International J. of Math. Sci. & Engg. Appls. (IJMSEA)
ISSN 0973-9424, Vol. 10 No. II (August, 2016), pp. 67-78

ON A SUBCLASS OF MEROMORPHIC p-VALENT FUNCTIONS WITH FRACTIONAL CALCULUS OPERATORS

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Abstract

In the present paper, a subclass of meromorphic multivalent functions is defined by using fractional differ-integral operators. Coefficients estimates, radii of starlikeness and convexity are obtained. Also convolution property, neighborhoods and convex linear combination for the class $G_p^+(\lambda, \mu, v, \eta, c, \alpha)$ are also established.

1. Introduction

Let $L(p)$ be the class of all functions of the form:

$$f(z) = z^{-p} + \sum_{n=p+1}^{\infty} a_n z^n, \quad p \in N = \{1, 2, \cdots\}, \tag{1.1}$$

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Key Words : *Meromorphic multivalent function, Differential operator, Convex combination, Convolution property, Neighborhoods, Radii of starlikeness and convexity.*

AMS Subject Classification : 30C45.

which are analytic and meromorphic multivalent in the punctured unit disk $U^* = \{z \in C : 0 < |z| < 1\}$ consider a subclass G_P of the class $L(p)$ of the functions of the form:

$$f(z) = z^{-p} - \sum_{n=p+1}^{\infty} a_n z^n, \quad (a_n \geq 0). \tag{1.2}$$

A function $f \in G_p$ is meromorphic p-valent starlike function of order $\varphi(0 \leq \varphi < p)$ if

$$-Re\left\{\frac{zf'(z)}{f(z)}\right\} > \varphi, \quad (0 \leq \varphi < p; z \in U^*).$$

A function $f \in G_p$ is meromorphic p-valent convex function of order $\varphi(0 \leq \varphi < p)$ if

$$-Re\left\{\frac{zf''(z)}{f'(z)} + 1\right\} > \varphi, \quad (0 \leq \varphi < p;\ z \in U^*).$$

The convolution (or Hadamard product) of two functions f given by (1.2) and

$$g(z) = z^{-p} - \sum_{n=p+1}^{\infty} b_n z^n, \quad (b_n \geq 0,\ p \in N),$$

is defined by

$$(f * g)(z) = z^{-p} - \sum_{n=p+1}^{\infty} a_n b_n = (g * f)(z). \tag{1.3}$$

In this paper, we discuss and study a subclass of meromorphic p-valent functions by making use of the fractional differ-integral operator contained in:

Definition 1 [1] : The fractional differ-integral operator is defined as follows:

$$W_{0,Z}^{\lambda,\mu,v,\eta} f(z) = \begin{cases} \frac{\Gamma(\mu+v+\eta-\lambda)\Gamma(\lambda)}{\Gamma(\mu+\eta)\Gamma(v+\eta)} z^{-p+\eta+1} J_{0,z}^{\lambda,\mu,v,\eta}[z^{\mu+p} f(z)](0 \leq \lambda < 1) \\ \\ \frac{\Gamma(\mu+v+\eta-\lambda)\Gamma(\lambda)}{\Gamma(\mu+\eta)\Gamma(v+\eta)} z^{-p+\eta+1} I_{0,z}^{-\lambda,\mu,v,\eta}[z^{\mu+p} f(z)](-\infty \leq \lambda < 0), \end{cases} \tag{1.4}$$

where $J_{0,z}^{\lambda,\mu,v,\eta}$ is the generalized fractional derivative operator of order λ defined

$$J_{0,z}^{\lambda,\mu,v,\eta} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \left\{ z^{\lambda-\mu} \int_0^z t^{\eta-1} (z-t)^{\lambda}\ {}_2F_1(\mu, -\lambda, 1-v; 1-\lambda; 1-\frac{t}{2}) f(t) dt \right\}$$

$$(0 \leq \lambda < 1, \mu, \eta \in R,\ r \in R^+ \ \text{ and } \ r > (\max\{0, \mu\} - \eta)), \tag{1.5}$$

where f is an analytic function in a simply-connected region of the z-plane containing the origin and multiplicity of $(z-t)$ is removed by requiring $\log(z-t)$ to be real when $(z-t) > 0$, provided further that

