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GENERALIZED TOTALLY BOUNDED FUNCTIONS

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Abstract

A real-valued function defined on $[-1, 1]$ is said to be totally bounded if all of its Lagrange interpolants are bounded. In this paper we introduce a generalization of total boundedness using generalized Lagrange interpolants. Also it will be proved that the collection of all generalized totally bounded functions forms a Banach space under the induced norm.

1. Introduction

In [3] T. Popoviciu introduced generalized divided differences based on any complete Tchebycheff system. Interpolation of functions in an extended complete Tchebycheff Sapce (ECT-space) can be done by using generalized divided differences. If f is any function defined on a closed interval $[a, b]$, then an explicit expression for the generalized polynomial interpolating to f at given points can be derived in a way similar to the Newton form for Lagrange interpolating polynomials.

Key Words : *Generalized polynomial, Generalized Lagrange interpolants, Generalized totally bounded functions.*

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Alan. L. Horwitz and Lee. A. Rubel in [1] introduced totally bounded functions on [-1,1]. We generalize totally bounded functions by using generalized Lagrange interpolants. Definitions and results from theory of Tchebycheff spaces and generalized divided differences are discussed in the next section. In section 3, we introduce generalized totally bounded functions and prove our main result there. Throughout this paper, we denote the closed bounded interval $[a,b]$ by I.

2. Preliminaries

We refer to chapters 2 and 9 of [4] for the definitions and results in this section.

Definiton 2.1 : Let u_1, u_2, ..., u_m be real valued functions defined on $I=[a,b]$ and let $x_1 \leq x_2 \leq \ldots \leq x_m$ be points in I. The collocation matrix associated with $\{u_i\}_1^m$ and $\{x_i\}_1^m$ is denoted by $M\begin{pmatrix} x_1, & \ldots, & x_{m-1}, & x_m \\ u_1, & \ldots, & u_{m-1}, & u_m \end{pmatrix}$ and is defined by

$$M\begin{pmatrix} x_1, & \ldots, & x_{m-1}, & x_m \\ u_1, & \ldots, & u_{m-1}, & u_m \end{pmatrix} = [D^{d_i}u_j(x_i)]_{i,j=1}^m$$

where $d_i = max\{j : x_i = x_{i-j}\}, i = 1,2,\ldots m$, provided the ${d_i}^{th}$ derivative of u_j exists at the points x_i, $i,j=1,2,\ldots m$

Remark 2.1 : In the above definition, if the points x_1, x_2, ..., x_m are all distinct,then the collocation matrix becomes

$$M\begin{pmatrix} x_1, & \ldots, & x_{m-1}, & x_m \\ u_1, & \ldots, & u_{m-1}, & u_m \end{pmatrix} = \begin{bmatrix} u_1(x_1) & \ldots & u_{m-1}(x_1) & u_m(x_1) \\ u_1(x_2) & \ldots & u_{m-1}(x_2) & u_m(x_2) \\ \ldots & \ldots & \ldots & \ldots \\ u_1(x_m) & \ldots & u_{m-1}(x_m) & u_m(x_m) \end{bmatrix}$$

The Determinant associated with the matrix $M\begin{pmatrix} x_1, & \ldots, & x_{m-1}, & x_m \\ u_1, & \ldots, & u_{m-1}, & u_m \end{pmatrix}$ is denoted by $D\begin{pmatrix} x_1, & \ldots, & x_{m-1}, & x_m \\ u_1, & \ldots, & u_{m-1}, & u_m \end{pmatrix}$

Definiton 2.2 : Let $U_m = \{u_i\}_{i=1}^m$ be any collection of functions in $C^{m-1}(I)$, the space of all $(m-1)$-times continuously differentiable functions defined on I. U_m is called an extended Tchebycheff system (ET-system) on I if the determinants associated with the collocation matrix $M\begin{pmatrix} x_1, & \ldots, & x_{m-1}, & x_m \\ u_1, & \ldots, & u_{m-1}, & u_m \end{pmatrix}$ is positive for all $x_1 \leq x_2 \leq \ldots \leq x_m$ in I.

Definition 2.3 : Let $\{u_1, u_2, \ldots\}$ be any finite or infinite sequence of functions in I. If for each k, $\{u_1, \dots, u_k\}$ forms an ET-system on I, then $\{u_1, u_2, \dots\}$ is called an extended complete Tchebycheff system (ECT-system) on I.

