International J. of Math. Sci. & Engg. Appls. (IJMSEA)
ISSN 0973-9424, Vol. 10 No. II (August, 2016), pp. 55-65

SOLVING LINEAR SYSTMES OF PARTIAL DIFFERENTIAL EQUATIONS OF THE SECOND ORDER BY USING AL-TEMEME TRANSFORM

ALI HASSAN MOHAMMED[1] AND BANEEN SADEQ MOHAMMED ALI[2]
[1,2] Department of Mathematics,
Faculty of Education for Girls,
University of Kufa,

Abstract

Our aim in this paper is to find the solution of Linear Systems Partial Differential Equations (LSPDE) of the second order with variable coefficients subjected to some initial conditions by using Al-Tememe transform ($T.T$) through generalized the method that found in [2].

1. Introduction

Integral transformations are an important role to solve the linear partial differential equations (LPDE) of the second order with constant coefficients and variable coefficients. We will use Al-Tememe Transform ($\mathcal{T}.T$) to solve systems of linear partial differential

---

Key Words : *$T.T$ Al-Tememe transform, $T^{-1}.T$ inverse of Al-Tememe transform, LPDE linear partial differential equation, Partial functions, Second order with variable coefficients.*

equations of second order with variable coefficients and the method summarized by taking ($\mathcal{T}.T$) to both sides of the equations. Then we take ($\mathcal{T}^{-1}.T$) to both sides of the equations and by using the given initial conditions, we find the functions.

2. Preliminaries

Definition 1 [1] : Let f is defined function at period (a, b) then the integral transformation for f whose it's symbol $F(s)$ is defined as

$$F(s) = \int_a^b k(s, x) f(x) dx$$

where k is a fixed function of two variables, called the kernel of the transformation and a, b are real numbers or $\mp\infty$, such that the above integral is convergent.

Definition 2 [3] : The Al-Tememe transformation for the function $f(x) : x > 1$ is defined by the following integral:

$$\mathcal{T}[f(x)] = \int_1^\infty x^{-s} f(x) dx = F(s)$$

such that this integral is convergent, s is positive constant. From the above definition we can write

$$\begin{aligned} T(u(x,t)) &= \int_1^\infty t^{-s} u(x,t) dt \\ &= v(x,s) \end{aligned}$$

such that $u(x, t)$ is a function of x and t.

Property 1 [3] : This transformation is characterizedby the linear property, that is

$$\mathcal{T}[Au_1(x,t) + Bu_2(x,t)] = A\mathcal{T}[u_1(x,t)] + B\mathcal{T}[u_2(x,t)],$$

where A and B are constants while the functions $u_1(x, t), u_2(x, t)$ are defined when $(t > 1)$.

The Al-Tememe transform for some fundamental functions are given in Table 1 [3].

Table 1

ID	Function $f(x)$	$F(s)=\int_1^\infty x^{-s}f(x)dx=T[f(x)]$	Regional of convergence
1	$k; k=$ constant	$\frac{k}{(s-1)}$	$s>1$
2	$x^n, n\in R$	$\frac{1}{(s-(n+1))}$	$s>n+1$
3	lnx	$\frac{1}{(s-1)^2}$	$s>1$
4	$x^n lnx, n\in R$	$\frac{1}{[s-(n+1)]^2}$	$s>n+1$
5	$\sin a(lnx)$	$\frac{a}{(s-1)^2+a^2}$	$s>1$
6	$\cos a(lnx)$	$\frac{s-1}{(s-1)^2+a^2}$	$s>1$
7	$\sinh a(lnx)$	$\frac{a}{(s-1)^2-a^2}$	$\lvert s-1\rvert>a$
8	$\cosh a(lnx)$	$\frac{s-1}{(s-1)^2-a^2}$	$\lvert s-1\rvert>a$

From the Al-Tememe definition and the abvoe table, we get

Theorem 1 ; If $\mathcal{T}(u(x,t))=v(x,s)$ and a is constant, then $\mathcal{T}(u(x,t^{-a}))=v(x,s+a)$ see [3].

