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GENERALIZATION OF RIEMANN-STIELTJES INTEGRAL BASED ON GENERALIZED g-SEMIRING $([a,b],\oplus^{\beta},\odot)$

M. S. MISHA
Senior Research Fellow, Department of Mathematics,
University of Kerala, Kariavattom, India

Abstract

In this paper, we make a study of the generalization of classical Riemann-Stieltjes integral in the pseudo-analysis framework. Its construction is based on the following form of generalized generated pseudo-operations:

$$x \oplus^{\beta} y = g^{(-1)}(\beta g(x) + g(y)) \quad \text{and} \quad x \odot y = g^{(-1)}(g(x)g(y)),$$

where β is an arbitrary but fixed positive real number, g is a positive strictly monotone generating function defined on $[a,b] \subseteq [-\infty,+\infty]$ such that $0 \in Ran(g)$ and $g^{(-1)}$ is the pseudo-inverse function for function g.

1. Introduction

Pseudo-analysis is the generalization of the classical analysis, where instead of the field of real numbers a semiring is defined on a real interval $[a,b] \subseteq [-\infty,+\infty]$ with pseudo-addition $\oplus$ and with pseudo-multiplication $\odot$ (cf. [2]). Pseudo-analysis uses many mathematical tools from different fields as measure theory, functional analysis,

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Key Words : *Generalized g-semiring, Generalized generated pseudo-operations, g_{β}-set function, Pseudo-Riemann-Stieltjes integral based on generalized g-semiring* $([a,b],\oplus^{\beta},\odot)$.

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functional equations, variational calculus, optimization theory, semiring theory etc. and still is in the developing form (cf. [3]). Ivana Stajner-Papuga, T. Grbic and M. Dankova introduced the generalization of Riemann Stieltjes integral based on the generalized g-semiring $([a,b],\oplus,\odot)$ in 2006 (cf. [4], [5], [6]). In this paper, we make a study of the generalization of classical Riemann-Stieltjes integral based on the generalized g-semiring $([a,b],\oplus^{\beta},\odot)$. Here β is arbitrary but fixed positive real number and $g:[a,b]\to[0,+\infty]$ is a strictly monotone generating function defined on $[a,b]\subseteq[-\infty,+\infty]$ such that $0\in Ran(g)$. Using the generating function g, first, we define a set function and call g_{β}-set function. Using the g_{β}-set function and the generalized generated pseudo-operations from the generalized g-semiring $([a,b],\oplus^{\beta},\odot)$, we define the generalization of Riemann-Stieltjes integral in the pseudo-analysis framework, and call pseudo-Riemann-Stieltjes integral based on the generalized g-semiring $([a,b],\oplus^{\beta},\odot)$. We then prove various properties of the new integral.

2. Preliminary Notion

In this section, some of the important required concepts necessary to go further this paper are shown. They are taken from [1], [4], [5] and [6].

Definition 2.1 : Let $[a,b]$ be a closed subinterval of $[-\infty,+\infty]$ (in some cases semi closed subintervals will be considered) and let $\leq$ be a total order on $[a,b]$. The operation $\oplus$ is called a **pseudo-addition** if it is a function $\oplus:[a,b]\times[a,b]\to[a,b]$ which satisfies the following axioms: associativity, non-decreasing, a left neutral or zero element 0; that is $0\oplus x=x$, for all $x\in[a,b]$ and commutativity. The operation $\odot$ is called a **pseudo-multiplication** if it is a function $\odot:[a,b]\times[a,b]\to[a,b]$ which satisfies the following conditions: associativity, positively non-decreasing: ie if $x\leq y$ implies $x\odot z\leq y\odot z$, where $z\in[a,b]_{+}$ and $[a,b]_{+}=\{x/x\in[a,b],\mathbf{0}\leq x\}$, $\mathbf{1}$ is unit element: that is $\mathbf{1}\odot x=x$, for all $x\in[a,b]$ and commutativity. A **semiring** is the srtructure $([a,b],\oplus,\odot)$ having the following properties: $\oplus$ is pseudo-addition; $\odot$ is psuedo- multiplication; $\mathbf{0}\odot x=\mathbf{0}$ and $x\odot(y\oplus z)=(x\odot y)\oplus(x\odot z)$, that is $\odot$ is a distributive pseudo-multiplication with respect to $\oplus$. **g-semiring** is a semiring with strict pseudo-operations defined by strictly monotone and continuous generator function $g:[a,b]\to[0,+\infty]$. Here the operations are given by $x\oplus y=g^{-1}(g(x)+g(y))$ and $x\odot y=g^{-1}(g(x)g(y))$, where g^{-1} is the classical inverse function for function g.

