International J. of Math. Sci. & Engg. Appls. (IJMSEA) ISSN 0973-9424, Vol. 11 No. III (December, 2017), pp. 123-127

A NOTE ON BETA FUNCTION AND LAPLACE TRANSFORM

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Abstract

In this article, we use the Laplace Transform and Inverse Laplace Transform to prove the identities of beta function. It is possible that this technique of proof may be applied to solve the other problems involving beta function.

1. Introduction

The beta function [1], also called the Euler integral of the first kind, is a special function defined by $B(p,q) = \int_0^1 x^{p-1} (1-x)^{q-1} dx$, for $\operatorname{Re}(p) > 0$, $\operatorname{Re}(q) > 0$. The gamma function [2], also called the Euler integral of the second kind, is defined as convergent improper integral $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$ for $\operatorname{Re}(n) > 0$.

Properties of beta function:

1.
$$B(p,q) = B(q,p)$$

2.
$$B(p,q) = \int_0^\infty \frac{x^{p-1}}{(1+x)^{p+q}} dx$$
, $\operatorname{Re}(p) > 0$, $\operatorname{Re}(q) > 0$

3.
$$B(p,q) = 2 \int_0^{\frac{\pi}{2}} \sin^{2p-1} \cos^{2q-1} dx$$
, $\operatorname{Re}(p) > 0$, $\operatorname{Re}(q) > 0$

Key Words : *Beta function, Gamma function, Laplace Transform, Convolution Theorem.* 2010 AMS Subject Classification : 33B15, 44A10.

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The key property of beta function is $B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$, which is proved by using Laplace transform by Charng-Yih Yu in [3].

Identities of beta and gamma function:

- 1. B(p,q) = B(p+1,q) + B(p,q+1), p,q > 0
- 2. $B(p,q+1) = \frac{q}{p}B(p+1,q) = \frac{q}{p+q}B(p,q), p,q > 0$
- 3. $\Gamma(p+1) = p\Gamma(p), p > 0$

The proof of identities of beta function requires the following definitions and results.

Definition 1.1 [4] : If f(t) is defined for all positive values of t, then the Laplace transform of f(t) is defined as the integral, $L[f(t)] = F(s) = \int_0^\infty e^{-st} f(t) dt$, provided that integral exists for complex parameter s, and $f(t) = L^{-1}[F(s)]$ is called as the inverse Laplace transform of F(s).

Lemma 1.2 [4]: Convolution Theorem: $L^{-1}[F(s)] = f(t)$, and $L^{-1}[G(s)] = g(t)$, then $L^{-1}[F(s)G(s)] = \int_0^t f(x)g(t-x)dx = f(t) * g(t)$, where * denotes the convolution of f(t) and g(t).

Lemma 1.3 [3] : If p > 0, q > 0, then $\int_0^1 (1-x)^{p-1} x^{q-1} dx = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$.

2. Main Results

Lemma 2.1 [4] : Relation between Gamma function and Laplace transform: If s > 0, then $L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}$. **Proof** : Consider $L[t^n] = \int_0^\infty e^{-st} t^n dt$ put st = x, $t = \frac{x}{s}$ and $dt = \frac{dx}{s}$ (new limits are x = 0 to $x = \infty$), therefore $L[t^n] = \int_0^\infty e^{-x} (\frac{x}{s})^n \frac{dx}{s} = \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^{(n+1)-1} dx = \frac{\Gamma(n+1)}{s^{n+1}}$

Theorem 2.2 : If p > 0, q > 0, then

1.
$$B(p,q) = B(p+1,q) + B(p,q+1)$$

2. $B(p,q+1) = \frac{q}{p}B(p+1,q) = \frac{q}{p+q}B(p,q)$

 $\mathbf{Proof}: 1.$

$$Consider R.H.S. = B(p+1,q) + B(p,q+1)$$
(1)

$$= \int_0^1 x^{(p+1)-1} (1-x)^{q-1} dx + \int_0^1 x^{p-1} (1-x)^{(q+1)-1} dx \qquad (2)$$

put $x = \frac{y}{t}$, $dx = \frac{dy}{t}$, in above integrals. (new limits are y=0, y=t)

$$R.H.S. = \int_0^t (\frac{y}{t})^{(p+1)-1} (1-\frac{y}{t})^{q-1} \frac{dy}{t} + \int_0^t (\frac{y}{t})^{p-1} (1-\frac{y}{t})^{(q+1)-1} \frac{dy}{t}$$
(3)

$$= \int_0^t \frac{y^{(p+1)-1}(t-y)^{q-1}}{t^{p+q}} dy + \int_0^t \frac{y^{p-1}(t-y)^{(q+1)-1}}{t^{p+q}} dy$$
(4)

$$=\frac{1}{t^{p+q}}\int_0^t y^{(p+1)-1}(t-y)^{q-1}dy + \frac{1}{t^{p+q}}\int_0^t y^{p-1}(t-y)^{(q+1)-1}dy$$
(5)