Remark 2.2 : The determinant of the collocation matrix arising from an ECT-system $U_m = \{u_1, u_2, \dots u_m\}$ is denoted by $D_{U_m}(x_1, x_2, \dots, x_m)$. That is,

$$D_{U_m}(x_1, \dots, x_m) = D\begin{pmatrix} x_1, & \dots, & x_{m-1}, & x_m \\ u_1, & \dots, & u_{m-1}, & u_m \end{pmatrix}$$

Definitiion 2.4 : A subspace of $C(I)$, finite or infinite dimensional, is called an extended complete Tchebycheff space (ECT-space) if it has an ordered basis which is an ECT-system.

Elements of an ECT-space are called generalized polynomials.

Definition 2.5 :Let $U_m = \{u_1, u_2, \dots u_m\}$ be an ECT-system on I, and let f be a sufficiently differentiable function defined on I. Associated with the points $x_1 \leq x_2 \leq \dots \leq x_m$ in I, we define a function on I as follows:

$$D\begin{pmatrix} x_1, & \dots, & x_{m-1}, & x_m; & x \\ u_1, & \dots, & u_{m-1}, & u_m, & f \end{pmatrix} = \det \begin{bmatrix} u_1^{(d_1)}(x_1) & \dots & u_m^{(d_1)}(x_1) & f^{(d_1)}(x_1) \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ u_1^{(d_i)}(x_i) & \dots & u_m^{(d_i)}(x_i) & f^{(d_i)}(x_i) \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ u_1^{(d_m)}(x_m) & \dots & u_m^{(d_m)}(x_m) & f^{(d_m)}(x_m) \\ u_1(x) & \dots & u_m(x) & f(x) \end{bmatrix}$$

where $d_i = max\{j : x_i = x_{i-j}\}, i = 1, 2, \dots m$.

Remark 2.3 : If $U_{m+1} = \{u_1, u_2, \dots u_m, u_{m+1}\}$ is an ECT-system on $I = [a, b]$, then the function in Definition 2.5 with f replaced by u_{m+1} is denoted by $D_{U_{m+1}}(x_1, \dots, x_m; x)$

Remark 2.4 : A well-known example of an infinite ECT-system on any interval $I = [a, b]$ is $V = \{1, x, x^2, \dots\}$. In this case V forms an ordered basis for $\mathcal{P}$, the space of all polynomials on I. For each n, $V_n = \{1, x, \dots, x^{n-1}\}$ is an ECT-system forming a basis for $\mathcal{P}_n$, the space of all polynomials of degree atmost $n-1$.

Definition 2.6 : Suppose $U_m = \{u_1, u_2, \dots, u_m\}$ is an $ECT-$system in I. Given any function f defined on I, its $(m-1)$th order divided difference with respect to U_m at m distinct points x_1, x_2, …, x_m in I is defined as

$$[x_1, \dots, x_{m-1}, x_m]_{U_m} f = \frac{D\begin{pmatrix} x_1, & \dots, & x_{m-1}, & x_m \\ u_1, & \dots, & u_{m-1}, & f \end{pmatrix}}{D\begin{pmatrix} x_1, & \dots, & x_{m-1}, & x_m \\ u_1, & \dots, & u_{m-1}, & u_m \end{pmatrix}}$$

Analogous of the recurrance relation for ordinary divided diiferences, generalized divided differences also has a recursion relation given by Muhlbach in [2].

Theorem 2.1 [cf. [4], p. 369; [2]] : Suppose $x_1 \neq x_{m+1}$. Then

$$[x_1, \ldots, x_m, x_{m+1}]_{U_{m+1}} f = \frac{[x_2, \ldots, x_m, x_{m+1}]_{U_m} f - [x_1, \ldots, x_{m-1}, x_m]_{U_m} f}{[x_2, \ldots, x_m, x_{m+1}]_{U_m} u_{m+1} - [x_1, \ldots, x_{m-1}, x_m]_{U_m} u_{m+1}}$$

As an application of generalized divided differences, interpolation by functions in an ECT-space can be done. That is, if $\mathcal{U}_m$ is an ECT-space on $I = [a, b]$ and if $x_1 < \ldots < x_m$ are prescribed points in I, then for any given function f defined on I, there corresponds a unique generalized polynomial in $\mathcal{U}_m$ which interpolates to f at the points $x_1 < \ldots < x_m$ (cf. [4]). An explicit expression for this unique generalized polynomial p_m in $\mathcal{U}_m$ and also an expression for the error $f - p_m$ can be derived.

