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ON LEFT α-CENTRALIZER OF PRIME RINGS WITH INVOLUTIONS

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Abstract

Let R be a ring with involution $*$. An additive mapping $T : R \to R$ is called a left(respectively right) α-centralizer if $T(xy) = T(x)\alpha(y)$ (respectively $T(xy) = \alpha(x)T(y)$), for all $x, y \in R$. The purpose of this paper is to examine the commutativity of prime rings with involution satisfying certain identities involving left α-centralizers.

1. Introduction

Throughout this article, R will represent an associative ring with centre $Z(R)$. A ring R is said to be 2-torsion free if $2a = 0$ (where $a \in R$) implies $a = 0$. A ring R is called a prime ring if $aRb = (0)$ (where $a, b \in R$) implies $a = 0$ or $b = 0$. We write $[x, y] = xy - yx$. An additive map $x \to x^*$ of R into itself is called an involution if (i) $(xy)^* = y^*x^*$ and (ii) $(x^*)^* = x$ holds for all $x, y \in R$. A ring equipped with an involution $*$ is known as ring with involution or $*$-ring. An element x in a ring with involution $*$ is said to be hermitian if $x^* = x$ and skew-hermitian if $x^* = -x$. The sets

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of all hermitian and skewhermitian elements of R will be denoted by $H(R)$ and $S(R)$, respectively. If R is 2-torsion free then every $x \in R$ can be uniquely represented in the form $2x = h + k$, where $h \in H(R)$ and $k \in S(R)$. Note that x is normal, i.e., $xx^* = x^*x$, if and only if h and k commute. If all elements in R are normal, then R is called a normal ring. An example of the normal ring is the ring of quaternions. A description of such rings can be found in [7].

Following [16], an additive mapping $T : R \to R$ is said to be a left (respectively right) centralizer (multiplier) of R if $T(xy) = T(x)y$ (respectively $T(xy) = xT(y)$), for all $x, y \in R$. An additive mapping T is called centralizer in case T is a left and a right centralizer of R. Considerable work has been done on left (respectively right) centralizers in prime and semiprime rings during the last few decades (see for example [1-3, 6, 9, 10, 13-15 and 16]) where further references can be found.

Recently, E. Albas [1] introduced the notion of α-centralizers of R, i.e., an additive mapping $T : R \to R$ is called a left (resp. right) α-centralizer of R if $T(xy) = T(x)\alpha(y)$ (resp. $T(xy) = \alpha(x)T(y)$) holds for all $x, y \in R$, where α is an endomorphism of R. If T is both left as well as right α-centralizer, then we call T an α-centralizer. It is clear that for an additive mapping $T : R \to R$ associated with a homomorphism $\alpha : R \to R$, if $L_a(x) = a\alpha(x)$ and $R_a(x) = \alpha(x)a$ for a fixed element $a \in R$ and for all $x \in R$, then L_a is a left α-centralizer and R_a is a right α-centralizer. Clearly every multiplier is a special case of an α-multiplier with $\alpha = id_R$, the identity map on R. Recently Ali and Dar [2] proved that if a prime ring with involution admits a left centralizer $T : R \to R$ such that $T([x, x^*]) \pm [x, x^*] = 0$, for all $x \in R$, then R is commutative. Moreover, in [3] some related results involving left centralizers have also been discussed. In [9] Oukhtite established similar problems for certain situations involving left centralizers acting on Lie ideals.

In this paper, we shall consider similar problems when the ring R is equipped with a fixed involution $*$ even in more general setting by replacing centralizer with α centralizer. More precisely, we prove that if a prime ring with involution such that char $(R) \neq 2$ admits a left α-centralizer $T : R \to R$ satisfying any one of the following conditions: (i) $T(xx^*) \pm \alpha(xx^*) = 0$, (ii) $\alpha(x)T(x^*) \pm T(x)\alpha(x^*) = 0$ and (iii) $T(x)T(x^*) \pm \alpha(xx^*) = 0$, for all $x \in R$, then either T is a centralizer or R is normal.

Main Results

Theorem 2.1 : Let R be a prime ring with involution $*$ and α be an automorphism. If T is a left α-centralizer of R such that $T(xx^*) \pm \alpha(xx^*) = 0$, for all $x \in R$, then either T is a centralizer or R is normal.