Definition 2.2 : For **non-decreasing function** $f:[a,b]\rightarrow[c,d]$, where $[a,b]$ and $[c,d]$ are closed subintervals of extended real line $[-\infty,+\infty]$, **pseudo-inverse** is $f^{(-1)}(y)=\sup\{x\in[a,b]/f(x)<y\}$. If f is a **non-increasing function**, its **pseudo-inverse** is $f^{(-1)}(y)=\sup\{x\in[a,b]/f(x)>y\}$. For strictly monotone function f, $f^{(-1)}|_{Ran(f)}$ is also strictly monotone and following identities hold:

(1) $fof^{(-1)}|_{Ran(f)}=id|_{Ran(f)}$ and (2) $f^{(-1)}of=id|_{[a,b]}$.

Definition 2.3 : It is possible to define a metric using the generating function g. Let $d:[a,b]\times[a,b]\rightarrow[0,\infty]$ be a function defined by $d(x,y)=|g(x)-g(y)|$, where $x,y\in[a,b]$ and $g:[a,b]\rightarrow[0,\infty]$ is strictly monotone function defined on $[a,b]\subseteq[-\infty,+\infty]$ such that $0\in Ran(g)$.

3. Generalized g-Semiring $([a,b],\oplus^{\beta},\odot)$

Structure essential for this paper is the generalization of the previously mentioned semiring.

Definition 3.1 : Let β be arbitrary but fixed positive real number, $g:[a,b]\rightarrow[0,+\infty]$ be a strictly monotone generating function defined on $[a,b]\subseteq[-\infty,+\infty]$ such that $0\in Ran(g)$. The structure $([a,b],\oplus^{\beta},\odot)$ is called **generalized g-semiring** if operations $\oplus^{\beta}$ and $\odot$ are given by $x\oplus^{\beta}y=g^{(-1)}(\beta g(x)+g(y))$ and $x\odot y=g^{(-1)}(g(x)g(y))$, where $g^{(-1)}$ is the pseudo-inverse function for function g.

If the generating function g is continuous or bijection, then the operations are given by $x\oplus^{\beta}y=g^{-1}(\beta g(x)+g(y))$ and $x\odot y=g^{-1}(g(x)g(y))$, where g^{-1} is the classical inverse function for function g.

Properties of generalized g-semiring

Let $([a,b],\oplus^{\beta},\odot)$ be generalized g-semiring from the above definition. Then

1. If $\beta g(x)+g(y),g(z)g(x),g(z)g(y)\in Ran(g)$, $\odot$ is left distributive over $\oplus^{\beta}$, that is, $z\odot(x\oplus^{\beta}y)=(z\odot x)\oplus^{\beta}(z\odot y)$.
2. Neutral element from the left for $\oplus^{\beta}$ is $g^{(-1)}(0)$.
3. If $1\in Ran(g)$, the neutral element from the left for $\odot$ is $g^{(-1)}(1)$.
4. $g^{(-1)}(0)\odot x=x\odot g^{(-1)}(0)=g^{(-1)}(0)$ for all $x\in[a,b]$.
5. $\oplus^{\beta}$ is non-decreasing function.

6. $\odot$ is non-decreasing function.

7. In the general case, the cancellation law does not hold for $\oplus^{\beta}$.

We can prove the properties of generalized g-semiring directly from the above definition, the properties of the generating function and its pseudo-inverses.

