International J. of Math. Sci. & Engg. Appls. (IJMSEA)
ISSN 0973-9424, Vol. 11 No. III (December, 2017), pp. 129-144

ON CHROMATIC TOPOLOGICAL INDICES OF CERTAIN WHEEL RELATED GRAPHS

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Abstract

The notion of chromatic topological and irregularity indices has been defined and studied in recent literature as an extended coloring version of some Zagreb indices. This paper deals with the chromatic topological and irregularity indices of certain cycle related graphs such as wheels, double wheels, helms and closed helms.

1. Introduction

Being a real number preserved under graph isomorphism, a *topological index* of graphs (see [10]) are extensively studied in recent literature on graph theory. These numerical quantities representing the structure of a graph has contributed much to the progress of mathematical chemistry as molecular descriptors and also have a plethora of other applications. A new research area has been initialized recently in [8] by interchanging

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Key Words : *Chromatic Zagreb indices, Chromatic irregularity indices.*
2010 AMS Subject Classification : 05C15,05C38.

the vertex degrees with minimal coloring, keeping up certain additional coloring conditions. The graphs considered here are finite, non-trivial, undirected, connected and without loops or multiple edges. For notation and terminology not explicitly defined here, see [3, 4, 7, 11].

Graph coloring is a mapping of the vertices of a graph under consideration to a set of colors $\mathcal{C} = \{c_1, c_2, \dots, c_\ell\}$. A *proper vertex coloring* of a graph G is a coloring in which adjacent vertices of G have different colors. The minimum number of colors required to apply a proper vertex coloring to G is called the *chromatic number* of G and is denoted $\chi(G)$. The set of all vertices of G which have the color c_i is named as the color class of that color c_i in G. The strength of the color class, denoted by $\theta(c_i)$ is the cardinality of each color class of color c_i. A vertex coloring consisting of the colors having minimum subscripts may be called a *minimum parameter coloring* (see [8]). A φ^--*coloring* of a graph G is a minimum parameter coloring $\mathcal{C} = \{c_1, c_2, c_3, \dots c_\ell\}$ of G in which maximum possible number of vertices are colored with c_1, maximum possible number of remaining uncolored vertices are colored with c_2, then the maximum possible number of remaining uncolored vertices are colored with c_3 and proceed in this manner until all vertices are colored (see [8]). In a similar manner, if c_ℓ is assigned to maximum possible number of vertices first, then $c_{\ell-1}$ is assigned to the maximum possible number of remaining uncolored vertices and proceed in this manner until all vertices are colored, then such a coloring is called φ^+-*coloring* of G (see [8]).

For computational convenience, we define function $\zeta : V(G) \to \{1, 2, 3, \dots, \ell\}$ such that $\zeta(v_i) = s \iff \varphi(v_i) = c_s, c_s \in \mathcal{C}$. The total number of edges with end points having colors c_t and c_s is denoted by η_{ts}, where $t < s, 1 \le t, s \le \chi(G)$.

Analogous to the notions of Zagreb and irregularity indices of graphs (see [1, 6, 12, 13]), the two chromatic Zagreb indices $M_1^{\varphi_t}(G)$ and $M_2^{\varphi_t}(G)$ and the chromatic irregularity indices $M_3^{\varphi_t}(G)$ of a graph G corresponding to a proper coloring $\mathcal{C} = \{c_i : 1 \le i \le n\}$ have been defined in (see [8]) as follows:

(i) $M_1^{\varphi_t}(G) = \sum\limits_{i=1}^{n} (\zeta(v_i))^2$;

(ii) $M_2^{\varphi_t}(G) = \sum\limits_{i=1}^{n-1} \sum\limits_{j=2}^{n} (\zeta(v_i) \cdot \zeta(v_j)),\ v_iv_j \in E(G)$;

(iii) $M_3^{\varphi_t}(G) = \sum_{i=1}^{n-1} \sum_{j=2}^{n} |\zeta(v_i) - \zeta(v_j)|, \ v_i v_j \in E(G).$

The *chromatic total irregularity index* of a graph G has been defined in [9] as

$$M_4^{\varphi_t}(G) = \frac{1}{2} \sum_{i=1}^{n-1} \sum_{j=2}^{n} |\zeta(v_i) - \zeta(v_j)|, \ v_i, v_j \in V(G).$$

The minimum and maximum values of the above chromatic topological indices are denoted by $M_i^{\varphi^-}(G)$ and $M_i^{\varphi^+}(G)$ respectively.

Motivated by the studies mentioned above, we study the chromatic Zagreb indices and chromatic irregularity indices of certain fundamental graph classes in the following discussion.

2. New Results

A wheel graph is defined as $W_n = C_n + K_1$. The following theorem determines the chromatic topological indices of a wheel graph.