$$f(z) = 0 \ \ (|z|^r) \ (z \to 0), \tag{1.6}$$

and $I_{0,z}^{-\lambda,\mu,v,\eta}$ is the generalized fractional integral operator of order $-\lambda$ $(-\infty < \lambda < 0)$ defined by

$$I_{0,z}^{-\lambda,\mu,v,\eta} f(z) = \frac{z^{-(\lambda+\mu)}}{\Gamma(\lambda)} \int_0^z t^{\eta-1}(z-t)^{\lambda-1} \, {}_2F_1(\lambda+\mu, -v; \lambda; 1-\frac{t}{2}) f(t)dt$$

$$(\lambda > 0, \mu, \eta \in R, \quad r \in R^+ \text{ and } r > (\max\{0,\mu\} - \eta)), \tag{1.7}$$

where f is constrained and the multiplicity of $(z-t)^{\lambda-1}$ is removed as above and r is given by the order estimate (1.6). It follows from (1.5) and (1.6) that

$$J_{0,z}^{\lambda,\mu,v,1} f(z) = J_{0,z}^{\lambda,\mu,v} f(z), \tag{1.8}$$

and

$$I_{0,z}^{\lambda,\mu,v,1} f(z) = I_{0,z}^{\lambda,\mu,v} f(z), \tag{1.9}$$

where $J_{0,z}^{\lambda,\mu,v}$ and $I_{0,z}^{\lambda,\mu,v}$ are the familiar Owa-Saigo-Srivastava generalized fractional derivative and integral operators (see, e.g., [4] and [9] see also [8]). Also

$$J_{0,z}^{\lambda,\lambda,v,1} f(z) = D_Z^{\lambda} f(z), \quad (0 \leq \lambda < 1), \tag{1.10}$$

and

$$I_{0,z}^{\lambda,-\lambda,v,1} f(z) = D_z^{-\lambda} f(z), \quad (\lambda > 0), \tag{1.11}$$

where D_z^{λ} and $D_z^{-\lambda}$ are the familiar Owa-Srivastava fractional derivative and integral of order λ, respectively (cf, Owa[3]; see also Srivastava and Owa[7]).
Furthermore, in terms of Gamma function, we have

$$J_{0,z}^{\lambda,\mu,v,\eta} z^k = \frac{\Gamma(k+\eta)\Gamma(k+\eta-\mu+v)}{\Gamma(k+\eta-\mu)\Gamma(k+\eta-\lambda+v)} z^{k+\eta-\mu-1},$$

$$(0 \leq \lambda < 1, \mu, \eta \in R, v \in R^+ \text{ and } k > (\max\{0,\mu\} - \eta)), \tag{1.12}$$

and

$$I_{0,z}^{\lambda,\mu,v,\eta} z^k = \frac{\Gamma(k+\eta)\Gamma(k+\eta-\mu+v)}{\Gamma(k+\eta-\mu)\Gamma(k+\eta+\lambda+v)} z^{k+\eta-\mu-1},$$

$$(\lambda > 0, \mu, \eta \in R, v \in R^+ \text{ and } k > (\max\{0,\mu\} - \eta)), \tag{1.13}$$

Now using (1.2), (1.12) and (1.13) in (1.4), we find that

$$W_{0,Z}^{\lambda,\mu,v,\eta} f(z) = \frac{1}{z^p} - \sum_{n=p+1}^{\infty} \Gamma_n^{\lambda,\mu,v,\eta} a_n z^n, \tag{1.14}$$

provided that $-\infty < \lambda < 1, \mu + v + \eta > \lambda, \mu > -\eta, v > -\eta, \eta > 0, p \in N, f \in G_p$ and

$$\Gamma_n^{\lambda,\mu,v,\eta} = \frac{(\mu+\eta)_{n+p}(v+\eta)_{n+p}}{(\mu+v+\eta-\lambda)_{n+p}(\eta)_{n+p}}. \tag{1.15}$$

It may be worth noting that, by choosing $\mu = \lambda, \eta = 1$ and $p = 1$, the operator $W_{0,Z}^{\lambda,\mu,v,\eta} f(z)$ reduces to the well-known Ruscheweyh derivative $D^{\lambda} f(z)$ for meromorphic univalent function [6].

