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K-COLOURABLE S-VALUED GRAPHS

T. V. G. SHRIPRAKASH[1] AND M. CHANDRAMOULEESWARAN[2]
[1] Kurinji College of Engineering and Technology, Manapparai. Ibdia
[2] Saiva Bhanu Ksatriya College, Aruppukottai, India

Abstract

In [3], the authors introduced the notion of semiring valued graphs. In [1], the authors introduced the notion of Regularity on S-valued graphs. In [4], we have introduced the notion of K-Coloring on S-valued graphs. In this paper, we studied the upper bounds of K-Colorable S-valued graphs.

1. Introduction

The first known mention of coloring problems was in 1852., when August De Morgan wrote Sir William Rowan Hamilton about the problem: Whether four colors would be sufficient for all possible decompositions of the plane into regions. In the most common kind of graph coloring, colors are assigned to the vertices.

In any proper vertex coloring of G, the vertices that receive the same color are independent. The vertices that receive a particular color make up a color class. Thus, in

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any chromatic partition of $V(G)$, the parts of the partition constitute the color classes. This allows an equivalent way of defining the chromatic number.Although the chromatic number is one of the most studied parameters in graph theory, no formula exists for the chromatic number of an arbitrary graph. It is straight forward to establish the chromatic number of graphs in some of the most common graph families.

In [3], the authors introduced the notion of semiring valued graphs. In [1], the authors introduced the notion of Regularity on S-valued graphs. In [4], we have introduced the notion of K-Coloring on S-valued graphs. In this paper, we study the upper bounds of K-Colorable S-valued graphs.

2. Preliminaries

In this section, we recall some basic de

nitions that are needed for our work.

Definition 2.1 : A semiring $(S, +, \cdot)$ is an algebraic system with a non-empty set S together with two binary operations $+$ and $\circ$ such that

(1) $(S, +, 0)$ is a monoid.

(2) $(S, \cdot)$ is a semigroup.

(3) $\forall \;\; a, b, b, c \in S, a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdots c = a \cdots c + b \cdot c$.

(4) $0 \cdot x = x \cdot 0 = 0 \;\; \forall \;\; x \in S$.

Definition 2.2 : Let $(S, +, \cdot)$ be a semiring. A canonical Pre-order $\preceq$ in S is defined as follows : $\forall \; a, b \in S, a \preceq b$ if and only if, there exist an element $c \in S$ such that $a + c = b$.

Definition 2.3 [2] : A k-vertex colouring of a graph G is an assignment of k - colours to the vertices of G such that no two adjacent vertices receive the same colour.

Definition 2.4 [2] : A graph G that required k-different colours for its colouring and not less number of colours is called a k-chromatic graph and the number k is called the chromatic number of G, denoted by $\chi(G)$. That is, $\chi(G) = k$.

Definition 2.5 [3] : Let $G = (V, E)$ be a given graph with $V, E \neq \phi$. For any semiring $(S, +, \cdot)$, a semiring-valued graph (or a S valued graph), G^S is defined to be the graph

$G^S = (V, E, \sigma, \psi)$ where $\sigma : V \to S$ and $\psi : E \to S$ is defined to be

$$\psi(x, y) = \begin{cases} \min\{\sigma(x), \sigma(y)\}, & \text{if} \quad \sigma(x) \prec \sigma(y) \quad \text{or} \quad \sigma(y) \prec \sigma(x) \\ 0 & \text{if} \quad \text{otherwise} \end{cases}$$

for every unordered pair (x, y) of $E \subseteq VXV$. We call σ, a S vertex set and ψ, a S-edge set of G^S.

Definition 2.6 [4] : Consider the S-valued graph $G^S = (V, E, \sigma, \psi)$. A colouring of G^S is given by a function $f : VXV \to SXC$ such that for all $v \in V$ we have $f(v, v) = (\sigma(v); c(v))$, where $c(v) \in C$.

