International J. of Math. Sci. & Engg. Appls. (IJMSEA)
ISSN 0973-9424, Vol. 11 No. I (April, 2017), pp. 11-17

# PROXIMATE GOL'DBERG ORDER OF AN ENTIRE FUNCTION IN SEVERAL COMPLEX VARIABLES

**BIBHAS CHANDRA MONDAL[1], MONALISA MIDDYA [2] AND CHETNA SINGH[3]**

[1] Department of Mathematics, Surendranath College,
24/2, M.G. Road, Kolkata 700009, India
[2] Department of Mathematics, St. Xavier's College,
30, Mother Teresa Sarani, Kolkata 700016, India
[3] Department of Basic Science and Humanities,
Institute of Engineering & Management, D-1, Sector 5,
Salt Lake Electronics Complex, Kolkata 700091, India

**Abstract**

In this paper the concept of Proximate Gol'dberg order in several complex variables has been introduced and the existence has been proved.

## 1. Introduction

Let $f : \mathcal{C}^n \to \mathcal{C}$ be an entire function and $D \subset \mathcal{C}^n$ be an arbitrary bounded complete n-circular domain, centered at the origin, that is $D = \{z = (z_1, z_2, ..., z_n) \in \mathcal{C}^n : |z_1| \leq r_1, ..., |z_n| \leq r_n\}$ for some $r_1, ..., r_n > 0$ For $R > 0$, $R$ Real, the maximum modulus function is $M_{f,D}(R) = sup\{|f(z)| : z \in D_R\}$ where $D_R = \{z : \frac{z}{R} \in D\}$ The Gol'dberg order $\rho$ of $f$ with respect to the domain $D$ is defined as [1]

---

Key Words : *Gol'dberg order, Proximate Gol'dberg order.*

AMS Subject Classification : Primary 30B50, Secondary 30D99.

$$\rho = \limsup_{R\to\infty} \frac{\log\log M_{f,D}(R)}{\log R}$$

The growth rate of entire functions are measured by order and type of the function. To refine the growth of functions whose orders are same but are of infinite type, the concept of proximate order was introduced by G. Valiron[3]. Proximate order is considered as the intermediate comparison function which refines the growth scale of functions with same order but of types at infinity and therefore it makes no sense to consider the maximal or minimal types of a function. The proof of existence of Proximate order, given by G. Valiron [3] was simplified by S. M. Shah [2], without using any special properties of maximum modulus function and hence that proof has wider scope.

In this paper we have proved the existence of proximate order of an entire function in several complex variables with finite Gol'dberg order. To prove this, we follow the path shown by S. M. Shah [2] in which he simplified the proof of existence of proximate order in single variable.

**Definition 1.1** : The positive continuous function $\rho_D(R)$, satisfying the following properties:

1. $\rho_D(R)$ is differentiable for all large $R$ except for some isolated points where $\rho'_D(R-0)$ and $\rho'_D(R+0)$ exists.
2. $\lim_{R\to\infty} \rho_D(R) = \rho$
3. $\lim_{R\to\infty} R\rho'_D(R)\log R = 0$
4. $\limsup_{R\to\infty} \frac{\log M_{f,D}(R)}{R^{\rho_D(R)}} = 1,$

is called the proximate Gol'dberg order of an entire function $f$ in several complex variables.

Although the order of a function independent of domain $D$, proximate Gol'dberg order depends on the domain $D$. We now give proof of the existence of such proximate Gol'dberg order in several variables.

**Proof** : The Gol'dberg order $\rho$ of a multiple entire function $f$ is given by

$$\rho = \limsup_{R\to\infty} \frac{\log logM_{f,D}(R)}{\log R}$$

Let us take $\sigma_D(R) = \frac{\log\log M_{f,D}(R)}{\log R}$.
Then $\limsup_{R\to\infty} \sigma_D(R) = \rho$.
There are two possible cases:

- Case (A) : $\sigma_D(R) > \rho$ for a sequence of values of $R$ tending to infinity
- Case (B) : $\sigma_D(R) \le \rho$ for all large $R$

**Case (A)** : $\sigma_D(R) > \rho$ for a sequence of values of $R$ tending to infinity. Here we define, for all real $R > 0$,

$$\phi_D(R) = \max_{X \ge R}\{\sigma_D(X)\}$$

Such maximum value exists because $\sigma_D(R)$ is continuous by it's definition, $\limsup\limits_{R\to\infty} \sigma_D(R) = \rho$ exists finitely and also $\sigma_D(R) > \rho$ for a sequence of values of $R$ tending to infinity. Now

$$\max_{X \ge X_1}\{\sigma_D(X)\} \ge \max_{X \ge X_2}\{\sigma_D(X)\} \quad for \quad X_1 \le X_2$$