Proof : First we consider the case $T(xx^*) - \alpha(xx^*) = 0$ for all $x \in R$. This can be further written as

$$T(x)\alpha(x^*) - \alpha(xx^*) = 0, \quad \text{for all} \;\; x \in R. \tag{2.1}$$

Linearizing the above relation, we get

$$T(x)\alpha(y^*) + T(y)\alpha(x^*) - \alpha(xy^*) - \alpha(yx^*) = 0, \quad \text{for all} \;\; x, y \in R. \tag{2.2}$$

Replacing y by yx in (2.2), we obtain

$$T(x)\alpha(x^*y^*) + T(y)\alpha(xx^*) - \alpha(xx^*y^*) - \alpha(yxx^*) = 0.$$

Application of (2.1) yields that

$$T(y)\alpha(xx^*) - \alpha(yxx^*) = 0, \quad \text{for all} \;\; x, y \in R. \tag{2.3}$$

Substituting zy for y in (2.3), we have

$$T(z)\alpha(yxx^*) - \alpha(zyxx^*) = 0, \quad \text{for all} \;\; x, y, z \in R. \tag{2.4}$$

Left multiplication to (2.3) by $\alpha(z)$ yields that

$$\alpha(z)T(y)\alpha(xx^*) - \alpha(zyxx^*) = 0, \quad \text{for all} \;\; x, y, z \in R. \tag{2.5}$$

Subtracting (2.4) from (2.5), we obtain

$$\alpha(z)T(y)\alpha(xx^*) - T(z)\alpha(yxx^*) = 0, \quad \text{for all} \;\; x, y, z \in R. \tag{2.6}$$

Substituting yr for y in (2.6) to get $\alpha(z)T(y)\alpha(rxx^*) - T(z)\alpha(yrxx^*) = 0$, for all $x, y, z, r \in R$ which can be further written as

$$(\alpha(z)T(y) - T(z)\alpha(y))\alpha(rxx^*) = 0, \quad \text{for all} \;\; x, y, z, r \in R. \tag{2.7}$$

Replacing x by x^* in (2.7), we find that

$$(\alpha(z)T(y) - T(z)\alpha(y))\alpha(rx^*x) = 0, \quad \text{for all} \;\; x, y, z, r \in R. \tag{2.8}$$

Subtracting (2.8) from (2.7), we obtain

$$(\alpha(z)T(y) - T(z)\alpha(y))\alpha(r[x, x^*] = 0, \quad \text{for all} \quad x, y, z, r \in R.$$

This implies that $(\alpha(z)T(y) - T(z)\alpha(y))R\alpha([x, x^*] = (0)$, for all $x, y, z \in R$. Thus by the primeness of R, we have either $\alpha(z)T(y) - T(z)\alpha(y) = 0$, for all $y, z \in R$ or $\alpha([x; x^*] = 0$, for all $x \in R$. Now if $\alpha(z)T(y) = T(z)\alpha(y) = 0$, for all $y, z \in R$. That is, $\alpha(z)T(y) = T(z)\alpha(y)$. Then T is also a right α-centralizer of R and hence a α-centralizer of R. On the other hand, if $\alpha([x; x^*]) = 0$, for all $x \in R$, that is $[x, x^*] = 0$, then R is normal.

By the same arguments, we obtain the same conclusion in case $T(xx^*) + \alpha(xx^*) = 0$, for all $x \in R$. This proves the theorem.

By similar arguments as above with necessary variation, we can prove the following theorem.

Theorem 2.2 : Let R be a prime ring with involution $*$ and α be an automorphism. If T is a left α-centralizer of R such that $T(x^*x) \pm \alpha(x^*x) = 0$, for all $x \in R$, then either T is a centralizer or R is normal.

Theorem 2.3 : Let R be a prime ring with involution $*$ and α is an automorphism. If T is a left α-centralizer of R such that $\alpha(x)T(x^*) \pm T(x)\alpha(x^*) = 0$, for all $x \in R$, then either T is a centralizer or R is normal.