Definition 3.2 : Let $\oplus^{\beta}$ and $\odot$ be operations from Definition 3.1 and $\alpha_i \in [a,b]$ where $i \in \{1,2,\cdots,n\}$. Then $\oplus_{i=1}^{n} = \alpha_i = (\cdots((\alpha_1 \oplus^{\beta} \alpha_2) \oplus^{\beta} \alpha_3) \oplus^{\beta} \cdots) \oplus^{\beta} \alpha_n$.

Proposition 3.3 : (1) If $g : [a,b] \to [0,+\infty]$ is either strictly increasing left continuous or strictly decreasing right continuous generating function such that $0 \in Ran(g)$, then the pseudo-sum of n elements and pseudo-sum of pseudo-products satisfy the following:

(i) $\oplus_{i=1}^{n}\alpha_i \leq g^{(-1)}(\sum_{i=1}^{n} \beta^{n-1} g(\alpha_i))$

(ii) $\oplus_{i=1}^{n}(\alpha_i \odot \eta_i) \leq g^{(-1)}(\sum_{i=1}^{n} g(\alpha_i).g(\eta_i))$.

(2) If $g : [a,b] \to [0,+\infty]$ is either strictly decreasing left continuous or strictly increasing right continuous generating function such that $+\infty \in Ran(g)$, then the pseudo-sum of n elements and pseudo-sum of pseudo-products satisfy the following:

(i) $\oplus_{i=1}^{n}\alpha_i \geq g^{(-1)}(\sum_{i=1}^{n} \beta^{n-1} g(\alpha_i))$

(ii) $\oplus_{i=1}^{n}(\alpha_i \odot \eta_i) \geq g^{(-1)}(\sum_{i=1}^{n} \beta^{n-1} g(\alpha_i).g(\eta_i))$.

(3) If $g : [a,b] \to [0,+\infty]$ is a strictly monotone bijection, then the pseudo-sum of n elements and pseudo-sum of pseudo-products satisfy the following:

(i) $\oplus_{i=1}^{n}\alpha_i = g^{(-1)}(\sum_{i=1}^{n} \beta^{n-i} g(\alpha_i))$

(ii) $\oplus_{i=1}^{n}(\alpha_i \odot \eta_i) = g^{(-1)}(\sum_{i=1}^{n} \beta^{n-1} g(\alpha_i).g(\eta_i))$.

Proof 1 :.(i) Let $g : [a,b] \to [0,+\infty]$ be strictly increasing left continuous. Since $0 \in Ran(g)$, $g(a) = 0$ and since g is strictly increasing, we can write $\alpha = g^{(-1)}(x) = \sup\{y \in [a,b]/g(y) < x\}$. Therefore, $\lim_{y\to\alpha} -g(y) \leq x$. Since g is left continuous, $\lim_{y\to\alpha} -g(y) = g(\alpha)$. That is $g(\alpha) \leq x$. That is go $g^{(-1)}(x) \leq x$ for all $x \in [0,+\infty]$. Similarly, we can prove if g is strictly decreasing right continuous,go $g^{(-1)}(x) \leq x$ for all $x \in [0,+\infty]$.

Now,

$$\oplus_{i=1}^{n}\alpha_i = (\cdots((\alpha_1 \oplus^\beta \alpha_2) \oplus^\beta \alpha_3) \oplus^\beta \cdots) \oplus^\beta \alpha_n$$

$$(\alpha_1 \oplus^\beta \alpha_2) = g^{(-1)}(\beta g(\alpha_1) + g(\alpha_2))$$

$$((\alpha_1 \oplus^\beta \alpha_2) \oplus^\beta \alpha_3) = g^{(-1)}(\beta g(g^{(-1)}(\beta g(\alpha_1) + g(\alpha_2)) + g(\alpha_3))$$

$$\leq g^{(-1)}(\beta^2 g(\alpha_1) + \beta g(\alpha_2) + g(\alpha_3)) \quad [\text{since } \mathrm{go}g^{(-1)}(x) \leq x].$$

Proceeding like this, we get $\oplus_{i-1}^{n}\alpha_i \leq g^{(-1)}(\sum_{i=1}^{n} \beta^{n-1} g(\alpha_i))$.