Theorem 2.2 [[4], p. 370] : Let $\mathcal{U}_m$be an ECT-space on $I = [a, b]$ and let x_1, x_2, $\ldots$, x_m be distinct points in I. Let f be any function defined on I, which is sufficiently differentiable.

(i) Then an explicit expression for the unique generalized polynomial p_m in $\mathcal{U}_m$ satisfying the condition

$$p_m(x_i) = f(x_i) \qquad i = 1, 2, \ldots, m.$$

is given by

$$\begin{aligned} p_m(x) = [x_1]_{U_1} f . D_{U_1}(x) + [x_1, x_2]_{U_2} f . \frac{D_{U_2}(x_1; x)}{D_{U_1}(x_1)} + \\ \cdots + [x_1, \ldots, x_m]_{U_m} f . \frac{D_{U_m}(x_1, \ldots, x_{m-1}; x)}{D_{U_{m-1}}(x_1, \ldots, x_{m-2}, x_{m-1})}. \end{aligned}$$

where $U_m = \{u_1, u_2, \ldots, u_m\}$ is an ECT-system forming a basis for $\mathcal{U}_m$ and $U_k = \{u_1, u_2, \ldots, u_k\}$, $k = 1, 2, \ldots, m$

(ii) The error is given by

$$f(x) - p_m(x) = [x_1, \ldots, x_m; x]_{U_{m+1}} f . \frac{D_{U_{m+1}}(x_1, \ldots, x_m; x)}{D_{U_m}(x_1, \ldots, x_m)}. \tag{2.1}$$

where $U_{m+1} = \{u_1, u_2, \ldots, u_m, u_{m+1}\}$ is an ECT-system on I containing U_m.

3. Generalized Totally Bounded Functions

Throughout this section, $\mathcal{U}$ is an infinite dimensional ECT-space on $I = [a, b]$ with an ordered basis $U = \{u_1, u_2, \ldots\}$ which is an ECT-system on I. For each m, $\mathcal{U}_m$ is the ECT-space spanned by $U_m = \{u_1, u_2, \ldots, u_m\}$.

For a function f defined on I, we denote by $L(x_1, \ldots, x_m; f)$, the unique generalized polynomial in $\mathcal{U}_m$ interpolating to f at m distinct points x_1, x_2, …, x_m in I.

Definition 3.1 : Let f be a real valued function defined on I. A generalized polynomial p is called a generalized Lagrange interpolant to f from $\mathcal{U}$ if

$$p = L(x_1, \ldots, x_m; f)$$

for some distinct points x_1, x_2, …, x_m in I. The collection of all generalized Lagrange interpolants to f from $\mathcal{U}$ is denoted by $\mathcal{GL}_{\mathcal{U}}(f)$.

Definition 3.2 : A real valued function f defined in I is said to be generalized totally bounded on I if there exists an M such that

$$|p(x)| \leq M$$

for all $p \in \mathcal{GL}_{\mathcal{U}}(f)$ and for all $x \in I$.

The collection of all generalized totally bounded functions on I is denoted by $\mathcal{G}TB_{\mathcal{U}}$.

Definition 3.3 : For each $f \in \mathcal{G}TB_{\mathcal{U}}$, define

$$||f||_{\mathcal{G}TB_{\mathcal{U}}} = \sup_{\substack{p \in \mathcal{GL}_{\mathcal{U}}(f) \\ x \in I}} |p(x)|$$

Lemma 3.1 : $\mathcal{G}TB_{\mathcal{U}}$ is a normed linear space over the field of real numbers under the norm defined by

$$||f||_{\mathcal{G}TB_{\mathcal{U}}} = \sup_{\substack{p \in \mathcal{GL}_{\mathcal{U}}(f) \\ x \in I}} |p(x)| \tag{3.1.A}$$