Definition 3 [3] : Let $u(x,t)$ be a function where $(t>1)$ and $\mathcal{T}(u(x,t))=v(x,s), u(x,t)$ is said to be an inverse for the Al-Tememe transformation and written as $\mathcal{T}^{-1}(v(x,s))=u(x,t)$, where $\mathcal{T}^{-1}$ returns the transformation to the original function. For example

$$T^{-1}\left[\frac{2lnx}{((s-2)^2+4)}\right]=lnx\ t\sin 2lnt.$$

Propety 2 [3] : If $\mathcal{T}^{-1}(u_1(x,s))=v_1(x,t), \mathcal{T}^{-1}(u_2(x,s))=v_2(x,s),\cdots,\tau^{-1}(u_n(x,t))=v_n(x,s)$ and $a_1,a_2,\cdots,a_n$ are constants then

$$\begin{aligned}&\mathcal{T}^{-1}[a_1v_1(x,s)+a_2v_2(x,s)+\cdots+a_nv_n(x,s)]\\&=a_1u_1(x,t)+a_2u_2(x,t)=\cdots+a_nu_n(x,t).\end{aligned}$$

Theorem 2 [3] : If the function $u(x,t)$ is defined for $t>1$ and its derivatives $u_t(x,t),u_{tt}(x,t),\cdots,u_t^{(n)}(x,t)$ are exist then

$$\begin{aligned}&\mathcal{T}[t^nu_t^{(n)}(x,t)]=-u_t^{(n-1)}(x,1)-(s-n)u_t^{(n-2)}(x,1)-\cdots\\&-(s-n)(s-(n-1))\cdots(s-2)u(x,1)+(s-n)!v(x,s).\end{aligned}$$

3. Solving Linear Systems of Partial Differential Equations of the Second Order by Using Al-Tememe Transform

Let us consider we have a linear system of partial differential equations of second order with variable coefficients which we can write it by

$$\begin{aligned} at^2u_{1tt}(x,t)+btu_{1t}(x,t) &= a_1u_1(x,t)+a_2u_2(x,t)+g_1(x,t) \\ ct^2u_{2tt}(x,t)+dtu_{2t}(x,t) &= b_1u_1(x,t)+b_2u_2(x,t)+g_2(x,t) \end{aligned} \tag{1}$$

where $a, b, a_1, a_2, c, d, b_1$ and b_2 are constants, $u_{1t}(x,t)$ and $u_{1tt}(x,t)$ are derivatives of the function $u_1(x,t)$, $u_{2t}(x,t)$ and $u_{2tt}(x,t)$ are derivative of function $u_2(x,t)$, such that $u_1(x,t)$ and $u_2(x,t)$ are continuous functions and the $(\mathcal{T}.T)$ of $g_1(x,t)$ and $g_2(x,t)$ are known. For solving the system (1) we take $(\mathcal{T}.T)$ to both sides of it and after simplification we put $v_1(x,s)=\mathcal{T}(u_1(x,t))$, $v_2(x,s)=\mathcal{T}(u_2(x,t))$, $G_1(x,s)=\mathcal{T}(g_1(x,t))$, $G_2(x,s)=\mathcal{T}(g_2(x,t))$. So, we get:

$$\begin{aligned} &-au_{1t}(x,1)-a(s-2)u_1(x,1)+a(s-2)(s-1)v_1(x,s)-bu_1(x,1)+b(s-1)v_1(x,s) \\ &=a_1v_1(x,s)+a_2v_2(x,s)+G_1(x,s) \\ &((s-1)(a(s-2)+b)-a_1)v_1(x,s)-a_2v_2(x,s) \\ &=au_{1t}(x,1)+a(s-2)u_1(x,1)+bu_1(x,1)+G_1(x,s) \\ &-cu_{2t}(x,1)-c(s-2)u_2(x,1)+c(s-2)(s-1)v_2(x,s)-du_2(x,1)+d(s-1)v_2(x,s) \\ &-b_1v_1(x,s)-b_2v_2(x,s)=G_2(x,s) \end{aligned} \tag{2}$$

$$\begin{aligned} &((s-1)(c(s-2)+d)-b_2)v_2(x,s)-b_1v_1(x,s) \\ &=cu_{2t}(x,1)+c(s-2)u_2(x,1)+du_2(x,1)+G_2(x,s) \end{aligned} \tag{3}$$

