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$(p,q)^{th}$ RELATIVE GOL'DBERG ORDER OF ENTIRE FUNCTIONS OF SEVERAL VARIABLES

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Abstract

After the recent works of Prajapati and Rastogi [1] on the idea of p^{th} Gol'dberg relative order, we introduce in this paper $(p,q)^{th}$ relative Gol'dberg order of entire functions of several complex variables and extend their results for $(p+1,p)^{th}$ relative Gol'dberg order.

1. Introduction and Definitions

We denote the point $(z_1, z_2, ..., z_n) \in \mathbb{C}^n$ by z, where $\mathbb{C}^n$ denote the n-dimensional complex space. Let $D \subseteq \mathbb{C}^n$ be bounded complete n-circular domain with centre at the origin. For an entire function $f(z)$ of n complex variables, let $M_{f,D}(R) = \sup_{z \in D_R} |f(z)|$.
For $R > 0$, a point $z \in D_R$ if and only if $\frac{z}{R} \in D$.
If $f(z)$ is non-constant, then $M_{f,D}(R)$ is strictly increasing and its inverse

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$$M_{f,D}^{-1} : (|f(0)|, \infty) \to (0, \infty)$$

exists such that $\lim_{R\to\infty} M_{f,D}^{-1}(R) = \infty$.
The Gol'dberg order of an entire function of n complex variables is defined as follows.

Definition 1.1 [2] : The Gol'dberg order (briefly G-order) $\rho_{f,D}$ of f with respect to the domain D is defined as

$$\rho_{f,D} = \limsup_{R\to\infty} \frac{loglogM_{f,D}(R)}{logR}.$$

The lower Gol'dberg order $\lambda_{f,D}$ of f with respect to the domain D is defined as

$$\lambda_{f,D} = \liminf_{R\to\infty} \frac{loglogM_{f,D}(R)}{logR}.$$

It is known [2] that $\rho_{f,D}$ is independent of the choice of the domain D, so we write ρ_f instead of $\rho_{f,D}$.
In 2010, Mondal and Roy introduced the concept of relative order of an entire function in $\mathbb{C}^n$ with respect to another entire function of several variables.

Definition 1.2 [3]: Let f and g be entire functions of n-variables and D be a bounded complete n-circular domain with centre at the origin in $\mathbb{C}^n$. Then the relative order $\rho_{g,D}(f)$ of f with respect to g and the domain D is defined by

$$\begin{aligned}\rho_{g,D}(f) &= inf\{\mu > 0 : M_{f,D}(R) < M_{g,D}(R^\mu), \text{ for all } R > R_0(\mu) > 0\} \\ &= \limsup_{R\to\infty} \frac{logM_{g,D}^{-1}(M_{f,D}(R))}{logR}.\end{aligned}$$

In [3] Mondal and Roy proved that the relative order of f with respect to g is independent of the choice of the domain D. So the relative Gol'dberg order of f with respect to g will be denoted by $\rho_g(f)$.
In a recent paper, Prajapati and Rastogi [1] introduced the concept of p^{th} relative Gol'dberg order $\lambda_{g,D}^{[p]}(f)$ of f with respect to g in the domain D as

$$\lambda_{g,D}^{[p]}(f) = \liminf_{R\to\infty} \frac{log^{[p]}M_{g,D}^{-1}(M_{f,D}(R))}{logR}$$

where $p = 1, 2, 3, \ldots$.
In the case of relative order it therefore seems reasonable to define suitably $(p, q)^{th}$ Gol'dberg order and $(p, q)^{th}$ relative Gol'dberg order of an entire function with respect to another entire function of n complex variables in a domain D and to investigate its basic properties, which we attempt in this paper. With this in view we introduce the following definitions.