Theorem 2.1 : For a wheel $W_n = C_n + K_1$, we have

(i) $M_1^{\varphi^-}(W_n) = \begin{cases} \frac{5n+18}{2}; & \text{if } n \text{ is even} \\ \frac{5n+45}{2}; & \text{if } n \text{ is odd;} \end{cases}$

(ii) $M_2^{\varphi^-}(W_n) = \begin{cases} \frac{13n}{2}; & \text{if } n \text{ is even} \\ \frac{13n+31}{2}; & \text{if } n \text{ is odd;} \end{cases}$

(iii) $M_3^{\varphi^-}(W_n) = \begin{cases} \frac{5n}{2}; & \text{if } n \text{ is even} \\ \frac{5(n+1)}{2}; & \text{if } n \text{ is odd;} \end{cases}$

(iv) $M_4^{\varphi^-}(W_n) = \begin{cases} \frac{n^2+6n}{8}; & \text{if } n \text{ is even} \\ \frac{n^2+14n-11}{8}; & \text{if } n \text{ is odd.} \end{cases}$

Proof : Note that a wheel graph W_n has chromatic number 3 when n is even and chromatic number 4 when n is odd. Let $v_1, v_2, \cdots v_n$ be the vertices of C_n on the rim of the wheel and u be the central vertex.

Part (i): In order to calculate $M_1^{\varphi^-}$ of W_n, we consider the following cases.

Case-1: If n is even, then the rim vertices of W_n can be coloured using two colors, say c_1 and c_2 and the central vertex by c_3. Hence, we have $\theta(c_1) = \theta(c_2) = \frac{n}{2}$ and

$\theta(c_3) = 1$. Therefore, the corresponding chromatic Zagreb index is given by $M_1^{\varphi^-}(W_n) = \sum_{i=1}^{n}(\zeta(v_i))^2 = \frac{n}{2}(1^2+2^2) + 1 \cdot 3^2 = \frac{5n+18}{2}$.

Case-2: Let n be odd. Then, the $\frac{n-1}{2}$ rim vertices each can be colored using c_1 and c_2, the remaining single rim vertex gets the color c_3 and the central vertex gets the color c_4. Therefore, $\theta(c_1) = \theta(c_2) = \frac{n-1}{2}$ and $\theta(c_3) = \theta(c_4) = 1$. Then, we have $M_1^{\varphi^-}(W_n) = \sum_{i=1}^{4}\theta(c_i) \cdot i^2 = \frac{n-1}{2}(1^2+2^2) + (3^2+4^2) = \frac{5n+45}{2}$.

Part (ii): We color the vertices as mentioned in part(i). Now consider the following cases:

Case- 1: If n is even, we observe that $\eta_{12} = n, \eta_{23} = \eta_{13} = \frac{n}{2}$. Hence,

$$M_2^{\varphi^-}(W_n) = \sum_{1\le t,s\le 3}^{t<s} ts\eta_{ts} = 2n + 3n + \frac{3n}{2} = \frac{13n}{2}.$$

Case- 2: If n is odd, then $\eta_{12} = n-2$, $\eta_{13} = \eta_{23} = \eta_{34} = 1$, $\eta_{14} = \eta_{24} = \frac{n-1}{2}$. Hence, we have the sum $M_2^{\varphi^-}(W_n) = \sum_{1\le t,s\le 4}^{t<s} ts\eta_{ts} = 2(n-2) + \frac{3(n-1)}{2} + 3(n-1) + 4 + 8 + 12 = \frac{13n+31}{2}$.

Part (iii): We calculate the minimum irregularity measurement by considering the following cases:

Case- 1: Let n be even. Then, in this case, $\eta_{12} + \eta_{23} = \frac{3n}{2}$ edges contribute the distance 1 to the total sum, while $\eta_{13} = \frac{n}{2}$ edges contribute the distance 2. Then, we have $M_3^{\varphi^-}(W_n) = \sum_{i=1}^{n-1}\sum_{j=2}^{n}|\zeta(v_i) - \zeta(v_j)| = \frac{3n}{2} + \frac{n}{2} = \frac{5n}{2}$.

Case- 2: Let n be odd. Here, $\eta_{12}+\eta_{23}+\eta_{34}$ edges contribute 1 to the color distance, $\eta_{13}+\eta_{24}$ edges contribute 2, while η_{14} edges contribute 3. Then, $M_3^{\varphi^-}(W_n) = \sum_{i=1}^{n-1}\sum_{j=2}^{n}|\zeta(v_i) - \zeta(v_j)| = (n-2) + (n-1) + \frac{n-1}{2} + 3 + 2 + 1 = \frac{5(n+1)}{2}$.