So

$$k_1(s)v_1(x,s)-a_2v_2(x,s)=au_{1t}(x,1)+a(s-2)u_1(x,1)+bu_1(x,1)+G_1(x,s) \tag{4}$$

$$k_2(s)v_2(x,s)-b_1v_1(x,s)=cu_{2t}(x,1)+c(s-2)u_2(x,1)+du_2(x,1)+G_2(x,s) \tag{5}$$

where $k_1(s)=(s-1)(a(s-2)+b)-a_1$ and $k_2(s)=(s-1)(c(s-2)+d)-b_2$.

By multiplying equation (4) by $k_2(s)$ and equation (5) by a_2 and collecting the result terms we have :

$$v_1(x,s) = \frac{h_1(x,s)}{N_1(s)}; \quad N_1(s) \neq 0. \tag{6}$$

By the similar method we find

$$v_2(x,s) = \frac{h_2(x,s)}{N_2(s)}; \quad N_2(s) \neq 0 \tag{7}$$

where $h_1(x,s)$ and $h_2(x,s)$ are polynomials of s and x, $N_1(s)$ and $N_2(s)$ are polynomials of s, such that the degree of $h_1(x,s)$ is less than the degree of $N_1(s)$ and the degree of $h_2(x,s)$ is less than the degree of $N_2(s)$.

By taking the inverse of Al-Tememe transformation $(\mathcal{T}^{-1}.T)$ to both sides of equation (6) and (7) we get :

$$\begin{aligned} u_1(x,t) &= T^{-1}\left[\tfrac{h_1(x,s)}{N_1(s)}\right], \\ u_2(x,t) &= T^{-1}\left[\tfrac{h_2(x,s)}{N_2(s)}\right] \end{aligned} \tag{8}$$

Equation (8) represents the general solution of system (1) which we can be written it as follows:

$$\begin{aligned} u_1(x,t) &= \sum_i A_i(x)B_i(t) \\ u_2(x,t) &= \sum_i C_i(x)D_i(t) \end{aligned} \tag{9}$$

where as $B_1, B_2, \cdots, B_m$ are functions of t and $A_1, A_2, \cdots, A_m$ are functions of x, which is number equals to the degree of $h_1(x,s)$, also $C_1(x), C_2(x), \cdots, C_m(x)$ are functions of x which is number equals to the degree of $h_2(x,s)$.

To find the forms of functions of $A_1(x), A_2(x), \cdots, A_m(x)$ and $C_1(x), C_2(x), \cdots, C_m(x)$ we use partial fractions decomposition.

Example 1 : To solve the system of partial differential equation

$$t^2u_{1tt}(x,t) + 7tu_{1t}(x,t) = -9u_1(x,t) + u_2(x,t) + xt^2; u_1(x,1) = 0, u_{1t}(x,1) = 0$$

$$t^2u_{2tt}(x,t) - tu_{2t}(x,t) = u_1(x,t) - u_2(x,t) + x; \quad u_2(x,1) = 0, u_{2t}(x,1) = 0.$$