Proof : First we will show that $\mathcal{G}TB_{\mathcal{U}}$ is a linear space over the field $\mathbb{R}$ of real numbers. Let $f_1,\ f_2 \in \mathcal{G}TB_{\mathcal{U}}$. Let p be the unique generalized Lagrange interpolant to $f_1 + f_2$ from $\mathcal{U}$. Then p interpolates to $f_1 + f_2$ at some distinct points x_1, x_2, …, x_k in I. That is,

$$p = L(x_1, x_2, \ldots, x_k; f_1 + f_2).$$

Let

$$p_1 = L(x_1, x_2, \ldots, x_k; f_1).$$

and

$$p_2 = L(x_1, x_2, \ldots, x_k; f_2).$$

Now, for $i = 1, 2, \ldots, k$,

$$\begin{aligned}(p_1 + p_2)(x_i) &= p_1(x_i) + p_2(x_i) \\ &= f_1(x_i) + f_2(x_i) = (f_1 + f_2)(x_i)\end{aligned}$$

Now $p_1 + p_2 \in \mathcal{U}_k$ and it interpolates to $f_1 + f_2$ at the distinct points x_1, x_2, …, x_k. By uniqueness, $p_1 + p_2 = p$. Also for $x \in I$,

$$\begin{aligned}|p(x)| &\leq |p_1(x)| + |p_2(x)| \\ &\leq \sup_{\substack{p_1 \in \mathcal{GL}_{\mathcal{U}}(f_1) \\ x \in I}} |p_1(x)| + \sup_{\substack{p_2 \in \mathcal{GL}_{\mathcal{U}}(f_2) \\ x \in I}} |p_2(x)| \\ &\leq ||f_1||_{\mathcal{G}TB_{\mathcal{U}}} + ||f_2||_{\mathcal{G}TB_{\mathcal{U}}}\end{aligned}$$

Thus $f_1 + f_2 \in \mathcal{G}TB_{\mathcal{U}}$. Also

$$||f_1 + f_2||_{\mathcal{G}TB_{\mathcal{U}}} \leq ||f_1||_{\mathcal{G}TB_{\mathcal{U}}} + ||f_2||_{\mathcal{G}TB_{\mathcal{U}}}.$$

Let $f \in \mathcal{G}TB_{\mathcal{U}}$ and let $r \in \mathbb{R}$. Let q be the unique generalized Lagrange interpolant to rf from $\mathcal{U}$. Then q interpolates to rf at some distinct points x_1, x_2, …, x_k in I.That is,

$$q = L(x_1, x_2, \ldots, x_k; rf).$$

Let

$$q_1 = L(x_1, x_2, \ldots, x_k; f).$$

Now, for $i = 1, 2, \ldots, k$,

$$(rq_1)(x_i) = r(q_1(x_i)) = r(f(x_i)) = (rf)(x_i)$$

Now $rq_1 \in \mathcal{U}_k$ and it interpolates to rf at the distinct points x_1, x_2, …, x_k. Due to the uniqueness of generalized Lagrange interpolants, $rq_1 = q$. Also for $x \in I$,

$$\begin{aligned}|q(x)| &= |rq_1(x)| \\ &= |r||q_1(x)| \\ &\leq |r| \sup_{\substack{q_1 \in \mathcal{GL}_{\mathcal{U}}(f) \\ x \in I}} |q_1(x)| \leq |r|||f||_{\mathcal{G}TB_{\mathcal{U}}}\end{aligned}$$

Thus $rf \in \mathcal{G}TB_{\mathcal{U}}$. Also

$$||rf||_{\mathcal{G}TB_{\mathcal{U}}} \leq |r|||f||_{\mathcal{G}TB_{\mathcal{U}}}$$

Also, trivially, $||f||_{\mathcal{G}TB_{\mathcal{U}}} \geq 0$ and $||f|| = 0$ if and only if $f = 0$. Thus $\mathcal{G}TB_{\mathcal{U}}$ is a vector space over the field of real numbers under the ususal operations of addition and scalar multiplication of functions on which (3.1.A) defines a norm. Therefore, $\mathcal{G}TB_{\mathcal{U}}$ is a normed linear space. □

Remark 3.1 : For a real valued function f defined on I,

$$||f||_{\infty} \leq ||f||_{\mathcal{G}TB_{\mathcal{U}}}$$

For if $x_1 \in I$ and if p is the unique generalized Lagrange interpolant from $\mathcal{U}$ interpolating to f at x_1, then

$$|f(x_1)| = |p(x_1)| \leq ||f||_{\mathcal{G}TB_{\mathcal{U}}}$$

so that $||f||_{\infty} \leq ||f||_{\mathcal{G}TB_{\mathcal{U}}}$

In order to show that the collection of all generalized totally bounded functions form a Banach space, we need the following result.