Therefore $\phi_D(R)$ is a non-increasing function of $R$.
Let $R_1$ be a sufficiently large positive real number such that $\phi_D(R_1) = \sigma_D(R_1)$ that is $\max\limits_{X \ge R_1}\{\sigma_D(X)\}$ occurs at $R_1$. Such $R_1$ will exist for a sequence of values tending to infinity.
We now define the function $\rho_D(R)$ by

$$\rho_D(R_1) = \phi_D(R_1)$$

Let $S_1$ be the smallest integer not less than $1 + R_1$ such that $\phi_D(R_1) > \phi_D(S_1)$
For $R_1 \le R \le S_1$, let

$$\rho_D(R) = \rho_D(R_1) = \phi_D(R_1)$$

Define $U_1 > S_1$ as follows

$$\rho_D(R) = \rho_D(R_1) - \log\log\log R + \log\log\log S_1 \quad for \quad S_1 \le R \le U_1$$

$$\rho_D(R) = \phi_D(R) \quad at \quad R = U_1$$

Now for $S_1 \le R \le U_1$, $\log\log\log S_1 \le \log\log\log R$

$$\rho_D(R) > \phi_D(R) \quad on \quad S_1 \le R < U_1 \tag{1.1}$$

Let $R_2$ be the smallest value of $R$ for which $R_2 \geq U_1$ and $\phi_D(R_2) = \sigma_D(R_2)$. that is $\max\limits_{X \geq R_2} \{\sigma_D(X)\}$ has been attained at $R_2$.
If $R_2 > U_1$, then let

$$\rho_D(R) = \phi_D(R) \;\; on \;\; U_1 \leq R \leq R_2$$

Now $\phi_D(R)$ is constant on $U_1 \leq R \leq R_2$ therefore $\rho_D(R)$ also will be constant on $U_1 \leq R \leq R_2$
Repeating this argument, we obtain that $\rho_D(R)$ is differentiable except for a countable number of isolated points and in the adjacent intervals $\rho'_D(R) = 0$ or $\rho'_D(R) = -\frac{1}{R \log R \log\log R}$.
By construction of $\rho_D(R)$ and the argument in definition (1.1), we have $\rho_D(R) \geq \phi_D(R)$.
Also since $\phi_D(R) = \max\limits_{X \geq R}\{\sigma_D(X)\}$, therefore $\rho_D(R) \geq \phi_D(R) \geq \sigma_D(R)$ for all $R \geq R_1$
Further we choose $R_1, R_2, ...$ tending to infinity such that

$$\rho_D(R) = \phi_D(R) = \sigma_D(R)$$

where $\rho_D(R)$ is non-increasing and $\lim\limits_{R \to \infty} \sigma_D(R) = \rho$.
Therefore

$$\lim_{R \to \infty} \rho_D(R) = \rho$$

Thus proposition (2) of definition of proximate order is proved.
Now if $\rho'_D(R) = 0$,

$$\lim_{R \to \infty} R\rho'_D(R) \log R = 0$$

. Otherwise $\rho'_D(R) = -\frac{1}{R \log R \log\log R}$ implies

$$\lim_{R \to \infty} R\rho'_D(R) \log R = \lim_{R \to \infty} -\frac{1}{\log\log R} = 0$$

Thus proposition (3) of definition of proximate order is also proved.
Since $\sigma_D(R) = \frac{\log\log M_{f,D}(R)}{\log R}$ that implies

$$\begin{aligned} \log(R)^{\sigma_D(R)} &= \log\log M_{f,D}(R) \\ &\Rightarrow \log M_{f,D}(R) = R^{\sigma_D(R)} \\ &= R^{\rho_D(R)}, \end{aligned}$$

for an infinite number of values of $R$ tending to infinity and for remaining $R$,

$$\log M_{f,D}(R) < R^{\rho_D(R)}$$

Therefore

$$\limsup_{R\to\infty} \frac{\log M_{f,D}(R)}{R^{\rho_D(R)}} = 1$$

**Case (B)** : $\sigma_D(R) \le \rho$ for all large $R$. There are two subcases under this case.

- **subcase (i)** : $\sigma_D(R) = \rho$ for at least a sequence of values of $R$ tending to infinity.
- **subcase(ii)** : $\sigma_D(R) < \rho$ for all large $R$