We take Al-Tememe transform to both sides of the system we get

$$-u_{1t}(x,1) - (s-2)u_1(x,1) + (s-2)(s-1)v_1(x,s) - 7u_1(x,1)$$

$$+7(s-1)v_1(x,s) + 9v_1(x,s) - v_2(x,s) = \tfrac{x}{(s-3)}$$

$$(s-1)(s+5)v_1(x,s) + 9v_1(x,s) - v_2(x,s) - 1 = \tfrac{x}{s-3}$$

$$(s+2)^2 v_1(x,s) - v_2(x,s) = \frac{x}{s-3} \tag{10}$$

$$-u_{2t}(x,1) - (s-2)u_2(x,1) + (s-2)(s-1)v_2(x,s) + u_2(x,1)$$

$$-(s-1)v_2(x,s) + v_2(x,s) - v_1(x,s) = \tfrac{x}{s-1}$$

$$(s-2)(s-1)v_2(x,s) - (s-1)v_2(x,s) - v_1(x,s) + v_2(x,s) = \tfrac{x}{s-1}$$

$$(s-2)^2 v_2(x,s) - v_1(x,s) = \frac{x}{(s-1)} \tag{11}$$

By multiplying eq. (10) by $(s-2)^2$ and eq. (11) 1, where we get,

$$(s-2)^2(s+2)^2 v_1(x,s) - (s-2)^2 v_2(x,s) = \frac{x(s-2)^2}{s-3} \tag{12}$$

$$(s-2)^2 v_2(x,s) - v_1(x,s) = \frac{x}{s-1}. \tag{13}$$

From eq. (3) and eq. (4) we get

$$\begin{aligned} v_1(x,s) &= \frac{x(s-2)^2}{(s-3)(s^2-5)(s^2-3)} + \frac{x}{(s-1)(s^2-5)(s^2-3)} \\ &= \left[\frac{A(x)}{s-3} + \frac{B(x)s+C(x)}{s^2-5} + \frac{D(x)s+E(x)}{s^2-3}\right] \\ &\quad + \left[\frac{A_1(x)}{s-1} + \frac{B_1(x)s+C_1(x)}{s^2-5} + \frac{D_1(x)s+E_1(x)}{s^2-3}\right] \end{aligned}$$

$$A(x) + B(x) + D(x) = 0 \tag{14}$$

$$-3B(x) + C(x) - 3D(x) + E(x) = 0 \tag{15}$$

$$-8A(x) - 3B(x) - 3C(x) - 5D(x) - 3E(x) = x \tag{16}$$

$$9B(x) - 3C(x) + 15D(x) - 5E(x) = -4x \tag{17}$$

$$15A(x) + 9C(x)m + 15E(x) = 4x. \tag{18}$$

After we solve the system of equations (14), (15), (16), (17) and (18) we get

$$\Rightarrow A(x) = \frac{x}{24}, \quad B(x) = \frac{3x}{8}, \quad C(x) = \frac{-7x}{8}, \quad D(x) = \frac{-5x}{12}, \text{‘} \; E(x) = \frac{3x}{4}.$$

Also

$$A_1(x) + B_1(x) + D_1(x) = 0 \tag{19}$$

$$-B_1(x) + C_1(x) - D_1(x) + E_1(x) = 0 \tag{20}$$

$$-8A_1(x) - 3B_1(x) - C_1(x) - 5D_1(x) - E_1(x) = 0 \tag{21}$$

$$3B_1(x) - 3C_1(x) + 5D_1(x) - 5E_1(x) = 0 \tag{22}$$

$$15A_1(x) + 3C_1(x) + 5E_1(x) = x. \tag{23}$$

After we solve the system of equations (19), (20), (21), (22) and (23), we get :

$$\Rightarrow A_1(x) = \frac{x}{8}, \quad B_1(x) = \frac{x}{8}, \quad C_1(x) = \frac{x}{8}, \quad D_1(x) = \frac{-x}{4}, \quad E_1(x) = \frac{-x}{4}$$

Therefore, (after using $\mathcal{T}^{-1}.T$) we get :

$$\begin{aligned} v_1(x,s) &= \frac{x}{24}\frac{1}{s-3} + \frac{3x}{8}\frac{s}{s^2-5} - \frac{7x}{8}\frac{1}{s^2-5} - \frac{5x}{12}\frac{s}{s^2-3} + \frac{x}{8}\frac{1}{s-1} \\ &+ \frac{x}{8}\frac{s}{s^2-5} + \frac{x}{8}\frac{1}{s^2-5} - \frac{x}{4}\frac{s}{s^2-3} - \frac{x}{4}\frac{1}{s^2-3} \end{aligned}$$

$$\begin{aligned} u_1(x,t) &= \frac{x}{24}t^2 + \frac{x}{2}t^{-1}\cosh\sqrt{2}lnt - \frac{3}{4\sqrt{5}}t^{-1}\sinh\sqrt{5}lnt \\ &- \frac{2x}{3}t^{-1}\cosh\sqrt{3}lnt + \frac{x}{2\sqrt{3}}t^{-1}\sinh\sqrt{3}lnt \end{aligned}$$