Definition 1.3 : Let f and g be two non-constant entire functions of n-complex variables and D be a bounded complete n-circular domain with centre at the origin in $\mathbb{C}^n$. If p, q are positive integers such that $p > q \geq 1$ then the $(p, q)^{th}$ Gol'dberg order and $(p, q)^{th}$ Gol'dberg lower order are respectively denoted by $\rho_{f,D}^{[p,q]}$ and $\lambda_{f,D}^{[p,q]}$ and are defined by

$$\rho_{f,D}^{[p,q]} = \limsup_{R \to \infty} \frac{log^{[p]} M_{f,D}(R)}{log^{[q]} R}$$

and

$$\lambda_{f,D}^{[p,q]} = \liminf_{R \to \infty} \frac{log^{[p]} M_{f,D}(R)}{log^{[q]} R}, \ p = 2, 3, 4, \ldots .$$

When $p = 2$ and $q = 1$ then these are equivalent to the definition of Gol'dberg order and lower Gol'dberg order.

Definition 1.4 : Let f and g be entire functions of n-complex variables and D be a bounded complete n-circular domain with centre at the origin in $\mathbb{C}^n$. Then $(p, q)^{th}$ relative Gol'dberg order $\rho_{g,D}^{[p,q]}(f)$ of f with respect to g in the domain D is defined by

$$\rho_{g,D}^{[p,q]}(f) = \limsup_{R \to \infty} \frac{log^{[p-1]} M_{g,D}^{-1}(M_{f,D}(R))}{log^{[q]} R}.$$

Similarly $(p, q)^{th}$ relative Gol'dberg lower order $\lambda_{g,D}^{[p,q]}(f)$ with respect to g in the domain D is defined by

$$\lambda_{g,D}^{[p,q]}(f) = \liminf_{R \to \infty} \frac{log^{[p-1]} M_{g,D}^{-1}(M_{f,D}(R))}{log^{[q]} R}.$$

We say that f is of $(p, q)^{th}$ regular growth if $\rho_{g,D}^{[p,q]}(f) = \lambda_{g,D}^{[p,q]}(f)$.

Definition 1.5 : An entire function g is said to have property (A) if for any $\alpha > 1$ and for all large R,

$$\{M_{g,D}(exp^{[p-1]} R)\}^2 < \{M_{g,D}(exp^{[p-1]} R^{\alpha})\}.$$

2. Basic Results

The following theorem shows that $(p,q)^{th}$ relative Gol'dberg order is independent of the choice of the domain.

Theorem 2.1 : Let f and g be entire functions of n-complex variables then $(p,q)^{th}$ relative Gol'dberg order of f with respect to g is independent of the choice of the domain D.

Proof : Let D_1 and D_2 be any two bounded complete n-circular domains. Then there exist two real numbers $\alpha, \beta > 0$ such that $\alpha D_1 \subset D_2 \subset \beta D_1$ and so,

$$M_{f,\alpha D_1}(R) \leq M_{f,D_2}(R) \leq M_{f,\beta D_1}(R).$$

Hence for any bounded complete n-circular domain D

$$M_{g,D}^{-1}(M_{f,\alpha D_1}(R)) \leq M_{g,D}^{-1}(M_{f,D_2}(R)) \leq M_{g,D}^{-1}(M_{f,\beta D_1}(R)). \tag{2.1}$$

Since for any $\lambda > 0$ and D,

$$M_{f,\lambda D}(R) = M_{f,D}(\lambda R),$$

so we have

$$\begin{aligned}\limsup_{R\to\infty} \frac{log^{[p]} M_{g,D}^{-1}(M_{f,\lambda D}(R))}{log^{[q]} R} &= \limsup_{R\to\infty} \frac{log^{[p]} M_{g,D}^{-1}(M_{f,D}(\lambda R))}{log^{[q]} R} \\ &= \limsup_{R\to\infty} \frac{log^{[p]} M_{g,D}^{-1}(M_{f,D}(R))}{log^{[q]} \frac{R}{\lambda}} \\ &= \limsup_{R\to\infty} \frac{log^{[p]} M_{g,D}^{-1}(M_{f,D}(R))}{log^{[q]} R}.\end{aligned}$$