In this paper, we shall study a subclass of (1.2) define below.

Definition 2 : The function $f \in G_P$ is in the class $G_P(\lambda, \mu, v, \eta, c, \alpha)$ if it satisfies the condition:

$$\left| \frac{(-c)z - \frac{z^2 (W_{0,Z}^{\lambda,\mu,v,\eta} f(z))''}{(W_{0,Z}^{\lambda,\mu,v,\eta} f(z))'}}{\frac{z^2 (W_{0,Z}^{\lambda,\mu,v,\eta} f(z))''}{(W_{0,Z}^{\lambda,\mu,v,\eta} f(z))'} + (2-\alpha)z} \right| < 1, \tag{1.16}$$

where $0 \le c \le 1, 0 \le \alpha < 1, p \in N, -\infty < \lambda < 1, \mu + v + \eta > \lambda, \mu > -\eta, v > -\eta$ and $\eta > 0$.

Definition 3 : Let G_p^+ denote the subclass of G_p defined by (1.2). Then we define a subclass $G_p^+(\lambda, \mu, v, \eta, c, \alpha)$ by

$$G_p^+(\lambda, \mu, v, \eta, c, \alpha) = G_p^+ \cap G_P(\lambda, \mu, v, \eta, c, \alpha).$$

2. Coefficient Estimates

In the following theorem, we obtain the necessary and sufficient condition for the function to be $f \in G_P$ in the class $G_P(\lambda, \mu, v, \eta, c, \alpha)$.

Theorem 1 : Assume that $f \in G_P$ be given by (1.2) and

$$\sum_{n=p+1}^{\infty} n[2n - \alpha + c]\Gamma_n^{\lambda,\mu,v,\eta} a_n \le p[2p + \alpha - c], \tag{2.1}$$

where $0 \le c \le 1, 0 \le \alpha < 1, p \in N, -\infty < \lambda < 1, \mu + v + \eta > \lambda, \mu > -\eta, v > -\eta$ and $\eta > 0$. Then $f \in G_P(\lambda, \mu, v, \eta, c, \alpha)$.

Proof : Let us assume that inequality (2.1) is true and let $|z| = r < 1$, then, we have

$$\begin{aligned}
&|(-c)z(W_{0,Z}^{\lambda,\mu,v,\eta}f(z))' - z^2(W_{0,Z}^{\lambda,\mu,v,\eta}f(z))''| \\
&-|z^2(W_{0,Z}^{\lambda,\mu,v,\eta}f(z))'' + (2-\alpha)z(W_{0,Z}^{\lambda,\mu,v,\eta}f(z))'| \\
&= \left| p(c-p-1)z^{-p} + \sum_{n=p+1}^{\infty} n(n+c-1)\Gamma_n^{\lambda,\mu,v,\eta} a_n z^n \right| \\
&- \left| p(p+\alpha-1)z^{-p} + \sum_{n=p+1}^{\infty} n(n-\alpha+1)\Gamma_n^{\lambda,\mu,v,\eta} a_n z^n \right| \\
&\leq \sum_{n=p+1}^{\infty} n[2n-\alpha+c]\Gamma_n^{\lambda,\mu,v,\eta} a_n - p[2p+\alpha-c] \leq 0,
\end{aligned}$$

by hypothesis. Then by principle of maximum modulus theorem, $f \in G_P(\lambda, \mu, v, \eta, c, \alpha)$.

Theorem 2 : Let $f \in G_p^+$. Then $f \in G_p^+(\lambda, \mu, v, \eta, c, \alpha)$ if and only if

$$\sum_{n=p+1}^{\infty} n[2n-\alpha+c]\Gamma_n^{\lambda,\mu,v,\eta} a_n \leq p[2p+\alpha-c], \tag{2.2}$$

where $0 \leq c \leq 1, 0 \leq \alpha < 1, p \in N, -\infty < \lambda < 1, \mu + v + \eta > \lambda, \mu > -\eta, v > -\eta$ and $\eta > 0$.