and multiplying equation (10) by 1 and equatiion (11) by $(s+2)^2$ we get

$$(s+2)^2 v_1(x,s) - v_2(x,s) = \frac{x}{s-3} \tag{24}$$

$$(s+2)^2(s-2)^2 v_2(x,s) - (s+2)^2 v_1(x,s) = \frac{x(s+2)^2}{s-1} \tag{25}$$

From equation (24) and (25) we get

$$\begin{aligned} v_2(x,s) &= \frac{x}{(s-3)(s^2-5)(s^2-3)} + \frac{x(s+2)^2}{(s-1)(s^2-5)(s^2-3)} \\ v_2(x,s) &= \left[\frac{A(x)}{s-3} + \frac{B(x)s + C(x)}{s^2-5} + \frac{D(x)s + E(x)}{s^2-3}\right] \\ &+ \left[\frac{A_1(x)}{s-1} + \frac{B_1(x)s + C_1(x)}{s^2-5} + \frac{D_1(x)s + E_1(x)}{s^2-3}\right] \end{aligned}$$

$$A(x) + B(x) + D(x) = 0 \tag{26}$$

$$-3B(x) + C(x) - 3D(x) + E(x) = 0 \tag{27}$$

$$-8A(x) - 3B(x) - 3C(x) - 5D(x) - 3E(x) = 0 \tag{28}$$

$$9B(x) - 3C(x) + 15D(x) - 5E(x) = 0 \tag{29}$$

$$15A(x)+9C(x)+15E(x)=x. \tag{30}$$

After we solve the system of equations (26), (27), (28), (29) and (30) we get :

$$\Rightarrow A(x)=\frac{x}{24},\quad B(x)=\frac{-x}{8},\quad C(x)=\frac{-3x}{8},\quad D(x)=\frac{x}{12}, E(x)=\frac{x}{4}.$$

Also

$$A_1(x)+B_1(x)+D_1(x)=0 \tag{31}$$

$$-B_1(x)+C_1(x)-D_1(x)+E_1(x)=0 \tag{32}$$

$$-8A_1(x)-3B_1(x)-C_1(x)-5D_1(x)-E_1(x)=x \tag{33}$$

$$3B_1(x)-3C_1(x)+5D_1(x)-5E_1(x)=4x \tag{34}$$

$$15A_1(x)+3C_1(x)+5E_1(x)=4x. \tag{35}$$

After we solve the system of equations (31), (32), (33), (34) and (35) we get:

$$\Rightarrow A_1(x)=\frac{9x}{8},\quad B_1(x)=\frac{13x}{8},\quad C_1(x)=\frac{29x}{8},\quad D_1(x)=\frac{-11x}{4},\quad E_1(x)=\frac{-19x}{4}$$

$$\begin{aligned} v_2(x,s) \quad &= \quad \frac{x}{24}\frac{1}{s-3}-\frac{x}{8}\frac{s}{s^2-5}-\frac{3x}{8}\frac{1}{s^2-5}+\frac{x}{12}\frac{s}{s^2-3}+\frac{x}{4}\frac{1}{s^2-3} \\ &\quad +\frac{9x}{8}\frac{1}{s-1}+\frac{13x}{8}\frac{s}{s^2-5}+\frac{29x}{8}\frac{1}{s^2-5}-\frac{11x}{4}\frac{s}{s^2-3}-\frac{19x}{4}\frac{1}{s^2-3}. \end{aligned}$$

Therefore, (after using $\mathcal{T}^{-1}.T$) we get:

$$\begin{aligned} u_2(x,t) \quad &= \quad \frac{x}{24}t^2+\frac{3}{2}t^{-1}\cosh\sqrt{5}lnt+\frac{13x}{4\sqrt{5}}t^{-1}\sinh\sqrt{5}lnt \\ &\quad -\frac{8}{3}t^{-1}\cosh\sqrt{3}lnt-\frac{18x}{4\sqrt{3}}t^{-1}\sinh\sqrt{3}lnt+\frac{9x}{8}. \end{aligned}$$

Example 2 : To solve the system of partial differential equation

$$t^2u_{1tt}(x,t)+5tu_{1t}(x,t)=-4u_1(x,t)+u_2(x,t)+x\ lnt; u_1(X,1)=0, u_{1t}(x,1)=0$$

$$t^2u_{2tt}(x,t)+tu_{2t}(x,t)=u_1(x,t)+x\ \sin lnt;\quad u_2(x,1)=0, u_{2t}(x,1)=0.$$