**subcase (i)** : Here we define $\rho_D(R) = \rho$ for all values of $R$.
**subcase (ii)**: Let us take $X$, sufficiently large positive number such that $\sigma_D(R) < \rho$ whenever $R > X$.
Let

$$\xi_D(R) = \max_{X \le X' \le R} \{\sigma_D(X')\} \quad for \quad R > X.$$

Now for $X < R_1 \le R_2$

$$\max_{X \le X' \le R_1} \{\sigma_D(X')\} \le \max_{X \le X' \le R_2} \{\sigma_D(X')\}$$

Hence $\xi_D(R)$, $R > X$, is a non decreasing function. Now we take suitably large $R_1 > X$ and define

$$\rho_D(R_1) = \rho$$

Let us take a point $S_1 < R_1$ such that

$$\rho_D(R) = \rho + \log\log\log R - \log\log\log R_1 \quad for \quad S_1 \le R \le R_1$$

$$and \;\; \rho_D(R) = \xi_D(R) \;\; at \;\; R = S_1$$

If $\xi_D(S_1) \ne \sigma_D(S_1)$, that is the $\max_{X \le X' \le S_1} \{\sigma_D(X')\}$ is not attained at $X = S_1$, then there must be some point $T_1 < S_1$ nearest to $S_1$ at which $\max_{X \le X' \le T_1} \{\sigma_D(X')\}$ is attained *i.e.* $\xi_D(T_1) = \sigma_D(T_1)$.
We define

$$\rho_D(R) = \xi_D(R) \quad for \quad T_1 \le R \le S_1$$

Since $\xi_D(S_1) \ne \sigma_D(S_1)$ but $\xi_D(T_1) = \sigma_D(T_1)$, therefore $\rho_D(R)$ will be constant on $T_1 \le R \le S_1$, as $\xi_D(R)$ is so.
If $\xi_D(S_1) = \sigma_D(S_1)$ then let $T_1 = S_1$.

Choose $R_2 > R_1$ suitably large and let

$$\rho_D(R_2) = \rho$$

Similarly we take some point $S_2 < R_2$ such that

$$\rho_D(R) = \rho + \log\log\log R - \log\log\log R_2 \quad for \quad S_2 \le R \le R_2$$

$$\rho_D(R) = \xi_D(R) \quad at \quad R = S_2$$

If $\xi_D(S_2) \neq \sigma_D(S_2)$,then choose $T_2 < S_2$, the nearest point to $S_2$ at which $\xi_D(T_2) = \sigma_D(T_2)$.
We define

$$\rho_D(R) = \xi_D(R) \quad for \quad T_2 \le R \le S_2$$

If $\xi_D(S_2) = \sigma_D(S_2)$ then let $T_2 = S_2$.
For $R < T_2$, let

$$\rho_D(R) = \rho_D(T_2) + \log\log\log T_2 - \log\log\log R \quad for \quad U_1 \le R \le T_2$$

where $U_1 < T_2$ is the point of intersection of

$$Y = \rho \quad with$$

$$Y = \rho_D(T_2) + \log\log\log T_2 - \log\log\log R.$$

Set

$$\rho_D(R) = \rho \quad for \quad R_1 \le R \le U_1$$

It is always possible to choose $R_2$ so large that $R_1 < U_1$ and satisfies the above properties.
Repeating the argument we get

$$\rho_D(R) \ge \xi_D(R) \ge \sigma_D(R) \quad and$$

$$\rho_D(R) = \sigma_D(R) \quad for \quad R = T_1, T_2, T_3, ...$$

So,

$$\limsup_{R\to\infty} \sigma_D(R) = \rho$$

$$\Rightarrow \lim_{R\to\infty} \xi_D(R) = \rho$$

as $\xi_D(R)$ is non decreasing and $\xi_D(R) < \rho$ for all $R > X$.
Hence

$$\lim_{R\to\infty} \rho_D(R) = \rho$$

Also $\rho_D(R)$ is differentiable in adjacent intervals and $\rho'_D(R) = 0$ or $\rho'_D(R) = -\frac{1}{R\log R\log\log R}$ and hence $\lim_{R\to\infty} R\rho_D(R) = 0$ in both of these cases. Also by the similar argument in first case, we obtain that

$$\limsup_{R\to\infty} \frac{\log M_{f,D}(R)}{R^{\rho_D(R)}} = 1.$$

Hence all the propositions for definition of proximate order are proved in second case also and this completes the proof.

## References

[1] Fuks B. A., Introduction to the theory of analytic functions of several complex variables, (1965).

[2] Shah S. M., On proximate orders of integral functions. Bull. Amer. Math. Soc., 52, 326.

[3] Valiron G., Lectures on the general teory of integral functions, private, Toulouse (1922).