(ii) By what we first proved,

$$\oplus_{i=1}^{n}(\alpha_i \odot \eta_i) \leq g^{(-1)}(\sum_{i=1}^{n} \beta^{n-1} g(\alpha_i \odot \eta_i))$$

$$g^{(-1)}(\sum_{i=1}^{n} \beta^{n-1} g \circ g^{(-1)}(g(\alpha_i) g(\eta_i))) \leq g^{(-1)}(\sum_{i=1}^{n} \beta^{n-1} g(\alpha_i) \cdot g(\eta_i)).$$

Thus, we get $\oplus_{i=1}^{n}(\alpha_i \odot \eta_i) \leq g^{(-1)}(\sum_{i=1}^{n} \beta^{n-1} g(\alpha_i) \cdot g(\eta_i))$.

Similarly, we can prove part (2) and part (3) of the theorem by proving $gog^{(-1)}(x) \geq x$, for all $x \in [0, +\infty]$ and $gog^{-1}(x) = x$, for all $x \in [0, +\infty]$ respectively.

4. g_β - Set-Function

Another notion essential for the construction of pseudo-Riemann-Stieltjes integral based on the generalized g-semiring $([a,b], \oplus^\beta, \odot)$ is the notion of a set function introduced by means of generating function g and defined on family of subintervals of the real line in the following way.

Definition 4.1 : Let $([a,b], \oplus^\beta, \odot)$ be the generalized g-semiring. Let A be a compact subinterval of extended real line and let ψ be a monotonic increasing function on A. Let $\mathcal{C}$ be the collection of subintervals of A. For each positive integer n and each i $(i = 1, 2, \cdots, n)$, we define a g_β**-set-function** from $\mathcal{C}$ to $[a,b]$ as follows;

For any A in $\mathcal{C}$, $m_{n,i}(A) = g^{(-1)}\left(\frac{\psi(y)-\psi(x)}{\beta^{n-i}}\right)$, where x and y are the left and right end points of the interval A and $g^{(-1)}$ is the pseudo inverse of the generating function g. Using the Proposition 3.3, we can easily prove the following properties of the g_β-set function.

1. $m_{n,i}(\emptyset) = g^{(-1)}(0)$,

2. If $A_1, A_2, \cdots, A_n$ are pair wise disjoint members of $\mathcal{C}$ such that $A = \bigcup_{i=1}^{n} A_i$ is also in $\mathcal{C}$, then the following hold:

   (i) if $g : [a, b] \to [0, +\infty]$ is strictly monotone bijection, then $\oplus_{i=1}^{n} m_{n,i}(A_i) = g^{-1}(\psi(d) - \psi(c))$, where c and d are the left and right end points of A.

   (ii) if $g : [a, b] \to [0, +\infty]$ is either strictly increasing left-continuous or strictly decreasing right- continuous generating function then, $\oplus_{i=1}^{n} m_{n,i}(A_i) \leq g^{(-1)}(\psi(d) - \psi(c))$, where c and d are the left and right end points of A.

   (iii) if $g : [a, b] \to [0, +\infty]$ is either strictly decreasing left-continuous or strictly increasing right- continuous generating function such that $+\infty \in Ran(g)$ then, $\oplus_{i=1}^{n} m_{n,i}(A_i) \geq g^{(-1)}(\psi(d) - \psi(c))$, where c and d are the left and right end points of A.

5. Generalization of Riemann-Stieltjes Integral Based on Generalized g-Semiring $([a, b], \oplus^{\beta}, \odot)$

In this section, we define the generalization of Riemann-Stieltjes integral based on generalized g-semiring $([a, b], \oplus^{\beta}, \odot)$. For defining it, the generalized generated pseudo-operations from the generalized g-semiring $([a, b], \oplus^{\beta}, \odot)$, the g_{β}-set function and the metric presented in the Definition 2.3 will be used.