Part (iv): To calculate the chromatic total irregularity indices of wheel graphs, we have to consider all the possible vertex pairs and all color combinations contributing non zero distances are considered according to the following two cases:

Case- 1: Let n be even. The combinations possible are charted as $\{1,2\},\{2,3\}$ contributing a distance 1 and $\{1,3\}$ contributing 2. Observe that $\theta(c_1) = \theta(c_2) = \frac{n}{2}$ and

$\theta(c_3) = 1$. Thus, we have

$$\begin{aligned} M_4^{\varphi^-}(W_n) &= \frac{1}{2} \sum_{u,v \in V(W_n)} |\zeta(u) - \zeta(v)| \\ &= \frac{n^2}{4} + \frac{n}{2} + n = \frac{n^2 + 6n}{8}. \end{aligned}$$

Case- 2: Let n be odd. Here, the possible combinations which contributes to the color distances are $\{1,2\}, \{2,3\}$ and $\{3,4\}$ contributing 1, $\{1,3\}$ and $\{2,4\}$ contributing 2 and $\{1,4\}$ contributing 3. We calculate the chromatic total irregularity index as given below:

$$\begin{aligned} M_4^{\varphi^-}(W_n) &= \frac{1}{2} \sum_{u,v \in V(W_n)} |\zeta(u) - \zeta(v)| \\ &= \frac{(n-1)^2}{4} + 4(n-1) + 1 = \frac{n^2 + 14n - 11}{8} \end{aligned}$$

□

Instead of φ^- coloring, one can also work with a φ^+ coloring of wheels using minimum parameter coloring. The results obtained are charted below as next theorem.

Theorem 2.2 : For a wheel $W_n = C_n + K_1$, we have

(i) $M_1^{\varphi^+}(W_n) = \begin{cases} \frac{13n+2}{2}; & \text{if } n \text{ is even} \\ \frac{25n-15}{2}; & \text{if } n \text{ is odd;} \end{cases}$

(ii) $M_2^{\varphi^+}(W_n) = \begin{cases} \frac{17n}{2}; & \text{if } n \text{ is even} \\ 19n - 22; & \text{if } n \text{ is odd;} \end{cases}$

(iii) $M_3^{\varphi^+}(W_n) = \begin{cases} \frac{5n}{2}; & \text{if } n \text{ is even} \\ \frac{5n+7}{2}; & \text{if } n \text{ is odd;} \end{cases}$

(iv) $M_4^{\varphi^+}(W_n) = \begin{cases} \frac{n^2+6n}{8}; & \text{if } n \text{ is even} \\ \frac{n^2+14n-11}{8}; & \text{if } n \text{ is odd.} \end{cases}$

Proof : Here, we consider a φ_+ coloring of wheel to obtain desired results. When n is even, the vertices $S_1 = \{v_1, v_3, \cdots v_{n-1}\}$ and $S_2 = \{v_2, v_4, \cdots v_n\}$ forms the two maximum independent sets with same cardinality $\frac{n}{2}$. We color them with maximum colors c_3 and c_2 respectively. The remaining central vertex u is colored with c_1. Then $\eta_{12} = \eta_{13} = \frac{n}{2}$ and $\eta_{23} = n$.

Let n be odd, we have chromatic number 4. Here the maximum independent sets $S_1 = \{v_1, v_3, \cdots v_{n-1}\}$, $S_2 = \{v_2, v_4, \cdots v_{n-2}\}$ have same cardinality $\frac{n-1}{2}$ and colored with c_4 and c_3 where the vertices v_n and u are colored with c_1 and c_2 respectively to get maximum values. Thus $\eta_{12} = \eta_{13} = \eta_{14} = 1$, $\eta_{23} = \eta_{24} = \frac{n-1}{2}$ and $\eta_{34} = n-2$. The balance of the proof follows exactly as mentioned in the proof Theorem 2.1. □

Chromatic Topological Indices of Double Wheels

Joining all the vertices of two disjoint cycles to an external vertex will give us the double wheel graph. A *double wheel graph* DW_n is a graph defined by $2C_n + K_1$. The following result discusses the chromatic topological indices of a double wheel graph by using φ_- coloring.

Theorem 3.1 : For a double wheel $DW_n = 2C_n + K_1$, we have

(i) $M_1^{\varphi^-}(DW_n) = \begin{cases} 5n+9; & \text{if } n \text{ is even} \\ 5n+29; & \text{if } n \text{ is odd;} \end{cases}$

(ii) $M_2^{\varphi^-}(DW_n) = \begin{cases} 13n; & \text{if } n \text{ is even} \\ 16n+22; & \text{if } n \text{ is odd;} \end{cases}$

(iii) $M_3^{\varphi^-}(DW_n) = \begin{cases} 5n; & \text{if } n \text{ is even} \\ 7n-1; & \text{if } n \text{ is odd;} \end{cases}$

(iv) $M_4^{\varphi^-}(DW_n) = \begin{cases} \frac{n^2+3n}{2}; & \text{if } n \text{ is even} \\ \frac{n^2+9n-8}{2}; & \text{if } n \text{ is odd.} \end{cases}$

Proof : As we know, the double wheel $DW_n = 2C_n + K_1$ has chromatic number 3 when n is even and chromatic number 4 when n is odd. Let $v_1, v_2, \cdots v_n$ be the vertices on the outer cycle, $u_1, u_2, \cdots u_n$ be the vertices on the inner cycle and u be the central vertex. To obtain the minimum values of the chromatic topological indices we follow the φ^- coloring pattern to DW_n as described below.