Definition 2.7 [4] : If $|C| = k$ then f is called a k -colouring of G^S.

Definition 2.8 [4] : Consider the S-valued graph G^S. A colouring f on G^S is said to be equi-weight (or vertex regular) proper colouring if for all $v \in V$, $\sigma(v)$ have equal value in S and $c(v) \in C$ differ for adjacent vertices.

Definition 2.9 [4] : A colouring $f : VXV \to SXC$ is said to be total proper colouring, if $v \in V$ both $\sigma(v) \in S$ and $c(v) \in C$ have different value for adjacent vertices.

Definition 2.10 [4] : A S-valued graph G^S is said to be k-colourable, if it has a proper vertex regular or total proper colouring such that $|C| = k$.

Definition 2.11 [4] : Let G^S be k-colourable graph. The vertex chromatic number of G^S, denoted by $S(G^S)$ is defined to be $S(G^S) = (\min_{v \in V}(\sigma(v)), \min |C|)$.

Definition 2.12 [1] : Consider two S-valued graphs $G_1^S = (V_1, E_1, \sigma_1, \psi_1)$ and $G_2^S = (V_2, E_2, \sigma_2, \psi_2)$ such that $V_1 \cap V_2 = \phi$. The union of these two S-valued graphs is $G^S = G_1^S \cup G_2^S = (V, E, \sigma, \psi)$ where $V = V_1 \cup V_2, E = E_1 \cup E_2$ and the vertex and edge functions are de

ned as follows:

$$\sigma(v) = \begin{cases} \sigma_1(v) & \text{if} \quad v \in V_1 \\ \sigma_2(v) & \text{if} \quad v \in V_2 \end{cases}$$

and

$$\psi(v_1, v_2) = \begin{cases} \psi_1(v_i, v_j), & \text{if} \quad (v_i, v_j) \in E_1 \\ \psi_2(v_i, v_j) & \text{if} \quad (v_i, v_j) \in E_2 \end{cases}$$

3. K-Colourable S-valued Graphs

In this section, we derive the chromatic number for some special graphs and derive conditions for a S-valued graph G^s to be l-colorable.

Theorem 3.1 : Any complete graph K_n^S is l -colorable iff $l = n$.

Proof : Let $K_n^S = (V, E, \sigma, \psi)$ be l colorable. This implies $|V| = n$, $|E| = n(n-1)/2$. Let $C = \{C_1, C_2, \cdots, C_l\}$.

Case A : If K_n^S is vertex regular proper coloring, the $\sigma(v) = a$, for all $v \in K_n^S$.

Claim : $l = n$.

Suppose $l \neq n$, then $l > n$ or $l < n$. Since n is the total number of vertices $l > n$ is not possible. Hence assume that $l < n$ needs n distinct colors. Since $l < n$, then there exist atleast one vertex which has same color contradicting the fact K_n^S is l-colorable. Hence $l = n$.

Case B : If K_n^S is total proper colorable with $\sigma(v_i) \neq \sigma(v_j), \forall\ (v_i, v_j) \in E$ and $ij = 1, 2, \cdots, n$. Then $C(v_i) \neq C(v_j), i, j = 1, 2, \cdots, l$. Thus in both cases, $l = n$.

Conversely, if $l = n$ and $\sigma(v) = a$ for every $v \in V$ then all the n vertices are colored by $n = l$ distinct colors. Therefore by definition K_n^S is l-colorable.

Similarly if $l = n$ and $\sigma(v_i) \neq \sigma(v_j), \forall (v_i, v_j) \in E, i, j = 1, 2, \cdots, n$. Then by definition, K_n^S is l-colorable. Hence K_n^S is l-colorable.

Corollary 3.2 : The vertex chromatic number of K_n^S satisfies$\chi S(K_n^S) = (a, n)$ where $a = \min_{v \in V} \sigma(v)$.