Proof : In view of Theorem 1, it is sufficient to prove the "only if" part. Let us assume that $f \in G_p^+(\lambda, \mu, v, \eta, c, \alpha)$. Then

$$\left| \frac{(-c)z - \frac{z^2(W_{0,Z}^{\lambda,\mu,v,\eta}f(z))''}{(W_{0,Z}^{\lambda,\mu,v,\eta}f(z))'}}{\frac{z^2(W_{0,Z}^{\lambda,\mu,v,\eta}f(z))''}{(W_{0,Z}^{\lambda,\mu,v,\eta}f(z))'} + (2-\alpha)z} \right| = \left| \frac{p(c-p-1)z^{-p} + \sum_{n=p+1}^{\infty} n(n+c-1)\Gamma_n^{\lambda,\mu,v,\eta} a_n z^n}{p(p+\alpha-1)z^{-p} + \sum_{n=p+1}^{\infty} n(n-\alpha+1)\Gamma_n^{\lambda,\mu,v,\eta} a_n z^n} \right| < 1.$$

Since $Re(z) \leq |z|$ for all z, it follows that

$$Re\left\{ \frac{p(c-p-1)z^{-p} + \sum_{n=p+1}^{\infty} n(n+c-1)\Gamma_n^{\lambda,\mu,v,\eta} a_n z^n}{p(p+\alpha-1)z^{-p} + \sum_{n=p+1}^{\infty} n(n-\alpha+1)\Gamma_n^{\lambda,\mu,v,\eta} a_n z^n} \right\} < 1. \tag{2.3}$$

Now letting $z \to 1^-$, through real values, so we can write (2.3) as

$$\sum_{n=p+1}^{\infty} n[2n-\alpha+c]\Gamma_n^{\lambda,\mu,v,\eta} a_n \leq p[2p+\alpha-c].$$

Finally, sharpness following if we take

$$f(z) = \frac{1}{z^p} - \frac{p[2p+\alpha-c]}{n[2n-\alpha+c]\Gamma_n^{\lambda,\mu,v,\eta}} z^n, \quad n \geq p+1. \tag{2.4}$$

Corollary 1 : If f defined by (1.2) is in the class $G_p^+(\lambda, \mu, v, \eta, c, \alpha)$, then

$$a_n \leq \frac{p[2p+\alpha-c]}{n[2n-\alpha+c]\Gamma_n^{\lambda,\mu,v,\eta}}, \quad n \geq p+1, \ p \in N. \tag{2.5}$$

The equality in (2.5) is attained for the function f given by (2.4).

3. Convolution Property

In the following theorem, we obtain the Convolution (or Hadamard product) of the functions f and g in the class $G_p^+(\lambda, \mu, v, \eta, c, \alpha)$.

Theorem 3 : Let f and $g \in G_p^+(\lambda, \mu, v, \eta, c, \alpha)$. Then $(f * g) \in G_p^+(\lambda, \mu, v, \eta, \omega, \alpha)$ for

$$f(z) = \frac{1}{z^p} - \sum_{n=p+1}^{\infty} a_n z^n, \quad g(z) = \frac{1}{z^p} - \sum_{n=p+1}^{\infty} b_n z^n,$$

and

$$(f * g) = \frac{1}{z^p} - \sum_{n=p+1}^{\infty} a_n b_n z^n,$$

where

$$\omega \leq \frac{n(\alpha+2p)[2n-\alpha+c]^2\Gamma_n^{\lambda,\mu,v,\eta} + p(\alpha-2n)[2p+\alpha-c]^2}{n[2n-\alpha+c]^2\Gamma_n^{\lambda,\mu,v,\eta} + p[2p+\alpha-c]^2}.$$

Proof : Since f and g are in the class $G_p^+(\lambda, \mu, v, \eta, c, \alpha)$ then

$$\sum_{n=p+1}^{\infty} \frac{n[2n-\alpha+c]\Gamma_n^{\lambda,\mu,v,\eta}}{p[2p+\alpha-c]} a_n \leq 1 \tag{3.1}$$

and

$$\sum_{n=p+1}^{\infty} \frac{n[2n-\alpha+c]\Gamma_n^{\lambda,\mu,v,\eta}}{p[2p+\alpha-c]} b_n \leq 1. \tag{3.2}$$