Definition 5.1 : Let $g : [a, b] \to [0, +\infty]$ be strictly monotone function, where $[a, b]$ is a closed subinterval of $[-\infty, +\infty]$ and $\oplus^{\beta}$ and $\odot$ be the generalized generated pseudo - operations from the generalized g-semiring $([a, b], \oplus^{\beta}, \odot)$. Let $[c, d]$ be a compact subinterval of the extended real line and ψ be an increasing function on $[c, d]$. Let $P_n = \{(t_i, A_i) : 1 \leq i \leq n\}$ be a collection of pair wise disjoint subintervals of $[c, d]$ such that $t_i \in A_i$ and $[c, d] = \bigcup_{i=1}^{n} A_i$. The **Riemann-Stieltjes pseudo-sum of f with respect to ψ** for the tagged partition P_n is denoted by $\oplus^{\beta}(P_n, f, \psi)$ and defined as

$$\oplus^{\beta}(P_n, f, \psi) = \oplus_{i=1}^{n} f(t_i) \odot m_{n,i}(A_i), \quad \text{where} \quad f : [c, d] \to [a, b].$$

Definition 5.2 : The function $f : [c, d] \to [a, b]$ is said to be **pseudo-Riemann-Stieltjes integrable based on the generalized g-semiring** $([a, b], \oplus^{\beta}, \odot)$ with respect to ψ on $[c, d]$ if there exists a real number $I \in [a, b]$ satisfying the following

condition; for each $\epsilon > 0$ there exists a positive function δ defined on $[c, d]$ such that $d(\oplus^{\beta}(P_n, f, \psi), I) < \epsilon$, for each tagged partition P_n of $[c, d]$.
It can easily see that the number I, if it exists, is uniquely determined. This number I is called **pseudo-Riemnn-Stieltjes integral based on the generalized g-semiring** $([a, b], \oplus^{\beta}, \odot)$ of the function f with respect to ψ on $[c, d]$ and it will be denoted by $(pRS)\int_{[c,d]}^{(\oplus^{\beta},\odot)} f d\psi$. Specially, for $g(x) = x, [a, b] = [0, +\infty]$ and $\beta = 1$ the previous definition will give the definition of classical Riemann-Stieltjes integral $(RS)\int_c^d f d\psi$. Throughout this paper, we use the term pseudo-Riemann-Stieltjes integral instead of pseudo-Riemann-Stieltjes integral based on the generalized g-semiring $([a, b], \oplus^{\beta}, \odot)$.
Theorem 5.3 : Let $g : [a, b] \to [0, +\infty]$ be a strictly monotone function and let $f : [c, d] \to [a, b]$ be pseudo . Riemann-Stieltjes integrable with respect to ψ on $[c, d]$.

(1) If g is either strictly increasing right - continuous or strictly decreasing left - continuous generating function such that $+\infty \in Ran(g)$, then

$$g(pRS)\int_{[c,d]}^{(\oplus^{\beta},\odot)} f d\psi) \geq (RS)\int_c^d (gof) d\psi,$$

if the integral on the right hand side exists.

(2) If g is either strictly increasing left - continuous or strictly decreasing right - continuous generating function such that $0 \in Ran(g)$, then

$$g((pRS)\int_{[c,d]}^{(\oplus^{\beta},\odot)} f d\psi) \leq (RS)\int_c^d (gof) d\psi,$$

if the integral on the right hand side exists.

(3) If g is a strictly monotone bijection, then (gof) is a Riemann Stieltjes integrable with respect to ψ on $[c, d]$ and

$$(pRS)\int_{[c,d]}^{(\oplus^{\beta},\odot)} f d\psi = g^{-1}((RS)\int_c^d (gof) d\psi.$$