Let n be even. When n is even, we can find two maximum independent sets $S_1 = \{v_1, v_3, \cdots v_{n-1}, u_1, u_3, \cdots u_{n-1}\}$, $S_2 = \{v_2, v_4, \cdots v_n, u_2, u_4, \cdots u_n\}$ with same cardinality n taking alternative vertices from both cycles of DW_n. We color them with minimum colors c_1 and c_2 respectively. We color the central vertex u with c_3.

Let n be odd. Here $S_1 = \{v_1, v_3, \cdots v_{n-1}, u_1, u_3, \cdots u_{n-1}\}$, $S_2 = \{v_2, v_4, \cdots v_{n-2}, u_2, u_4, \cdots u_{n-2}\}$ are the two maximum independent sets. Since S_1, S_2 have same cardinality $n-1$ we color it with c_1 and c_2. The color c_3 is assigned to vertices v_n, u_n and the central vertex u is colored with c_4. Now we proceed to the following four parts of the theorem :

Part (i): In order to find $M_1^{\varphi^-}$ of DW_n, we first color the vertices as mentioned above and then proceed to consider the following cases.

Case-1: Let n be even, then we have $\theta(c_1) = \theta(c_2) = n$ and $\theta(c_3) = 1$. Therefore, the corresponding chromatic Zagreb index is given by

$$M_1^{\varphi^-}(DW_n) = \sum_{i=1}^{n}(\zeta(v_i))^2 = n + 4n + 9 = 5n + 9.$$

Case-2: Let n be odd. Then, we have $\theta(c_1) = \theta(c_2) = n-1, \theta(c_3) = 2$ and $\theta(c_4) = 1$. Now, by the definition of first chromatic Zagreb index, we have

$$M_1^{\varphi^-}(DW_n) = \sum_{i=1}^{4}(\theta(c_i))i^2 = (n-1) + 4(n-1) + 18 + 16 = 5n + 29.$$

Part (ii): We color the vertices as per the instructions in introductory part for even and odd cases of n. Now consider the following cases:

Case- 1: Let n be even. Here we see that $\eta_{12} = 2n, \eta_{23} = \eta_{13} = n$. The definition of second chromatic Zagreb index, gives the sum

$$M_2^{\varphi^-}(DW_n) = \sum_{1 \le t,s \le \chi(DW_n)}^{t<s} ts\eta_{ts} = 4n + 3n + 6n = 13n.$$

Case- 2: Let n be odd. Here we see that $\eta_{12} = 2(n-2), \eta_{13} = \eta_{23} = \eta_{34} = 2$, $\eta_{14} = \eta_{24} = n-1$. Hence, we have the sum

$$M_2^{\varphi^-}(DW_n) = \sum_{1 \le t,s \le \chi(W_n)}^{t<s} ts\eta_{ts} = 4(n-2) + 6 + 12 + 24 + 4(n-1) + 8(n-1) = 16n + 22.$$

Part (iii): To find the minimum irregularity measurement, consider the following cases:

Case- 1: Let n be even. Here $\eta_{12} + \eta_{23} = 3n$ edges contribute the distance 1 to the total summation while $\eta_{13} = n$ contribute the distance 2. The result follows from the following calculations:

$$M_3^{\varphi^-}(DW_n) = \sum_{i=1}^{n-1}\sum_{j=2}^{n}|\zeta(v_i) - \zeta(v_j)| = 2n + 2n + n = 5n.$$

Case- 2: Let n be odd. Here we see that, $\eta_{12}+\eta_{23}+\eta_{34}$ edges contribute 1 to the color distance, $\eta_{13}+\eta_{24}$ edges contribute 2, while η_{14} edges contribute 3. Then the result follows from the following calculations:

$$M_3^{\varphi^-}(DW_n)=\sum_{i=1}^{n-1}\sum_{j=2}^{n}|\zeta(v_i)-\zeta(v_j)|=2(n-2)+4+2+2+3(n-1)+2(n-1)=7n-1.$$

Part (iv): To calculate the total irregularity of DW_n, all the possible vertex pairs from DW_n have to be considered and their possible color distances are determined. The possibility of the vertex pairs which contribute to the color distance can be classified according to the following two cases.