We have to find the largest ω such that

$$\sum_{n=p+1}^{\infty} \frac{n[2n-\alpha+\omega]\Gamma_n^{\lambda,\mu,v,\eta}}{p[2p+\alpha-\omega]} a_n b_n \leq 1. \tag{3.3}$$

By Cauchy Schwarz inequality, we get

$$\sum_{n=p+1}^{\infty} \frac{n[2n-\alpha+c]\Gamma_n^{\lambda,\mu,v,\eta}}{p[2p+\alpha-c]} \sqrt{a_n b_n} \leq 1. \tag{3.4}$$

We want only to show that

$$\frac{n[2n-\alpha+\omega]\Gamma_n^{\lambda,\mu,v,\eta}}{p[2p+\alpha-\omega]} a_n b_n \leq \frac{n[2n-\alpha+c]\Gamma_n^{\lambda,\mu,v,\eta}}{p[2p+\alpha-c]} \sqrt{a_n b_n}.$$

This equivalently to

$$\sqrt{a_n b_n} \leq \frac{[2n-\alpha+c][2p+\alpha-\omega]}{[2n-\alpha+\omega][2p+\alpha-c]}.$$

From (3.4), we get

$$\sqrt{a_n b_n} \leq \frac{p[2p+\alpha-c]}{n[2n-\alpha+c]\Gamma_n^{\lambda,\mu,v,\eta}}.$$

Thus it is enough to show that

$$\frac{p[2p+\alpha-c]}{n[2n-\alpha+c]\Gamma_n^{\lambda,\mu,v,\eta}} \leq \frac{[2n-\alpha+c][2p+\alpha-\omega]}{[2n-\alpha+\omega][2p+\alpha-c]}$$

which simplifies to

$$\omega \leq \frac{n(\alpha+2p)[2n-\alpha+c]^2\Gamma_n^{\lambda,\mu,v,\eta} + p(\alpha-2n)[2p+\alpha-c]^2}{n[2n-\alpha+c]^2\Gamma_n^{\lambda,\mu,v,\eta} + p[2p+\alpha-c]^2}.$$

4. Neighborhoods

Following the earlier works on neighborhoods of analytic functions by Goodman [2] and Ruscheweyh [5], we begin by introducing here the δ-neighborhood of a function $f \in G_p^+$ of the form (1.2) by means of the definition below:

$$N_\delta(f) = \left\{ g \in G_p^+ : g(z) = z^{-p} - \sum_{n=p+1}^{\infty} b_n z^n \text{ and } \sum_{n=p+1}^{\infty} n|a_n - b_n| \leq \delta, 0 \leq \delta < 1 \right\}. \tag{4.1}$$

Particularly for the identity function $e(z) = z^{-p}$, we have

$$N_\delta(e) = \left\{ g \in G_p^+ : g(z) = z^{-p} - \sum_{n=p+1}^{\infty} b_n z^n \text{ and } \sum_{n=p+1}^{\infty} n|b_n| \leq \delta, 0 \leq \delta < 1 \right\}. \tag{4.2}$$

Definition 4 : A function $f \in G_p^+$ is said to be in the class $G_p^+(\lambda, \mu, v, \eta, c, \alpha)$ if there exists function $g \in G_p^+(\lambda, \mu, v, \eta, c, \alpha)$ such that

$$\left| \frac{f(z)}{g(z)} - 1 \right| < 1 - \sigma, \quad (z \in U^*, 0 \leq \sigma < 1).$$

Theorem 4 : If $g \in G_p^+(\lambda, \mu, v, \eta, c, \alpha)$ and

$$\sigma = 1 - \frac{\delta[2(p+1) + \alpha - c]\Gamma_n^{\lambda,\mu,v,\eta}}{(p+1)[2(p+1) + \alpha - c]\Gamma_n^{\lambda,\mu,v,\eta} - p[2p - \alpha + c]}. \tag{4.3}$$

Then $N_\delta(g) \subset G_p^+(\lambda, \mu, v, \eta, c, \alpha)$.