Proof : First we consider the case

$$\alpha(x)T(x^*) - T(x)\alpha(^*) = 0, \quad \text{for all} \quad x \in R. \tag{2.9}$$

Linearizing the above relation, we get

$$\alpha(x)T(y^*) - T(x)\alpha(y^*) + \alpha(y)T(x^*) - T(y)\alpha(x^*) = 0, \quad \text{for all} \quad x, y \in R. \tag{2.10}$$

Replacing y by yx in (2.10) and using (2.9), we obtain

$$T(y)\alpha(xx^*) - \alpha(yx)T(x^*) = 0, \quad \text{for all} \quad x, y \in R. \tag{2.11}$$

Substituting zy for y in (2.11), we have

$$T(z)\alpha(yxx^*) - \alpha(zyx)T(x^*) = 0, \quad \text{for all} \quad x, y, z \in R. \tag{2.12}$$

Left multiplying (2.11) by $\alpha(z)$ yields that

$$\alpha(z)T(y)\lambda(xx^*) - \alpha(zyx)T(x^*) = 0, \quad \text{for all} \quad x, y, z \in R. \tag{(2.13}$$

Subtracting (2.12) from (2.13), we obtain

$$T(z)\alpha(yxx^*) - \alpha(z)T(y)\alpha(xx^*) = 0, \quad \text{for all} \quad x, y, z \in R. \tag{2.14}$$

Substituting yr for y in (2.14) we get

$$(\alpha(z)T(y) - T(z)\alpha(y))\alpha(rxx^*) = 0, \quad \text{for all} \quad x, y, z \in R. \tag{2.15}$$

The above equation is same as (2.8) and henceforward using the same approach as we have used in the last paragraph of the proof of Theorem 2.1, we get the required result. By the same arguments, we obtain the same conclusion in case $\alpha(x)T(x^*)+T(x)\alpha(x^*) = 0$, for all $x \in R$. This proves the theorem.

Theorem 2.4 : Let R be a prime ring with involution $*$ and α is an automorphism. If T is a left α-centralizer of R such that $T(x)T(x^*) \pm \alpha(xx^*) = 0$, for all $x \in R$, then either T is a α-centralizer or R is normal.

Proof : First we consider the situation

$$T(x)T(x^*) - \alpha(xx^*) = 0, \quad \text{for all} \quad x \in R. \tag{2.16}$$

Replacing x by $x + y$, we get

$$T(x)T(y^*) - \alpha(xy^*) + T(y)T(x^*) - \alpha(yx^*) = 0, \quad \text{for all} \quad x, y \in R. \tag{2.17}$$

Substituting yx for y in (2.17), we obtain

$$T(x)T(x^*)\alpha(y^* - \alpha(xx^*y^*) + T(y)\alpha(x)T(x^*) - \alpha(yxx^*) = 0, \quad \text{or all} \quad x, y \in R.$$

In view of (2.16), the above expression reduces to

$$T(y)\alpha(x)T(x^*) - \alpha(yxx^*) = 0, \quad \text{for all} \quad x, y \in R. \tag{2.18}$$

Replace y by zy in (2.18) to get

$$T(z)\alpha(yx)T(x^*) - \alpha(zyxx^*) = 0, \quad \text{for all} \quad x, y, z \in R. \tag{2.19}$$

Left multiplying (2.18) by $\alpha(z)$, we get

$$\alpha(z)T(y)\alpha(x)T(x^*) - \alpha(zyxx^*) = 0, \quad \text{for all} \;\; x, y, z \in R. \tag{2.20}$$

Subtracting (2.20) from (2.19), we obtain

$$\alpha(z)T(y)\alpha(x)T(x^*) - T(z)\alpha(yx)T(x^*) = 0, \quad \text{for all} \;\; x, y, z \in R. \tag{2.21}$$

Substituting yr for y in (2.21), we find that

$$\alpha(z)T(y)\alpha(rx)T(x^*) - T(z)\alpha(yrx)T(x^*) = 0, \quad \text{for all} \;\; x, y, z, r \in R.$$

This implies that

$$(\alpha(z)T(y) - T(z)\alpha(y))\alpha(rx)T(x^*) = 0, \quad \text{for all} \;\; x, y, z, r \in R. \tag{2.22}$$