We take Al-Tememe transform to both sides of the system we get

$$\begin{aligned} &(s-2)(s-1)v_1(x,s)+5(s-1)v_1(x,s)+4v_1(x,s)-v_2(x,s)=\tfrac{x}{(s-1)^2} \\ &((s-1)(s+3)+4)v_1(x,s)-v_2(x,s)=x\tfrac{1}{(s-1)^2} \\ &(s+1)^2v_1(x,s)-v_2(x,s)=\tfrac{x}{(s-1)^2} \end{aligned} \tag{36}$$

$$\begin{aligned}(s-2)(s-1)v_2(x,s)+(s-1)v_2(x,s)-v_1(x,s)&=\tfrac{x}{(s-1)^2+1}\\(s-1)^2v_2(x,s)-v_1(x,s)&=\tfrac{x}{(s-1)^2+1}\end{aligned} \tag{37}$$

By multiplying eq. (36) by $(s-1)^2$ and eq. (37) 1, we get,

$$(s-1)^2(s+1)^2v_1(x,s)-(s-1)^2v_2(x,s)=x \tag{38}$$

$$(s-1)^2v_2(x,s)-v_1(x,s)=\frac{x}{(s-1)^2+1}. \tag{39}$$

From eq. (38) and eq. (39) we get

$$\begin{aligned}v_1(x,s)&=\frac{x}{s^2(s^2-2)}+\frac{x}{s^2(s^2-2)((s-1)^2+1)}=\frac{x((s-1)^2+1)+x}{s^2(s^2-2)((s-1)^2+1)}\\&=\frac{A(x)s+B(x)}{s^2}+\frac{C(x)s+D(x)}{s^2-2}+\frac{E(x)(s-1)+F(x)}{((s-1)^2+1)}\end{aligned}$$

$$A(x)+C(x)+E(x)=0 \tag{40}$$

$$-2A(x)+B(x)-2C(x)+D(x)-E(x)+F(x)=0 \tag{41}$$

$$-2B(x)+2C(x)-2D(x)-2E(x)=0 \tag{42}$$

$$4A(x)+2D(x)+2E(x)-2F(x)=x. \tag{43}$$

$$-4A(x)+4B(x)=-2x \tag{44}$$

$$-4B(x)=,3x. \tag{45}$$

After we solve the system of equations (40), (41), (42), (43), (44) and (45) we get

$$\Rightarrow A(x)=\frac{-x}{4},\quad B(x)=\frac{-3x}{4},\quad C(x)=\frac{x}{8},\quad D(x)=\frac{3x}{4},\quad E(x)=\frac{x}{8},\quad F(x)=\frac{-x}{8}.$$

$$v_1(x,s)=\frac{-x}{4}\frac{1}{s}-\frac{3x}{4}\frac{1}{s^2}+\frac{x}{8}\frac{s}{s^2-2}+\frac{3x}{4}\frac{1}{s^2-2}+\frac{x}{8}\frac{(s-1)}{(s-1)^2+1}-\frac{x}{8}\frac{1}{(s-1)^2+1}.$$

Therefore, (after using $\mathcal{T}^{-1}.T$) we get :

$$u_1(x,t)=\frac{-x}{4}t^{-1}-\frac{3x}{4}t^{-1}lnt+\frac{x}{8}t^{-1}\cosh\sqrt{2}\,lnt+\frac{3x}{4\sqrt{2}}t^{-1}\ \sinh\sqrt{2}\,lnt+\frac{x}{8}\cos lnt-\frac{x}{8}\sin lnt$$

and by multiplying equation (36) by 1 and equatiion (37) by $(s+1)^2$

$$(s+1)^2v_1(x,s)-v_2(x,s)=\frac{x}{(s-1)^2} \tag{46}$$

$$(s+1)^2(s-1)^2 v_2(x,s) - (s+1)^2 v_1(x,s) = \frac{x(s+1)^2}{(s-1)^2+1}. \tag{47}$$