Proof : Let $f \in N_\delta(g)$. Then we find from (4.1) that

$$\sum_{n=p+1}^{\infty} n|a_n - b_n| \leq \delta,$$

which implies the coefficient inequality

$$\sum_{n=p+1}^{\infty} |a_n - b_n| \leq \frac{\delta}{p+1}, \quad (n \geq p+1).$$

Since $g \in G_p^+(\lambda, \mu, v, \eta, c, \alpha)$, then by using Theorem 2

$$\sum_{n=p+1}^{\infty} b_n \leq \frac{p[2p - \alpha + c]}{(p+1)[2(p+1) + \alpha - c]\Gamma_n^{\lambda,\mu,v,\eta}}. \tag{4.4}$$

So that

$$\left| \frac{f(z)}{g(z)} - 1 \right| \leq \frac{\sum\limits_{n=p+1}^{\infty} |a_n - b_n|}{1 - \sum\limits_{n=p+1}^{\infty} b_n} \leq \frac{\delta[2(p+1) + \alpha - c]\Gamma_n^{\lambda,\mu,v,\eta}}{(p+1)[2(p+1) + \alpha - c][\Gamma_n^{\lambda,\mu,v,\eta}] - p[2p - \alpha + c]} = 1 - \sigma.$$

Hence by Definition 4 $f \in G_p^+(\lambda, \mu, v, \eta, c, \alpha)$ for σ given by (4.3). This completes the proof.

5. Convex Linear Combination

In next theorem, we obtain convex linear combination in the class $G_p^+(\lambda, \mu, v, \eta, c, \alpha)$.

Theorem 5 : The class $G_p^+(\lambda, \mu, v, \eta, c, \alpha)$ is closed under convex linear combination.

Proof : Let $f(z)$ and $g(z)$ be in the class $G_p^+(\lambda,\mu,v,\eta,c,\alpha)$, where

$$f(z)=z^{-p}-\sum_{n=p+1}^{\infty}a_nz^n \quad \text{and} \quad g(z)=z^{-p}-\sum_{n=p+1}^{\infty}b_nz^n.$$

We show the function

$$V(z)=\tau f(z)+(1-\tau)g(z), \quad (0\le\tau\le1),$$

is also in the class $G_p^+(\lambda,\mu,v,\eta,c,\alpha)$. Since for $0\le\tau\le1$,

$$V(z)=\frac{1}{z^p}-\sum_{n=p+1}^{\infty}[\tau a_n+(1-\tau)b_n]z^n.$$

Then by Theorem 2, we have

$$\begin{aligned}&\sum_{n=p+1}^{\infty}n[2n-\alpha+c]\Gamma_n^{\lambda,\mu,v,\eta}[\tau a_n+(1-\tau)b_n]\\&=\tau\sum_{n=p+1}^{\infty}n[2n-\alpha+c]\Gamma_n^{\lambda,\mu,v,\eta}a_n+(1-\tau)\sum_{n=p+1}^{\infty}n[2n-\alpha+c]\Gamma_n^{\lambda,\mu,v,\eta}b_n\\&\le\tau p[2p+\alpha-c]+(1-\tau)p[2p+\alpha-c]=p[2p+\alpha-c].\end{aligned}$$

Therefore, $V(z)\in G_p^+(\lambda,\mu,v,\eta,c,\alpha)$.

6. Radii of Starlikeness and Convexity

In the next theorems, we discuss the radii of starlikeness and convexity.

Theorem 6 : Let $f\in G_p^+(\lambda,\mu,v,\eta,c,\alpha)$, then f is p-valent meromorphic starlike of order $\varphi(0\le\varphi<p)$ in the disk $|z|<r=r_1$ where

$$r_1=\inf_{n\ge p+1}\left[\frac{n(p-\varphi)[2n-\alpha+c]\Gamma_n^{\lambda,\mu,v,\eta}}{p(n-\varphi+2p)[2p+\alpha-c]}\right]^{\frac{1}{n+p}}. \tag{6.1}$$

The result is sharp for the function f given by (2.2).