Case- 1: Let n be even. The combinations possible are charted as $\{1,2\},\{2,3\}$ contributing 1 and $\{1,3\}$ contributing 2. Observe that $\theta(c_1)=\theta(c_2)=n$ and $\theta(c_3)=1$. Thus, we have

$$\begin{aligned} M_4^{\varphi^-}(DW_n) &= \frac{1}{2}\sum_{u,v\in V(DW_n)}|\zeta(u)-\zeta(v)| \\ &= \frac{n^2+3n}{2}. \end{aligned}$$

Case- 2: Let n be odd. Here the possible combinations which contributes to the color distances are $\{1,2\},\{2,3\},\{3,4\}$ contributing 1, $\{1,3\},\{2,4\}$ contributing 2 and $\{1,4\}$ contributing 3. We calculate the total irregularity as given below:

$$\begin{aligned} M_4^{\varphi^-}(DW_n) &= \frac{1}{2}\sum_{u,v\in V(DW_n)}|\zeta(u)-\zeta(v)| \\ &= \frac{n^2+9n-8}{2} \end{aligned}$$

□

Using the minimum parameter coloring we can also work on φ_+ coloring of double wheels. Next theorem deals with this matter.

Theorem 3.2 : For a double wheel $DW_n=2C_n+K_1$, we have

(i) $M_1^{\varphi^+}(DW_n)=\begin{cases}13n+1; & \text{if } n \text{ is even}\\ 25n-16; & \text{if } n \text{ is odd;}\end{cases}$

(ii) $M_2^{\varphi^+}(DW_n)=\begin{cases}17n; & \text{if } n \text{ is even}\\ 31n-23; & \text{if } n \text{ is odd;}\end{cases}$

(iii) $M_3^{\varphi^+}(DW_n) = \begin{cases} 5n; & \text{if } n \text{ is even} \\ 7n-1; & \text{if } n \text{ is odd;} \end{cases}$

(iv) $M_4^{\varphi^+}(W_n) = \begin{cases} \frac{n^2+3n}{2}; & \text{if } n \text{ is even} \\ \frac{n^2+9n-8}{2}; & \text{if } n \text{ is odd.} \end{cases}$

□

Here, we follow φ_+ coloring of double wheel to obtain desired results.
When n is even, we have the color classes of c_2 and c_3 with same cardinality n. The remaining central vertex u is colored with c_1. Then $\eta_{12} = \eta_{13} = n$ and $\eta_{23} = 2n$
Let n be odd, we have chromatic number 4. Here we have, $\theta(c_4) = \theta(c_3) = n-1, \theta(c_2) = 2$ and $\theta(c_1) = 1$. Thus $\eta_{12} = \eta_{23} = \eta_{24} = 2$, $\eta_{13} = \eta_{14} = n-1$ and $\eta_{34} = 2(n-2)$.
The balance of the proof follows exactly as mentioned in the proof of Theorem 3.1. □

4. Chromatic Topological Indices of Helm Graph

A *helm graph* H_n is a graph obtained by attaching a pendant edge to every vertex of the rim C_n of a wheel graph W_n. The following result provides the chromatic indices of the helm graphs with φ_- coloring.

Theorem 4.1 : For a helm graph H_n, we have

(i) $M_1^{\varphi^-}(H_n) = \begin{cases} \frac{15n+2}{2}; & \text{if } n \text{ is even} \\ \frac{15n+21}{2}; & \text{if } n \text{ is odd;} \end{cases}$

(ii) $M_2^{\varphi^-}(H_n) = \begin{cases} 11n; & \text{if } n \text{ is even} \\ 11n+11; & \text{if } n \text{ is odd;} \end{cases}$

(iii) $M_3^{\varphi^-}(H_n) = \begin{cases} 4n; & \text{if } n \text{ is even} \\ 4n+4; & \text{if } n \text{ is odd;} \end{cases}$

(iv) $M_4^{\varphi^-}(H_n) = \begin{cases} \frac{7n^2+6n}{8}; & \text{if } n \text{ is even} \\ \frac{7n^2+18n-1}{8}; & \text{if } n \text{ is odd.} \end{cases}$

Proof : Let $v_1, v_2, \cdots v_n$ be the vertices on the rim of the wheel, $u_1, u_2, \cdots u_n$ be the pendant vertices and u be the central vertex. As we know, the helm graph has chromatic number 3 when n is even and chromatic number 4 when n is odd. To obtain the minimum values of the chromatic topological indices we follow the φ^- coloring pattern to H_n as described below.

