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RAPIDLY CONVERGENT SERIES FROM GREGORY SERIES

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Abstract

In this paper we shall extract a rapidly convergent series from a given series, by applying a correction function to the series. Hence the rate of convergence of the new series can be increased.

1. Introduction

The approximation of an alternating series can be done using remainder term of the series. The absolute value of the remainder term is the correction function. The introduction of correction function certainly improves the sum of the series and gives a better approximation for it. We can also deduce some rapidly convergent series using correction function and the corresponding error functions. The new series so extracted increases the rate of convergence of the series.

Key Words : *Remainder term, Alternating series, Madhava series, Rate of convergence, Rapidly convergent series.*

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2. Preliminary Definitions

Definition 1 : An alternating series is an infinite series of the form $\sum_{n=1}^{\infty}(-)^{n-1}a_n$ where the terms $a_n > 0$.

Definition 2 : The remainder term for an alternating series $\sum_{n=1}^{\infty}(-)^{n-1}a_n$ is the sum of the series after n terms. It is denoted by R_n.

i.e. $R_n = \sum_{k=n+1}^{\infty}(-)^{k-1}a_k$.

If S denote the sum of the series and Sn denote the sequence of partial sums of the series, then $R_n = S - S_n$.

Definition 3 : The correction function to an alternating series $\sum_{n=1}^{\infty}(-)^{n-1}a_n$ is denoted by G_n and it is defined as the absolute value of the remainder term.

If R_n denotes the remainder term of the series, then $R_n = (-1)^n G_n$ where G_n is the correction function.

i.e. $G_n = \sum_{k=1}^{\infty}(-1)^{k-1}a_{n+k}$.

If $\{a_n\}$ is monotonically decreasing, then $G_n = |S - S_n|$.

Definition 4 : An alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}C_n$ is said to be **rapidly convergent** than the series $\sum_{n=1}^{\infty}(-1)^{n-1}d_n$ if the ratio $\frac{c_n}{d_n} \to 0$ as $n \to \infty$.

3. Correction Function for Gregory Series

The Gregory series is convergent and converges to $\frac{\pi}{4}$. Thus

$$\frac{\pi}{4} = 1 \cdot \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + (-1)^{n-1}\frac{1}{2n-1} + \cdots .$$

Proposition 1 : The correction function for Gregory series is $G_n = \frac{1}{4n}$ and the corresponding error function is $|E_n| = \frac{4}{(32n^3+48n^2+16n)}$.

Proof : We have Gregory series is convergent and converges to $\frac{\pi}{4}$.

If G_n denotes the correction function after n terms of the series, then it follows that $G_n + G_{n+1} = \frac{1}{2n+1}$.

The error function is $E_n = G_n + G_{n+1} - \frac{1}{2n+1}$.

We may choose G_n in such a way that $|E_n|$ is a minimum.

Now for a fixed n and for $r \in R$.

Let $G_n(r) = \frac{1}{a4n+2-r}$.

Then the error function is $E_n(r) = G_n(r) + G_{n+1}(r) = \frac{1}{2n+1}$ is a rational function of r.

i.e. $E_n(r) = \frac{N_n(r)}{D_n(r)}$ where

$$\begin{aligned} D_n(r) &= (4n+2-r)(4n+6-r)(2n+1) \\ &\approx 32n^2 \quad \text{which is a maximum for large values of } n \end{aligned}$$

$$N_n(r) = \begin{cases} (4n+2-r)(r-2)+2r, & r \neq 2 \\ 4, & r = 2 \end{cases}$$

Hence $|N_n(r)|$ is minimum for $r = 2$.

So $|E_n(r)|$ is minimum for $r = 2$.

Hence the correction function for Gregory series is $G_n = 14$.

The absolute value of the corresponding error function is

$$|E_n| = \frac{4}{(32n^3 + 48n^2 + 16n)}.$$

Hence the proof.

3. Rapidly Convergent Series from Gregory Series

We have $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots + (-1)^{n-1}\frac{1}{2n-1} + \cdots$.

Let $\partial_n = 1 - \frac{1}{3} + \frac{1}{5} - \cdots + (-1)^{n-1}\frac{1}{2n-1} + (-1)^n G_n$.

Let the error $\epsilon_n = \partial_{n+1} - \partial_n$.

$$\partial_{n+1} = \partial_n + \epsilon_n.$$

Put $n = 1, 2, 3, \cdots, n-1$ in succession in the place of n and add to get

$$\begin{aligned} \partial_n &= \partial_1 + \epsilon_1 + \epsilon_2 + \epsilon_3 + \cdots \epsilon_{n-1} \\ &= 1 - G_1 + \epsilon_1 + \epsilon_2 + \epsilon_3 + \cdots + \epsilon_{n-1}, \quad \text{since} \quad \partial_1 = 1 - G_1. \end{aligned}$$

$$\lim_{n\to\infty} \partial_n = 1 - G_1 + \epsilon_1 + \epsilon_2 + \epsilon_3 + \cdots .$$

But $\lim_{n\to\infty} \partial_n = \frac{\pi}{4}$.

Hence $\frac{\pi}{4} = 1 - G_1 + \epsilon_1 + \epsilon_2 + \epsilon_3 + \cdots + \epsilon_n$.

Case 1 : For $G_n = \frac{1}{4n}$.

We have $\epsilon_n = (-1)^{n+1} \quad E_n = \frac{(-1)^{n+1}}{p^3-p}$ where $p = 2n+1$.

The new deduced series is

$$\begin{aligned} \frac{\pi}{4} &= 1 - G_1 + \epsilon_1 + \epsilon_2 + \epsilon_3 + \cdots \\ &= \frac{3}{4} + \frac{1}{3^3-3} - \frac{5^3-5}{+}\frac{1}{7^3-7} - \cdots \end{aligned}$$

If c_n denotes the n-th term of the Gregory series and if dn denotes the n-th term of the new deduced series, then $c_n = (-1)^{n-1}\frac{1}{2n-1}, \quad d_n = \epsilon_n$.
It is clear that $\frac{d_n}{c_n} = \longrightarrow 0$ as $n \to \infty$.
Hence the deduced series is rapidly convergent than the original series.
Hence the rate of convergence of new series is increased.
Thus the deduced series is a rapidly convergent series.

4. Application

1. We have $\pi = 3.1415926536$, using a calculator.
Using G_n, the approximation is given below.

Number of terms (n)	S_n	$S_n + (-1)^n G_n$
10	**3**.0418396189	**3.141**839619
100	**3.1**315929035	**3.141592**904
1000	**3.14**05926538	**3.141592654**
10000	**3.141**4926536	**3.141592654**

2. We have $\frac{\pi}{4} = 0.7853981634$.
If S_n denotes the sequence of partial sum of the original series and if S'_n denotes the sequence of partial sums of the deduced series, then the rapidity of convergence is shown in the following table.

Number of terms (n)	S_n	S'_n
10	**0.7**604599047	**0.785**6643357

With a pre-designed accuracy, the number of terms required for the original series and the deduced series is given below.

Accuracy	Number of terms required from the original series	Number of terms required from the deduced series
3.1405926538	1000	10

5. Conclusion

The correction functions and the corresponding error functions play a vital role in series approximation. We can deduce new series which are rapidly convergent than the original series.

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