Lemma 3.2 : Let $\mathcal{U}_m$ be an ECT-space on $I = [a, b]$ and let $x_1, x_2, \ldots, x_m$ be distinct points in I. If (f_n) is a sequence of functions defined on I such that f_n converges to f on I, then $L(x_1, \ldots, x_m; f_n)$ converges to $L(x_1, \ldots, x_m; f)$.

Proof : First we will show that $[x_1, \ldots, x_m]_{U_m} f_n \to [x_1, \ldots, x_m]_{U_m} f$. We will use the method of induction. For $m = 1$,

$$[x_1]_{U_1} f_n = \frac{f_n(x_1)}{u_1(x_1)} \to \frac{f(x_1)}{u_1(x_1)} = [x_1]_{U_1} f$$

since $u_1(x_1) > 0$. For $m = 2$,

$$[x_1, x_2]_{U_2} f_n = \frac{D\begin{pmatrix} x_1 & x_2 \\ u_1 & f_n \end{pmatrix}}{D_{U_2}(x_1, x_2)}$$

Now

$$\begin{aligned} D\begin{pmatrix} x_1 & x_2 \\ u_1 & f_n \end{pmatrix} &= u_1(x_1)f_n(x_2) - u_1(x_2)f_n(x_1) \\ &\to u_1(x_1)f(x_2) - u_1(x_2)f(x_1) = D\begin{pmatrix} x_1 & x_2 \\ u_1 & f \end{pmatrix} \end{aligned}$$

Since $D_{U_2}(x_1, x_2) > 0$, we have $[x_1, x_2]_{U_2} f_n \to [x_1, x_2]_{U_2} f$. Now suppose that $[x_1, \ldots, x_k]_{U_k} f_n \to [x_1, \ldots, x_k]_{U_k} f$ for any collection of k distinct points x_1, x_2, ..., x_k in I. Let $\{x_1, x_2, \ldots, x_{k+1}\}$ be any collection of $k+1$ distinct points in I. By Theorem 2.1

$$
\begin{aligned}
[x_1, \ldots, x_k, x_{k+1}]_{U_{k+1}} f_n &= \frac{[x_2, \ldots, x_k, x_{k+1}]_{U_k} f_n - [x_1, \ldots, x_{k-1}, x_k]_{U_k} f_n}{[x_2, \ldots, x_k, x_{k+1}]_{U_k} u_{k+1} - [x_1, \ldots, x_{k-1}, x_k]_{U_k} u_{k+1}} \\
&\to \frac{[x_2, \ldots, x_k, x_{k+1}]_{U_k} f - [x_1, \ldots, x_{k-1}, x_k]_{U_k} f}{[x_2, \ldots, x_k, x_{k+1}]_{U_k} u_{k+1} - [x_1, \ldots, x_{k-1}, x_k]_{U_k} u_{k+1}} \\
&\to [x_1, \ldots, x_k, x_{k+1}]_{U_{k+1}} f
\end{aligned}
$$

Hence, by induction, $[x_1, \ldots, x_m]_{U_m} f_n$ converges to $[x_1, \ldots, x_m]_{U_m} f$ for any collection of m distinct points x_1, x_2, ..., x_m. By applying (i) of Theorem 2.2, we have

$$
\begin{aligned}
L(x_1, \ldots, x_m; f_n) &= [x_1]_{U_1} f_n . D_{U_1}(x) + \cdots + [x_1, \ldots, x_m]_{U_m} f_n . \frac{D_{U_m}(x_1, \ldots, x_{m-1}; x)}{D_{U_{m-1}}(x_1, \ldots, x_{m-1})}. \\
&\to [x_1]_{U_1} f . D_{U_1}(x) + \cdots + [x_1, \ldots, x_m]_{U_m} f . \frac{D_{U_m}(x_1, \ldots, x_{m-1}; x)}{D_{U_{m-1}}(x_1, \ldots, x_{m-1})}. \\
&\to L(x_1, \ldots, x_m; f)
\end{aligned}
$$