Let n be even. When n is even, the pendant vertices along with the central vertex u comprises the largest independent set S_1 and it is colored with c_1. Now we can find two more independent sets S_2, S_3 with same cardinality $\frac{n}{2}$ taking alternative vertices on the rim of the helm graph, H_n. We color them with minimum colors c_2 and c_3 respectively. Let n be odd. Here again the set comprising of the pendant vertices and the central vertex form the largest independent set S_1 and it is colored with c_1. The balance vertices are on the rim of the wheel. We can find two more independent sets S_2, S_3 with same cardinality $\frac{n-1}{2}$ taking alternative vertices on the rim of the helm graph, H_n. Also, one more vertex forms S_4 and is colored with the color c_4. Now we proceed to the four parts of the theorem.

Part (i): In order to find $M_1^{\varphi^-}$ of H_n, we first color the vertices as mentioned above and then proceed to consider the following cases:

Case-1: Let n be even, then we have $\theta(c_1) = n+1$ and $\theta(c_2) = \theta(c_3) = \frac{n}{2}$. Therefore, the corresponding chromatic topological index is given by

$$M_1^{\varphi^-}(H_n) = \sum_{i=1}^{n} (\zeta(v_i))^2 = \frac{15n+2}{2}.$$

Case-2: Let n be odd. Then, we have $\theta(c_1) = n+1$, $\theta(c_2) = \theta(c_3) = \frac{n-1}{2}$ and $\theta(c_4) = 1$. Now, by the definition of first chromatic Zagreb index, we have

$$M_1^{\varphi^-}(H_n) = \sum_{i=1}^{4} (\theta(c_i))i^2 = \frac{15n+21}{2}.$$

Part (ii): We color the vertices as per the instructions in introductory part for even and odd cases of n. Now consider the following cases:

Case- 1: Let n be even. Here we see that $\eta_{12} = \eta_{23} = \eta_{13} = n$. The definition of second chromatic Zagreb index, gives the sum

$$M_2^{\varphi^-}(H_n) = \sum_{1 \le t,s \le \chi(H_n)}^{t<s} ts\eta_{ts} = 11n.$$

Case- 2: Let n be odd. Here we see that $\eta_{12} = \eta_{13} = n-1, \eta_{23} = n-2, \eta_{14} = 2,$
$\eta_{34} = \eta_{24} = 1$. Hence, we have the sum

$$M_2^{\varphi^-}(H_n) = \sum_{1 \le t,s \le \chi(H_n)}^{t<s} ts\eta_{ts} = 11n + 11.$$

Part (iii): To find the minimum irregularity measurement, consider the following cases:
Case- 1: Let n be even. Here $\eta_{12} + \eta_{23} = 2n$ edges contribute the distance 1 to the total summation while $\eta_{13} = n$ contribute the distance 2. The result follows from the following calculations:

$$M_3^{\varphi^-}(H_n) = \sum_{i=1}^{n-1}\sum_{j=2}^{n} |\zeta(v_i) - \zeta(v_j)| = 4n.$$

Case- 2: Let n be odd. Here we see that, $\eta_{12} + \eta_{23} + \eta_{34}$ edges contribute 1 to the color distance, $\eta_{13} + \eta_{24}$ edges contribute 2, while η_{14} edges contribute 3. Then the result follows from the following calculations:

$$M_3^{\varphi^-}(H_n) = \sum_{i=1}^{n-1}\sum_{j=2}^{n} |\zeta(v_i) - \zeta(v_j)| = 4n + 4.$$

Part (iv): To calculate the total irregularity of H_n, all the possible vertex pairs from H_n have to be considered and their possible color distances are determined. The possibility of the vertex pairs which contribute to the color distance can be classified according to the following two cases.
Case- 1: Let n be even. The combinations possible are charted as $\{1, 2\}, \{2, 3\}$ contributing 1 and $\{1, 3\}$ contributing 2. Observe that $\theta(c_1) = \theta(c_2) = \theta(c_3) = n$. Thus, we have

$$\begin{aligned} M_4^{\varphi^-}(H_n) &= \frac{1}{2}\sum_{u,v \in V(H_n)} |\zeta(u) - \zeta(v)| \\ &= \frac{7n^2 + 6n}{8}. \end{aligned}$$

Case- 2: Let n be odd. Here the possible combinations which contributes to the color distances are $\{1, 2\}, \{2, 3\}, \{3, 4\}$ contributing 1, $\{1, 3\}, \{2, 4\}$ contributing 2 and $\{1, 4\}$ contributing 3. We calculate the total irregularity as given below:

$$\begin{aligned} M_4^{\varphi^-}(H_n) &= \frac{1}{2}\sum_{u,v \in V(H_n)} |\zeta(u) - \zeta(v)| \\ &= \frac{7n^2 + 18n - 1}{8} \end{aligned}$$

□

Using the minimum parameter coloring we can also work on φ_+ coloring of helm graphs. Next theorem deals with this matter.