Hence from (2.1)

$$\begin{aligned}\limsup_{R\to\infty}\frac{log^{[p]}M_{g,D}^{-1}(M_{f,D_1}(R))}{log^{[q]}R} &= \limsup_{R\to\infty}\frac{log^{[p]}M_{g,D}^{-1}(M_{f,\alpha D_1}(R))}{log^{[q]}R}\\ &\le \limsup_{R\to\infty}\frac{log^{[p]}M_{g,D}^{-1}(M_{f,D_2}(R))}{log^{[q]}R}\\ &\le \limsup_{R\to\infty}\frac{log^{[p]}M_{g,D}^{-1}(M_{f,\beta D_1}(R))}{log^{[q]}R}\\ &\le \limsup_{R\to\infty}\frac{log^{[p]}M_{g,D}^{-1}(M_{f,D_1}(R))}{log^{[q]}R}.\end{aligned}$$

Thus

$$\limsup_{R\to\infty}\frac{log^{[p]}M_{g,D}^{-1}(M_{f,D_1}(R))}{log^{[q]}R} = \limsup_{R\to\infty}\frac{log^{[p]}M_{g,D}^{-1}(M_{f,D_2}(R))}{log^{[q]}R}.$$

Hence the theorem.

So after this we shall always write, $\rho_g^{[p,q]}(f)$ instead of $\rho_{g,D}^{[p,q]}(f)$.

Theorem 2.2 :Let f and g be entire functions of n complex variables such that $0<\lambda_f^{[p+1,p]}\le\rho_f^{[p+1,p]}$ and $0<\lambda_g^{[p+1,p]}\le\rho_g^{[p+1,p]}$.Then $\frac{\lambda_f^{[p+1,p]}}{\rho_g^{[p+1,p]}} \le \lambda_g^{[p+1,p]}(f) \le min\{\frac{\lambda_f^{[p+1,p]}}{\lambda_g^{[p+1,p]}},\frac{\rho_f^{[p+1,p]}}{\rho_g^{[p+1,p]}}\} \le max\{\frac{\lambda_f^{[p+1,p]}}{\lambda_g^{[p+1,p]}},\frac{\rho_f^{[p+1,p]}}{\rho_g^{[p+1,p]}}\} \le \rho_g^{[p+1,p]}(f) \le \frac{\rho_f^{[p+1,p]}}{\lambda_g^{[p+1,p]}}$.

Proof : From the definition of $(p+1,p)^{th}$ Gol'dberg order and Gol'dberg lower order we get for arbitrary $\epsilon>0$ and for all large values of R

$$M_{f,D}(exp^{[p-1]}R) < exp^{[p]}R^{\rho_f^{[p+1,p]}+\epsilon} \tag{2.2}$$

$$M_{g,D}(exp^{[p-1]}R) < exp^{[p]}R^{\rho_g^{[p+1,p]}+\epsilon} \tag{2.3}$$

$$M_{f,D}(exp^{[p-1]}R) > exp^{[p]}R^{\lambda_f^{[p+1,p]}-\epsilon} \tag{2.4}$$

$$M_{g,D}(exp^{[p-1]}R) > exp^{[p]}R^{\lambda_g^{[p+1,p]}-\epsilon}. \tag{2.5}$$

Also for a sequence $\{R_n\}$ tending to infinity we get that

$$M_{f,D}(exp^{[p-1]}R_n) > exp^{[p]}R_n^{\rho_f^{[p+1,p]}-\epsilon} \tag{2.6}$$

$$M_{g,D}(exp^{[p-1]}R_n) > exp^{[p]}R_n^{\rho_g^{[p+1,p]}-\epsilon} \tag{2.7}$$

$$M_{f,D}(exp^{[p-1]}R_n) < exp^{[p]}R_n^{\lambda_f^{[p+1,p]}+\epsilon} \tag{2.8}$$

$$M_{g,D}(exp^{[p-1]}R_n) < exp^{[p]}R_n^{\lambda_g^{[p+1,p]}+\epsilon} \tag{2.9}$$

Now from the definition of $(p+1,p)^{th}$ relative Gol'dberg order, we get for arbitrary $\epsilon_1 > 0$ and for all large values of R that

$$\rho_g^{[p+1,p]}(f) + \epsilon_1 > \frac{log^{[p]}M_{g,D}^{-1}(M_{f,D}(exp^{[p-1]}R))}{logR}.$$