Theorem 3.3 : Any (a, k) regular S valued graph G^S is l-colorable iff $l = k$ and G^S is complete.

Proof : Let $G^S = (V, E, \sigma, \psi)$ be (a, k) regular S valued graph, $\sigma(v) = a \forall\ v \in V$, for some $a \in S$ and $deg(v) = k\ \forall\ v \in V$. Let G^S be coloured by l distinct color $C_i = \{C_1, C_2, \cdots, C_l\}$ so that $|C_i| = l$ where the colouring is proper equi-weight vertex regular or total proper coloring.

Claim : $l = k$.

Since $l < k$ is not possible and the underlying graph G of G^S is k-regular, it requires atleast l colours to colour the vertices in both the cases of equi-weight or total proper colouring.

If the graph G^S is not complete, $l < k$, which is not possible. On the other hand, if the graph G^S is complete it requires only $l = k$ colours.

Corollary 3.4 : For any (a, k) regular S valued graph G^S, $\chi S(G^S) = (a, k)$ for some $a \in S$.

Any complete or complete bipartite S-valued graph on k vertices are (a, k) regular for some $a \in S$. Thus any complete or complete bipartite S-valued graph on k vertices are l-colourable if and only if $l = k$. This proves the following main theorem.

Theorem 3.5 : For any complete or complete bipartite S-valued graph on k vertices, $\chi S(G^S) = (a, k)$ where $a = \min_{v \in V} \sigma(v)$.

Theorem 3.6 : For two S-valued graphs $G_1^S = (V_1, E_1, \sigma_1, \psi_1)$ and $G_2^S = (V_2, E_2, \sigma_2, \psi_2)$ with $V_1 \cap V_2 = \phi$, $\chi S(G_1^S \cup G_2^S) \leq \chi S(G_1^S) + \chi S(G_2^S)$.

Proof : Consider two S-valued graphs $G_1^S = (V_1, E_1, \sigma_1, \psi_1)$ and $G_2^S = (V_2, E_2, \sigma_2, \psi_2)$. Let $G^S = G_1^S \cup G_2^S) = (V, E, \sigma, \psi)$ where $V = V_1 \cup V_2$ $E = E_1 \cup E_2$. The vertex and edge functions are de

ned as follows:

$$\sigma(v) = \begin{cases} \sigma_1(v), & \text{if} \quad v \in V_1 \\ \\ \sigma_2(v), & \text{if} \quad v \in V_2 \end{cases}$$

and

$$\psi(v_1, v_2) = \begin{cases} \psi_1(v_i, v_j), & \text{if} \quad (v_i, v_j) \in E_1 \\ \\ \psi_2(v_i, v_j), & \text{if} \quad (v_i, v_j) \in E_2 \end{cases}$$

Let $|V_1| = n_1$ and $|V_2| = n_2$. Similarly, let $|E_1| = m_1$ and $|E_2| = m_2$. Let $\min_{v \in V_1} \sigma(v) = a$ for some $a \in S$ and $\min_{v \in V_2} \sigma(v) = b$ for some $b \in S$. Let $\chi S(G_1^S) = (a, l)$, $C_1 = \{c_1, c_, \cdots, c_l\}$. Let $\chi S(G_2^S) = (b, k)$, $C_1 = \{c_1, c_2, \cdots, c_k\}$. Now,

$$\min_{v \in V} \sigma(v) = \min\{a, b\} = \begin{cases} a, & \text{if} \quad a \prec b \\ \\ b, & \text{if} \quad b \prec a \end{cases}$$

If $l \leq k$ and $a \preceq b$, we have

$$\chi S(G_1^S \cup G_1^S) = (a, k) \tag{1}$$

If $k \leq l$ and $b \preceq a$, we have

$$\chi S(G_1^S \cup G_2^S) = (b, l) \tag{2}$$

and

$$\chi S(G_1^S) + \chi S(G_2^S) = (a, l) + b, k) = (a + b, k + l).$$

There are three possibilities, namely $a+b=a, a+b=b, a+b=c$ for some $c \in S$. Hence,

$$\chi S(G_1^S) + \chi S(G_2^S) = \begin{cases} (a, l+k) & \text{if} \quad a+b=a \\ (b, l+k) & \text{if} \quad a+b=b \\ (c, l+k) & \text{if} \quad a+b=c \end{cases} \tag{3}$$

From (1), (2) and (3) the above, we conclude that $\chi S(G_1^S \cup G_2^S) \le \chi S(G_1^S) + \chi S(G_2^S)$.

