \leq \beta[\sigma(x, Tx, a) + \sigma(y, Ty, a)] \tag{3.11}$$

where β is a non-negative real number such that $\beta < 1 \quad \forall x, y \in X$

If we put $\beta = 0$ in Theorem 3.2 then we get following corollary.

Corollary 3.5 : Let (X, σ) be a 2-metric space and $T : X \to X$ be a continuous mapping such that

$$\sigma(Tx, Ty, a) \leq \alpha[\sigma(x, Ty, a) + \sigma(y, Tx, a)] \tag{3.12}$$

where α is non-negative real number such that $\alpha < 1 \quad \forall x, y \in X$.

Open Problems

Under what conditions Theorem 3.1 and 3.2 can be extended to a pair of two self mappings, producing common fixed point.

Competing Interests

The authors declare that they have no competing interests.

Author's contribution

All authors contributed equally and significantly in writing this article.

All authors read and approved final manuscript.

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