Now from (2.6) we get for a sequence $\{R_n\}$ tending to infinity that,

$$\begin{aligned}
\rho_g^{[p+1,p]}(f) + \epsilon_1 &> \frac{log^{[p]}M_{g,D}^{-1}(exp^{[p]}R_n^{\rho_f^{[p+1,p]}-\epsilon})}{logR_n} \\
&= \frac{log^{[p]}M_{g,D}^{-1}(exp^{[p]}(R_n^{\frac{\rho_f^{[p+1,p]}-\epsilon}{\rho_g^{[p+1,p]}+\epsilon}})^{\rho_g^{[p+1,p]}+\epsilon})}{logR_n} \\
&> \frac{log^{[p]}M_{g,D}^{-1}M_{g,D}(exp^{[p-1]}(R_n^{\frac{\rho_f^{[p+1,p]}-\epsilon}{\rho_g^{[p+1,p]}+\epsilon}}))}{logR_n}, \quad using (2.3) \\
&= \frac{\rho_f^{[p+1,p]}-\epsilon}{\rho_g^{[p+1,p]}+\epsilon}.
\end{aligned}$$

As $\epsilon_1(>0)$ and $\epsilon(>0)$ are arbitrary, we get

$$\rho_g^{[p+1,p]}(f) \geq \frac{\rho_f^{[p+1,p]}}{\rho_g^{[p+1,p]}}. \tag{2.10}$$

Also from (2.2) we get for arbitrary $\epsilon > 0$ and for all large values of R that

$$\begin{aligned}
\frac{log^{[p]}M_{g,D}^{-1}M_{f,D}(exp^{[p-1]}R)}{logR} &< \frac{log^{[p]}M_{g,D}^{-1}(exp^{[p]}R^{\rho_f^{[p+1,p]}+\epsilon})}{logR} \\
&= \frac{log^{[p]}M_{g,D}^{-1}(exp^{[p]}(R^{\frac{\rho_f^{[p+1,p]}+\epsilon}{\rho_g^{[p+1,p]}-\epsilon}})^{\rho_g^{[p+1,p]}-\epsilon})}{logR}.
\end{aligned}$$

Now from (2.7) we get for a sequence $\{R_n\}$ tending to infinity that

$$\frac{log^{[p]}M_{g,D}^{-1}M_{f,D}(exp^{[p-1]}R_n)}{logR_n} < \frac{log^{[p]}M_{g,D}^{-1}M_{g,D}(exp^{[p-1]}(R_n^{\frac{\rho_f^{[p+1,p]}+\epsilon}{\rho_g^{[p+1,p]}-\epsilon}}))}{logR_n}$$

Therefore
$$\liminf_{R_n\to\infty} \frac{log^{[p]} M_{g,D}^{-1} M_{f,D}(exp^{[p-1]} R_n)}{logR_n} \le \frac{\rho_f^{[p+1,p]} + \epsilon}{\rho_g^{[p+1,p]} - \epsilon}.$$
Since $\epsilon > 0$ is arbitrary, we have
$$\lambda_g^{[p+1,p]}(f) \le \frac{\rho_f^{[p+1,p]}}{\rho_g^{[p+1,p]}}. \tag{2.11}$$

Now from the definition of $(p+1,p)^{th}$ relative Gol'dberg lower order, we get for arbitrary $\epsilon_2 > 0$ and for all large values of R that
$$\lambda_g^{[p+1,p]}(f) - \epsilon_2 < \frac{log^{[p]} M_{g,D}^{-1}(M_{f,D}(exp^{[p-1]} R))}{logR}.$$
Now from (2.8) we get for a sequence $\{R_n\}$ tending to infinity
$$\begin{aligned}
\lambda_g^{[p+1,p]}(f) - \epsilon_2 &< \frac{log^{[p]} M_{g,D}^{-1}(exp^{[p]} R_n^{\lambda_f^{[p+1,p]}+\epsilon})}{logR_n} \\
&= \frac{log^{[p]} M_{g,D}^{-1}(exp^{[p]} (R_n^{\frac{\lambda_f^{[p+1,p]}+\epsilon}{\lambda_g^{[p+1,p]}-\epsilon}})^{\lambda_g^{[p+1,p]}-\epsilon})}{logR_n} \\
&< \frac{log^{[p]} M_{g,D}^{-1} M_{g,D}(exp^{[p-1]} R_n^{\frac{\lambda_f^{[p+1,p]}+\epsilon}{\lambda_g^{[p+1,p]}-\epsilon}})}{logR_n}, \quad using (2.5) \\
&= \frac{\lambda_f^{[p+1,p]} + \epsilon}{\lambda_g^{[p+1,p]} - \epsilon}.
\end{aligned}$$
Since $\epsilon_2 (> 0)$ and $\epsilon (> 0)$ are arbitrary, we obtain that
$$\lambda_g^{[p+1,p]}(f) \le \frac{\lambda_f^{[p+1,p]}}{\lambda_g^{[p+1,p]}}. \tag{2.12}$$