By Lemma 1.2

$$R.H.S. = \frac{1}{t^{p+q}} L^{-1} [L[t^{(p+1)-1}]L[t^{q-1}]] + \frac{1}{t^{p+q}} L^{-1} [L[t^{p-1}]L[t^{(q+1)-1}]]$$
(6)

By Lemma 2.1

$$R.H.S. = \frac{1}{t^{p+q}} L^{-1} \left[\frac{\Gamma(p+1)}{s^{p+1}} \frac{\Gamma(q)}{s^q} \right] + \frac{1}{t^{p+q}} L^{-1} \left[\frac{\Gamma(p)}{s^p} \frac{\Gamma(q+1)}{s^{q+1}} \right]$$
(7)

$$= \frac{1}{t^{p+q}} \Gamma(p+1) \Gamma(q) L^{-1}\left[\frac{1}{s^{p+q+1}}\right] + \frac{1}{t^{p+q}} \Gamma(p) \Gamma(q+1) L^{-1}\left[\frac{1}{s^{p+q+1}}\right]$$
(8)

Since $L^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{\Gamma(n)}$,

$$R.H.S. = \frac{1}{t^{p+q}}\Gamma(p+1)\Gamma(q)\frac{t^{p+q}}{\Gamma(p+q+1)} + \frac{1}{t^{p+q}}\Gamma(p)\Gamma(q+1)\frac{t^{p+q}}{\Gamma(p+q+1)}$$
(9)

$$=\frac{\Gamma(p+1)\Gamma(q)+\Gamma(p)\Gamma(q+1)}{\Gamma(p+q+1)}$$
(10)

By identity $\Gamma(p+1) = p\Gamma(p)$,

$$R.H.S. = \frac{(p+q)\Gamma(p)\Gamma(q)}{(p+q)\Gamma(p+q)}$$
(11)

$$= \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$
(12)
$$R(p,q)$$
(12)

$$=B(p,q) \tag{13}$$

2.

$$Consider B(p,q+1) = \int_0^1 x^{p-1} (1-x)^{(q+1)-1} dx$$
(14)

put $x = \frac{y}{t}$, $dx = \frac{dy}{t}$, in above integral. (new limits are y=0, y=t)

$$B(p,q+1) = \int_0^t \left(\frac{y}{t}\right)^{p-1} \left(1 - \frac{y}{t}\right)^{(q+1)-1} \frac{dy}{t}$$
(15)

$$= \int_0^t \frac{y^{p-1}(t-y)^{(q+1)-1}}{t^{p+q}} dy$$
(16)

$$=\frac{1}{t^{p+q}}\int_0^t y^{p-1}(t-y)^{(q+1)-1}dy$$
(17)

By Lemma 1.2

$$B(p,q+1) = \frac{1}{t^{p+q}} L^{-1}[L[t^{p-1}]L[t^{(q+1)-1}]]$$
(18)

By Lemma 2.1

$$B(p,q+1) = \frac{1}{t^{p+q}} L^{-1} \left[\frac{\Gamma(p)}{s^p} \frac{\Gamma(q+1)}{s^{q+1}} \right]$$
(19)

$$= \frac{1}{t^{p+q}} \Gamma(p) \Gamma(q+1) L^{-1}[\frac{1}{s^{p+q+1}}]$$
(20)

Since $L^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{\Gamma(n)}$,

$$B(p,q+1) = \frac{1}{t^{p+q}} \Gamma(p) \Gamma(q+1) \frac{t^{p+q}}{\Gamma(p+q+1)}$$
(21)

$$=\frac{q\Gamma\left(p\right)\Gamma\left(q\right)}{\left(p+q\right)\Gamma\left(p+q\right)}\tag{22}$$

$$=\frac{q}{(p+q)}B(p,q) \tag{23}$$

$$Also, B(p, q+1) = \frac{\Gamma(p)\Gamma(q+1)}{\Gamma(p+q+1)}$$
(24)

$$=\frac{qp\Gamma\left(p\right)\Gamma\left(q\right)}{p\Gamma\left(p+q+1\right)}\tag{25}$$

$$=\frac{q\Gamma\left(p+1\right)\Gamma\left(q\right)}{p\Gamma\left(p+1+q\right)}$$
(26)

$$=\frac{q}{p}B(p+1,q) \tag{27}$$

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3. Conclusion

The Laplace transform technique can be used to solve problems involving Beta function instead of probability theory.

References

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