Proof : (1) If $g : [a, b] \to [0, +\infty]$ is either strictly increasing right - continuous or strictly decreasing left - continuous generating function such that $+ \in Ran(g)$ then $gog^{-1}(x) \geq x$ for all $x \in [0, +\infty]$. Since $f : [c, d] \to [a, b]$ is pseudo Riemann-Stieltjes integrable with respect to ψ on $[c,d]$, then by the definition, there is a real number $I \in [a,b]$ satisfying the following: for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$d(\oplus^{\beta}(P_n, f, \psi), I) < \epsilon \cdots , \tag{5.1}$$

for all tagged partitions P_n of $[c,d]$. Suppose that (gof) is Riemann Stieltjes integrable with respect to ψ on $[c,d]$. Then by the definition, there is a real number $(RS)\int_c^d (gof)d\psi$ satisfying the following condition: for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$\left|\sum_{i=1}^{n}(gof)(t_i)(\psi(x_i) - \psi(x_{i-1})) - (RS)\int_c^d (gof)d\psi\right| < \epsilon \tag{5.2}$$

for all tagged partitions $P_n = \{(t_i, (x_{i-1}, x_i]), i = 1, 2, \cdots, n\}$ of $[c,d]$.
Now,

$$\begin{aligned} \oplus^{\beta}(P_n, f, \psi) &= \oplus_{i=1}^{n} f(t_i) \odot m_{n,i}(A_i) = \oplus_{i=1}^{n} f(t_i) \odot g^{(-1)}\left(\frac{\psi(x_i) - \psi(x_{i-1})}{\beta^{n-1}}\right) \\ &\geq g^{(-1)}\left(\sum_{i=1}^{n} g(f(t_i))(\psi(x_i) - \psi(x_{i-1}))\right). \end{aligned}$$

[By part(2) of Propositon 3.3]. Hence

$$\begin{aligned} g(\oplus^{\beta}(P_n, f, \psi)) &\geq g(g^{(-1)}(\sum_{i=1}^{n}(gof)(\psi(x_i) - \psi(x_{i-1}))) \\ &\geq \sum_{i=1}^{\infty}(gof)(t_i)(\psi(x_i) - \psi(x_{i-1})). \end{aligned}$$

$$g(\oplus^{\beta}(P_n, f, \psi)) - \sum_{i=1}^{n}(gof)(t_i)(\psi(x_i) - \psi(x_{i-1})) \geq 0 \tag{5.3}$$

From (5.1) we can write

$$\begin{aligned} |g(\oplus^{\beta}(P_n, f, \psi)) - g(I)| < \epsilon &\Rightarrow g(\oplus^{\beta}(P_n, f, \psi)) - \epsilon < g(I) \\ &\Rightarrow g(\oplus^{\beta}(P_n, f, \psi)) - \epsilon < g((pRS)\int_{[c,d]}^{(\oplus^{\beta}, \odot)} f d\psi) \end{aligned} \tag{5.4}$$

From (5.2) we get,

$$-\sum_{i=1}^{n}(gof)(t_i)(\psi(x_i) - \psi(x_{i-1})) - \epsilon < -(RS)\int_c^d (gof)d\psi. \tag{5.5}$$

From (5.3), (5.4) and (5.5), we can write

$$g((pRS)\int_{[c,d]}^{(\oplus^{\beta},\odot)} f d\psi) - (RS)\int_{c}^{d}(gof)d\psi$$
$$> g(\oplus^{\beta}(P_n, f, \psi)) - \epsilon - \sum_{i=1}^{n}(gof)(t_i)(\psi(x_i) - \psi(x_{i-1})) - \epsilon \geq -2\epsilon.$$

This holds for all $\epsilon > 0$ and after allowing $\epsilon \to 0$, then

$$g((pRS)\int_{[c,d]}^{(\oplus^{\beta},\odot)} f d\psi) - (RS)\int_{c}^{d}(gof)d\psi \geq 0.$$

That is

$$g((pRS)\int_{[c,d]}^{(\oplus^{\beta},\odot)} f d\psi) \geq (RS)\int_{c}^{d}(gof)d\psi.$$

Similarly, we can prove the second part of the theorem by using the part (1) of the Proposition 3.3 and the fact that $gog^{(-1)}(x) \leq x$, for all $x \in [0, +\infty]$. In addition, we can prove the third part of the theorem by using the part (3) of the Proposition 3.3 and the fact that $gog^{-1}(x) = x$, for all $x \in [0, +\infty]$. □

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