That is, $(\alpha(z)T(y) - T(z)\alpha(y))R\alpha(x)T(x^*) = (0)$, for all $x, y, z \in R$. Thus by the primeness of R, we find that either $\alpha(z)T(y) - T(z)\alpha(y) = 0$, for all $y, z \in R$ or $\alpha(x)T(x^*) = 0$, for all $x \in R$. If $\alpha(z)T(y) - T(z)\alpha(y) = 0$ i.e., $T(z)\alpha(y) = \alpha(z)T(y)$, for all $y, z \in R$, then T is also a right α-centralizer and hence a α-centralizer on R. On the other hand, suppose $\alpha(x)T(x^*) = 0$, for all $x \in R$. This gives $T(y)\alpha(x)T(x^*) = 0$, for all $x, y \in R$. Hence (2.16) reduces to $\alpha(yxx^*) = 0$, for all $x, y \in R$. Replacing y by yr in the above relation, we obtain

$$\alpha(yrxx^*) = 0, \quad \text{for all} \;\; x, y, r \in R. \tag{2.23}$$

Replacing x by x^* in (2.23), we have

$$\alpha(yrx^*x) = 0, \quad \text{for all} \;\; x, y, r \in R. \tag{2.24}$$

Subtracting (2.24) from (2.23), we obtain $\alpha(yr[x, x^*]) = 0$, for all $x, y, r \in R$. This implies that $\alpha([x, x^*])R\alpha([x, x^*]) = (0)$, for all $x \in R$. Since R is prime and α is automorphism, the last expression forces that $[x, x^*] = 0$, for all $x \in R$.

Similar conclusion holds for the case $T(x)T(x^*) + \alpha(xx^*) = 0$, for all $x \in R$. This finishes the second case, and so the theorem is proved.

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References

[1] Albash E., On τ-centralizers of semiprime rings, Siberian Math. J., 48(2) (2007), 191-196.

[2] Ali S. and Dar N. A., On centralizers of prime rings with involution, Bull. Iranian Math. Soc., 41(6) (2015), 1465-1475.

[3] Ashraf M. and Ali S., On left multipliers and the commutativity of prime rings, Demon-stratio Math., 41(4) (2008), 764-771.

[4] Ali A. and Yasen M., A note on automorphisms of prime and semiprime rings, J. Math. Kyoto Univ., 45(2) (2005), 243-246.

[5] Divinsky N., On commuting automorphisms of rings, Trans. Roy. Soc. Canada. Sect. III. 49(3) (1955), 19-22.

[6] Hentzel I. R. and El-Sayiad T., Left centralizers of rings that are not semiprime, Rocky Mountain J. Math., 41(5) (2011), 1471-1482.

[7] Herstein I. N., Rings with Involution, Chicago Lectures in Mathematics, The University of Chicago Press, Chicago, (1976).

[8] Luh J., A note on commuting automorphisms of rings, Amer. Math. Monthly, 77 (1970), 61-62.

[9] Oukhtite L., Left multipliers and Lie ideals in rings with involution, Int. J. Open Probl. Comput. Sci. Math., 3(3) (2010), 267-277.

[10] Oukhtite L., Left multipliers and Jordan ideals in rings with involution, Afr. Diaspora J. Math., 11(1) (2011), 24-28.

[11] Posner E. C., Derivations in prime rings, Proc. Amer. Math. Soc., 8 (1957), 1093-1100.

[12] Smiley M. E., Remarks on commuting automorphisms, Amer. Math. Monthly, 63 (1956), 466-470.

[13] Vukman J., An identity related to centralizer in semiprime rings, Comment. Math. Univ. Carolin, 40(3) (1999), 447-456.

[14] Vukman J., Centralizers on semiprime rings, Comment. Math. Univ. Carolinae, 42(2) (2001), 237-245.

[15] Vukman J. and Irena K. U., On centralizers of semiprime rings with involution, Studia Scientiarum Math. Hungarica, 43(1) (2006), 61-67.

[16] Zalar B., On centralizer of semiprime rings, Comment. Math. Univ. Carolinae, 32(4) (1991), 609-614.