From equation (46) and (47) we get

$$\begin{aligned} v_2(x,s) &= \frac{x}{(s-1)^2(s^2-2)s^2} + \frac{x(s+1)^2}{s^2(s^2-2)((s-1)^2+1)} \\ v_2(x,s) &= \left[\frac{A(x)s+B(x)}{s^2} + \frac{C(x)(s-1)+D(x)}{(s-1)^2} + \frac{E(x)s+F(x)}{s^2-2}\right] \\ &\quad + \left[\frac{A_1(x)s+B_1(x)}{s^2} + \frac{C_1(x)s+D_1(x)}{s^2-2} + \frac{E_1(x)(s-1)+F_1(x)}{((s^-1)^2+1)}\right] \end{aligned}$$

$$A(x)+C(x)+E(x)=0 \tag{48}$$

$$-2A(x)+B(x)-C(x)+D(x)-2E(x)+F(x)=0 \tag{49}$$

$$-A(x)-2B(x)-2C(x)+E(x)-2F(x)=0 \tag{50}$$

$$4A(x)-B(x)+2C(x)-2D(x)+F(x)=0 \tag{51}$$

$$-2B(x)=x. \tag{52}$$

After we solve the system of equations (48), (49), (50), (51) and (52) we get :

$$\Rightarrow B(x) = \frac{-x}{2}, \quad A(x) = -x, \quad C(x) = 0, \quad D(x) = -x, E(x) = x, F(x) = \frac{3x}{2}.$$

$$A_1(x)+C_1(x)+E_1(x)=0 \tag{53}$$

$$-2A_1(x)+B_1(x)-2C_1(x)+D_1(x)-E_1(x)+F_1(x)=0 \tag{54}$$

$$-2B_1(x)+2C_1(x)-2D_1(x)-2E_1(x)=0 \tag{55}$$

$$4A_1(x)+2D_1(x)+2E_1(x)-2F_1(x)=x \tag{56}$$

$$-4A_1(x)+4B_1(x)=2x. \tag{35}$$

$$-4B_1(x)=x \Rightarrow B_1(x)=\frac{-x}{4} \tag{58}$$

After we solve the system of equations (53), (54), (55), (56), (57) and (58) we get:

$$A_1(x) = \frac{-3x}{4}, \quad C_1(x) = \frac{7x}{8}, \quad D_1(x) = \frac{5x}{4}, \quad E_1(x) = \frac{-x}{8}, \quad F_1(x) = \frac{-7x}{8}$$

$$\begin{aligned} v_2(x,s) &= \frac{-x}{s} - \frac{x}{2}\frac{1}{s^2} - x\frac{1}{(s-1)^2} + x\frac{s}{s^2-2} + \frac{3x}{2}\frac{1}{s^2-2} - \frac{3x}{4}\frac{1}{s} - \frac{x}{4}\frac{1}{s^2} + \frac{7x}{8}\frac{s}{s^2-2} \\ &\quad + \frac{5x}{4}\frac{1}{s^2-2} - \frac{x}{8}\frac{(s-1)}{(s-1)^2+1} - \frac{7x}{8}\frac{1}{(s-1)^2+1} \end{aligned}$$

Therefore, (after using $\mathcal{T}^{-1}.T$) we get:

$$\begin{aligned} u_2(x,t) &= \frac{-7x}{4}t^{-1} - \frac{3x}{4}t^{-1}lnt - xlnt + \frac{15x}{8}t^{-1}\cosh\sqrt{2}lnt + \frac{11x}{4\sqrt{2}}t^{-1}\sinh\sqrt{2}lnt \\ &\quad -\frac{x}{8}\cos lnt - \frac{7x}{8}\sin lnt. \end{aligned}$$

References

[1] Gabriel Nagy, Ordinary Differential Equations, Mathematics Department, Michigan State University, East Lansing, MI, 48824, (October 14, 2014).

[2] Herbert Kreyszig, Edward J. Norminton, Advanced Engineering Mathematics, Ohio State University, Columbus, Ohio, Tenth Edition, (2011).

[3] Mohammed A. H., Athera Nema Kathem, Solving Euler's equation by using new transformation, Karbala University Magazine for Completely Sciences, 6(4) (2008).