Hence the proof. □

Theorem 3.1 : $\mathcal{G}TB_{\mathcal{U}}$ is a Banach space under the norm defined by

$$
||f||_{\mathcal{G}TB_{\mathcal{U}}} = \sup_{\substack{p \in \mathcal{GL}_{\mathcal{U}}(f) \\ x \in I}} |p(x)| \qquad (3.1.A)
$$

Proof : From the lemma 3.1, we obtain that $\mathcal{G}TB_{\mathcal{U}}$ is a normed linear space under norm given by (3.1.A). Now we have to show that $\mathcal{G}TB_{\mathcal{U}}$ is complete in the metric induced by norm given by (3.1.A). Let $\{f_n\}$ be a Cauchy sequence in $\mathcal{G}TB_{\mathcal{U}}$. By Remark 3.1, $\{f_n\}$ is a Cauchy sequence in the uniform norm on I. Since $C[a, b]$ with the uniform norm is complete, there exists a continuous function f in $[a, b]$ such that $f_n \to f$ uniformly on $[a, b]$. We claim that f is in $\mathcal{G}TB_{\mathcal{U}}$. Since a Cauchy sequence in a normed linear space is bounded, there exists a constant M such that

$$
||f_n||_{\mathcal{G}TB_{\mathcal{U}}} \leq M \qquad \text{for} \quad n = 1, 2, \ldots
$$

Let L be any generalized Lagrange interpolation operator. That is, for any function f defined on $[a, b]$, $L(f)$ is the unique generalized polynomial in $\mathcal{U}$ interpolating to f at some distinct points $x_1, x_2, \ldots, x_k$ and belonging to $\mathcal{U}_k$. Thus for $x \in I$

$$|L(f_n)(x)| \leq M \quad \text{and} \quad \text{all} \quad L \in \mathcal{L}$$

where $\mathcal{L}$ is the collection of all generalized Langrange interpolation operators. Since f_n converges to f on I, by Lemma 3.2, we have

$$(L(f_n))(x) \to (L(f))(x).$$

Let $\epsilon > 0$. Fix $x \in I$. Then there exists an N(depending on n) such that

$$|L(f_n)(x) - L(f)(x)| < \epsilon \quad \text{for} \quad \text{all} \quad n \geq N$$

so

$$|(L(f))(x)| \leq |(L(f_N))(x)| + |(L(f))(x) - (L(f_N))(x)| \leq \epsilon + M.$$

Thus

$$|(L(f))(x)| \leq \epsilon + M \quad \text{for} \quad \text{all} \quad x \in I$$

Hence

$$||f||_{\mathcal{G}TB_{\mathcal{U}}} \leq \epsilon + M$$

That is, f is in $\mathcal{G}TB_{\mathcal{U}}$ .Now we claim that $f_n \to f$ in the $\mathcal{G}TB_{\mathcal{U}}$ norm. Since $\{f_n\}$ is a Cauchy sequence in the $\mathcal{G}TB_{\mathcal{U}}$ norm, given $\epsilon > 0$, we can choose an N_1 such that

$$\|f_m - f_n||_{\mathcal{G}TB_{\mathcal{U}}} < \frac{\epsilon}{2} \quad \text{for} \quad \text{every} \quad m, n \geq N_1.$$

Therefore, for every $x \in I$,

$$|(L(f_m - f_n))(x)| < \frac{\epsilon}{2}, \quad \text{for} \quad \text{every} \quad m, n \geq N_1.$$

Also, let $\epsilon > 0$. Fix $x \in I$. Then we can choose an integer N_2 (depending on x) such that

$$|(L(f - f_m))(x)| < \frac{\epsilon}{2}, \quad \text{for} \quad \text{every} \quad m \geq N_2.$$

Let $N = Max\{N_1, N_2\}$. Then

$$(L(f - f_n))(x) = (L(f - f_N))(x) + (L(f_N - f_n))(x)$$

Therefore, for $n \geq N$,

$$|(L(f - f_n))(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Thus for all $x \in I$

$$|(L(f - f_n))(x)| < \epsilon \text{ whenever } \quad n \geq N.$$

Thus

$$||L(f - f_n)|| < \epsilon \quad \text{whenever} \quad n \geq N.$$

So

$$\sup_{L \in \mathcal{L}} ||L(f - f_n)|| < \epsilon$$

So $||f - f_n||_{\mathcal{G}TB_{\mathcal{U}}} < \epsilon$. Thus $\mathcal{G}TB_{\mathcal{U}}$ is a Banach space. □

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