Proof : It is sufficient to show that

$$\left|\frac{zf'(z)}{f(z)}+p\right|\le p-\varphi, \tag{6.2}$$

but

$$\left|\frac{zf'(z)+pf(z)}{f(z)}\right|\le\frac{\sum\limits_{n=p+1}^{\infty}(n+p)a_n|z|^{n+p}}{1-\sum\limits_{n=p+1}^{\infty}a_n|z|^{n+p}}.$$

Thus, (6.2) will be satisfied if

$$\frac{\sum\limits_{n=p+1}^{\infty}(n+p)a_n|z|^{n+p}}{1-\sum\limits_{n=p+1}^{\infty}a_n|z|^{n+p}} \leq p-\varphi,$$

or if

$$\sum_{n=p+1}^{\infty}\frac{(n-\varphi+p)}{p-\varphi}a_n|z|^{n+p} \leq 1. \tag{6.3}$$

Since $f \in G_p^+(\lambda,\mu,v,\eta,c,\alpha)$, we have

$$\sum_{n=p+1}^{\infty}\frac{n[2n-\alpha+c]\Gamma_n^{\lambda,\mu,v,\eta}}{p[2p+\alpha-c]}a_n \leq 1.$$

Hence, (6.3) will true if

$$\frac{(n-\varphi+2p)}{p-\varphi}|z|^{n+p} \leq \frac{n[2n-\alpha+c]\Gamma_n^{\lambda,\mu,v,\eta}}{p[2p+\alpha-c]},$$

or equivalently

$$|z| \leq \left[\frac{n(p-\varphi)[2n-\alpha+c]\Gamma_n^{\lambda,\mu,v,\eta}}{p(n-\varphi+2p)[2p+\alpha-c]}\right]^{\frac{1}{n+p}}, \quad n \geq p+1$$

which follows the result.

Theorem 7 : Let $f \in G_p^+(\lambda,\mu,v,\eta,c,\alpha)$. Then f is p-valent meromorphic convex of order φ $(0 \leq \varphi < p)$ in the disk $|z| < r = r_2$, where

$$r_2 = \inf_{n\geq p+1}\left[\frac{(p-\varphi)[2n-\alpha+c]\Gamma_n^{\lambda,\mu,v,\eta}}{(n-\varphi+2p)[2p+\alpha-c]}\right]^{\frac{1}{n+p}}. \tag{6.4}$$

The result is sharp for the function f given by (2.2).

Proof : It is sufficient to show that

$$\left|\frac{zf''(z)}{f'(z)}+1+p\right| \leq p-\varphi, \tag{6.5}$$

but

$$\left|\frac{zf''(z)+(p+1)f'(z)}{f'(z)}\right| \leq \frac{\sum\limits_{n=p+1}^{\infty}n(n+p)a_n|z|^{n+p}}{p-\sum\limits_{n=p+1}^{\infty}a_n|z|^{n+p}}.$$

Thus (6.5) will be satisfied if

$$\frac{\sum\limits_{n=p+1}^{\infty} n(n+p)a_n|z|^{n+p}}{p-\sum\limits_{n=p+1}^{\infty} a_n|z|^{n+p}} \leq p-\varphi,$$

or if

$$\sum_{n=p+1}^{\infty} \frac{n(n-\varphi+p)}{p(p-\varphi)} a_n|z|^{n+p} \leq 1. \tag{6.6}$$

Since $f \in G_p^+(\lambda, \mu, v, \eta, c, \alpha)$, we have

$$\sum_{n=p+1}^{\infty} \frac{n[2n-\alpha+c]\Gamma_n^{\lambda,\mu,v,\eta}}{p[2p+\alpha-c]} a_n \leq 1.$$

Hence, (6.6) will true if

$$\frac{n(n-\varphi+2p)}{p(p-\varphi)}|z|^{n+p} \leq \frac{n[2n-\alpha+c]\Gamma_n^{\lambda,\mu,v,\eta}}{p[2p+\alpha-c]},$$

or equivalently

$$|z| \leq \left[\frac{(p-\varphi)[2n-\alpha+c]\Gamma_n^{\lambda,\mu,v,\eta}}{(n-\varphi+2p)[2p+\alpha-c]}\right]^{\frac{1}{n+p}}, \quad n \geq p+1$$

which follows the result.

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