Theorem 4.2 : For a helm graph H_n, we have

(i) $M_1^{\varphi^+}(H_n) = \begin{cases} \frac{23n+18}{2}; & \text{if } n \text{ is even} \\ \frac{45n+21}{2}; & \text{if } n \text{ is odd;} \end{cases}$

(ii) $M_2^{\varphi^+}(H_n) = \begin{cases} 11n; & \text{if } n \text{ is even} \\ 26n-19; & \text{if } n \text{ is odd;} \end{cases}$

(iii) $M_3^{\varphi^+}(H_n) = \begin{cases} 4n; & \text{if } n \text{ is even} \\ 4n+4; & \text{if } n \text{ is odd;} \end{cases}$

(iv) $M_4^{\varphi^+}(H_n) = \begin{cases} \frac{7n^2+6n}{8}; & \text{if } n \text{ is even} \\ \frac{7n^2+16n+1}{8}; & \text{if } n \text{ is odd.} \end{cases}$

Proof : The proof follows exactly as mentioned in the proof Theorem 4.1. □

5. Chromatic Topological Indices of Closed Helm Graphs

A *closed helm graph* CH_n is a graph obtained from the helm graph H_n, by joining a pendant vertex v_i to the pendant vertex v_{i+1}, where $1 \leq i \leq n$ and $v_{n+i} = v_i$. That is, the pendant vertices in H_n induce a cycle in CH_n. Then, we have the following results about chromatic topological indices of the closed helm graphs.

Theorem 5.1 : For the closed helm graph $CH_n =$, we have

(i) $M_1^{\varphi^-}(CH_n) = \begin{cases} \frac{5n+9}{2}; & \text{if } n \text{ is even} \\ \frac{5n+29}{2}; & \text{if } n \text{ is odd;} \end{cases}$

(ii) $M_2^{\varphi^-}(CH_n) = \begin{cases} \frac{21n}{2}; & \text{if } n \text{ is even} \\ \frac{21n+53}{2}; & \text{if } n \text{ is odd;} \end{cases}$

(iii) $M_3^{\varphi^-}(CH_n) = \begin{cases} \frac{9n}{2}; & \text{if } n \text{ is even} \\ \frac{9n+11}{2}; & \text{if } n \text{ is odd;} \end{cases}$

(iv) $M_4^{\varphi^-}(CH_n) = \begin{cases} \frac{n^2+3n}{2}; & \text{if } n \text{ is even} \\ \frac{n^2+9n-8}{2}; & \text{if } n \text{ is odd.} \end{cases}$

Proof : It is so clear that the closed helm graph CH_n has chromatic number 3 and 4 as n possess values odd and even respectively. In CH_n let's put $v_1, v_2, \cdots v_n$ be the

vertices on the outer cycle , $u_1, u_2, \cdots u_n$ be the vertices on the inner cycle and u be the central vertex. Now we apply the φ^- coloring pattern to CH_n as described below.
When n is even, both the outer and inner cycles can be colored with c_1 and c_2 alternatively such that both color classes have cardinality n and we color the central vertex u with color c_3. Now let n be odd. Here we color the vertices $\{v_1, v_3, \cdots v_{n-2}, u_2, u_4, \cdots u_{n-2}\}$ with color c_1 and $\{v_2, v_4, \cdots v_{n-1}, u_3, u_5, \cdots u_n\}$ with color c_2. The vertices $\{v_n, u\}$ are colored with color c_3 and the vertex u_1 with color c_4. Now we proceed to the four parts of the theorem.
Part (i): In order to find $M_1^{\varphi^-}$ of CH_n, we first color the vertices as mentioned above and then proceed to consider the following cases.
Case-1: Let n be even, then we have $\theta(c_1) = \theta(c_2) = n$ and $\theta(c_3) = 1$. Therefore, the corresponding chromatic topological index is given by

$$M_1^{\varphi^-}(CH_n) = \sum_{i=1}^{n} (\zeta(v_i))^2 = 5n + 9.$$

Case-2: Let n be odd. Then, we have $\theta(c_1) = \theta(c_2) = n-1, \theta(c_3) = 2$ and $\theta(c_4) = 1$. Now, by the definition of first chromatic Zagreb index, we have

$$M_1^{\varphi^-}(CH_n) = \sum_{i=1}^{4} (\theta(c_i))i^2 = 5n + 29.$$

Part (ii): We color the vertices as per the instructions in the introductory part for even and odd cases of n and consider the following cases:
*Case- 1:*Let n be even. Here we see that $\eta_{12} = 3n, \eta_{23} = \eta_{13} = \frac{n}{2}$. The definition of second chromatic Zagreb index, gives the sum

$$M_2^{\varphi^-}(CH_n) = \sum_{1 \le t,s \le \chi(CH_n)}^{t<s} ts\eta_{ts} = 6n + \frac{3n}{2} + \frac{6n}{2} = \frac{21n}{2}.$$