Corollary 3.7 : For any S-valued graph G^S, $\chi S(G^S \cup \overline{G^S}) \le \chi S(G^S) + \chi S(\overline{G^S})$.

Proof : By taking $G_1^S = G^S$ and $G_2^S = \overline{G^S}$ in the above theorem we obtain the result.

Theorem 3.8 : If G_1^S and G_2^S are l and k-colorable S-regular graph with S-vertex set $\{a\}$ and $\{b\}$ respectively and $l \le k$ then

$$\chi S(G_1^S) \cup \chi S(G_2^S) = \begin{cases} (a, k) & \text{if} \quad a \le b \\ (b, k) & \text{if} \quad b \le a \end{cases}$$

Proof : Let G_1^S be l colorable S regular graph with S vertex set $\{a\}$ and G_2^S be k colorable S regular graph with S vertex set $\{b\}$. Then $\sigma_1(v) = a \;\; \forall v \in V_1$. Therefore, $\min \sigma_1(v) = a, v \in V_1$. $\sigma(v) = b \;\forall\; v \in V_2$. Therefore, $\min \sigma_2(v) = b \;\forall\; v \in V_2$. Now consider $G_1^S \cup G_2^S = (V, E, \sigma, \psi)$. If $a \le b$, then $\min \sigma_1(v) = a, v \in V$. Since $V_1 \cap V_2 = \phi$, $V_1 \cup V_2 = V$ needs l and k-colors, therefore $\chi(G_1^S \cup G_2^S) = (a, l+k)$ if $b \le a$, then $\chi(G_1^S \cup G_2^S) = (b, l+k)$.

Theorem 3.9 :

$$W_s(G_1^S \cup G_2^S) = \begin{cases} W_s(G_1^S), & \text{if} \quad |W_s(G_1^S)| < |W_s(G_2^S)| \\ W_s(G_2^S), & \text{if} \quad |W_s(G_2^S)| < |W_s(G_1^S)| \end{cases}$$

where $|W_s(G_i^S)|$ is the cardinality of the mini clique of the S-valued graph.

Proof : Let $G_1^S = (V_1, E_1, \sigma_1, \psi_1)$ both $W_s(G_1^S) = (a, n)$ where $a =$ minimum $\sigma(v)$ v-clique minimum, n-number of vertices in minimum clique. Let $G_2^S = (V_2, E_2, \sigma_2, \psi_2)$ with $W_s(G_2^S) = (b, m)$. Let $G_1^S \cup G_2^S) = (V, E, \sigma, \psi)$. Since the minimum clique in G_1^S and G_2^S are cliques in $(G_1^S \cup G_2^S)$ any which, for which number of vertices is less became a minimum clique in $(G_1^S \cup G_2^S)$. Therefore, if $n \le m$, then $W_s(G_1^S \cup G_2^S) = (a, n) = W_s(G_1^S)$. If $m \le n$, then $W_s(G_1^S \cup G_2^S) = (b, m) = W_s(G_2^S)$.

$$W_s(G_1^S \cup G_2^S) = \begin{cases} W_s(G_1^S), & \text{if} \quad G_1^S \; \textit{contain the minimal clique} \;\; G_2^S \\ W_s(G_2^S), & \text{if} \quad G_2^S \; \textit{contain the minimal clique} \;\; G_1^S. \end{cases}$$

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