Case- 2: Let n be odd. Here we see that $\eta_{12} = 3(n-2), \eta_{13} = \frac{n+1}{2}, \eta_{23} = \frac{n+3}{2}$,
$\eta_{14} = 2, \eta_{34} = \eta_{24} = 1$. Hence, we have the sum

$$M_2^{\varphi^-}(CH_n) = \sum_{1 \le t,s \le \chi(CH_n)}^{t<s} ts\eta_{ts} = \frac{21n + 53}{2}.$$

Part (iii): To find the minimum irregularity measurement, consider the following cases:

Case- 1: Let n be even. Here $\eta_{12} + \eta_{23}$ edges contribute the distance 1 to the total summation while η_{13} contribute the distance 2. The result follows from the following calculations:

$$M_3^{\varphi^-}(CH_n) = \sum_{i=1}^{n-1}\sum_{j=2}^{n} |\zeta(v_i) - \zeta(v_j)| = 3n + n + \frac{n}{2} = \frac{9n}{2}.$$

Case- 2: Let n be odd. Here we see that, $\eta_{12} + \eta_{23} + \eta_{34}$ edges contribute 1 to the color distance, $\eta_{13} + \eta_{24}$ edges contribute 2, while η_{14} edges contribute 3. Then the result follows from the following calculations:

$$M_3^{\varphi^-}(CH_n) = \sum_{i=1}^{n-1}\sum_{j=2}^{n} |\zeta(v_i) - \zeta(v_j)| = 3(n-2) + (n+1) + \frac{n+3}{2} + 9 = \frac{9n+11}{2}.$$

Part (iv): To calculate the total irregularity of CH_n, all the possible vertex pairs from CH_n have to be considered and their possible color distances are determined. The possibility of the vertex pairs which contribute to the color distance can be classified according to the following two cases.

Case- 1: Let n be even. The combinations possible are charted as $\{1, 2\}, \{2, 3\}$ contributing 1 and $\{1, 3\}$ contributing 2. Observe that $\theta(c_1) = \theta(c_2) = n$ and $\theta(c_3) = 1$. Thus, we have

$$\begin{aligned} M_4^{\varphi^-}(CH_n) &= \frac{1}{2}\sum_{u,v\in V(CH_n)} |\zeta(u) - \zeta(v)| \\ &= \frac{n^2+3n}{2}. \end{aligned}$$

Case- 2: Let n be odd. Here the possible combinations which contributes to the color distances are $\{1, 2\}, \{2, 3\}, \{3, 4\}$ contributing 1, $\{1, 3\}, \{2, 4\}$ contributing 2 and $\{1, 4\}$ contributing 3. We calculate the total irregularity as given below:

$$\begin{aligned} M_4^{\varphi^-}(CH_n) &= \frac{1}{2}\sum_{u,v\in V(CH_n)} |\zeta(u) - \zeta(v)| \\ &= \frac{n^2+9n-8}{2} \end{aligned}$$

□

Using the minimum parameter coloring we can also work on φ_+ coloring of closed helm graphs. Next theorem deals with this matter.

Theorem 5.2 : For a closed helm graph CH_n, we have

(i) $M_1^{\varphi^+}(CH_n) = \begin{cases} 13n+1; & \text{if } n \text{ is even} \\ 25n-16; & \text{if } n \text{ is odd;} \end{cases}$

(ii) $M_2^{\varphi^+}(CH_n) = \begin{cases} \frac{23n}{2}; & \text{if } n \text{ is even} \\ 43n-46; & \text{if } n \text{ is odd;} \end{cases}$

(iii) $M_3^{\varphi^+}(CH_n) = \begin{cases} \frac{9n}{2}; & \text{if } n \text{ is even} \\ \frac{9n+11}{2}; & \text{if } n \text{ is odd;} \end{cases}$

(iv) $M_4^{\varphi^+}(CH_n) = \begin{cases} \frac{n^2+3n}{2}; & \text{if } n \text{ is even} \\ \frac{n^2+9n-8}{2}; & \text{if } n \text{ is odd.} \end{cases}$

Proof : The proof follows exactly as mentioned in the proof Theorem 5.1. □

6. Conclusion

Chromatic topological indices can find a variety of applications in mathematical chemistry, optimization techniques, distribution theory and even in sociology. An overview of chromatic Zagreb indices and irregularity indices of some cycle related graphs are provided in this paper. More research areas will be opened if other graph classes like antiprisms and antiladders are considered. Also, comparative study on chromatic Zagreb indices and irregularity indices of graph classes and their operations will be interesting. One can also work on chromatic Zagreb indices and irregularity indices of some associated graphs such as line graphs, subdivision of graphs, total graphs, etc. Even the chromatic version of other topological indices gives fresh areas of research with tremendous applications.

Acknowledgment

The first author would like to acknowledge the academic help rendered by Centre for Studies in Discrete Mathematics, Vidya Academy of Science and Technology, Thrissur, Kerala, India.

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