Now from (2.4) we get for arbitrary $\epsilon > 0$ and for large values of R that
$$\begin{aligned}
\frac{log^{[p]} M_{g,D}^{-1} M_{f,D}(exp^{[p-1]} R)}{logR} &> \frac{log^{[p]} M_{g,D}^{-1}(exp^{[p]} R^{\lambda_f^{[p+1,p]}-\epsilon})}{logR} \\
&= \frac{log^{[p]} M_{g,D}^{-1}(exp^{[p]} (R^{\frac{\lambda_f^{[p+1,p]}-\epsilon}{\lambda_g^{[p+1,p]}+\epsilon}})^{\lambda_g^{[p+1,p]}+\epsilon})}{logR}.
\end{aligned}$$

Now from (2.9) we obtain for a sequence $\{R_n\}$ tending to infinity that

$$\frac{log^{[p]}M_{g,D}^{-1}M_{f,D}(exp^{[p-1]}R_n)}{logR_n} > \frac{log^{[p]}M_{g,D}^{-1}M_{g,D}(exp^{[p-1]}R_n^{\frac{\lambda_f^{[p+1,p]}-\epsilon}{\lambda_g^{[p+1,p]}+\epsilon}})}{logR_n}.$$

So,

$$\limsup_{R_n\to\infty}\frac{log^{[p]}M_{g,D}^{-1}M_{f,D}(exp^{[p-1]}R_n)}{logR_n} \geq \frac{\lambda_f^{[p+1,p]}-\epsilon}{\lambda_g^{[p+1,p]}+\epsilon}.$$

Since $\epsilon > 0$ is arbitrary, we have

$$\rho_g^{[p+1,p]}(f) \geq \frac{\lambda_f^{[p+1,p]}}{\lambda_g^{[p+1,p]}}. \tag{2.13}$$

Again from definition, we get for arbitrary $\epsilon_3 > 0$ and for a sequence $\{R_n\}$ tending to infinity that

$$\begin{aligned}
\rho_g^{[p+1,p]}(f)-\epsilon_3 &< \frac{log^{[p]}M_{g,D}^{-1}(M_{f,D}(exp^{[p-1]}R_n))}{logR_n} \\
&< \frac{log^{[p]}M_{g,D}^{-1}(exp^{[p]}R_n^{\rho_f^{[p+1,p]}+\epsilon})}{logR_n}, \quad using(2.2) \\
&= \frac{log^{[p]}M_{g,D}^{-1}(exp^{[p]}(R_n^{\frac{\rho_f^{[p+1,p]}+\epsilon}{\lambda_g^{[p+1,p]}-\epsilon}})^{\lambda_g^{[p+1,p]}-\epsilon})}{logR_n} \\
&< \frac{log^{[p]}M_{g,D}^{-1}M_{g,D}(exp^{[p-1]}R_n^{\frac{\rho_f^{[p+1,p]}+\epsilon}{\lambda_g^{[p+1,p]}-\epsilon}})}{logR_n}, \quad using(2.5) \\
&= \frac{\rho_f^{[p+1,p]}+\epsilon}{\lambda_g^{[p+1,p]}-\epsilon}.
\end{aligned}$$

Since $\epsilon_3(>0)$ and $\epsilon(>0)$ are arbitrary, we have

$$\rho_g^{[p+1,p]}(f) \leq \frac{\rho_f^{[p+1,p]}}{\lambda_g^{[p+1,p]}}. \tag{2.14}$$

Also from definition, we get for arbitrary $\epsilon_4 > 0$ and for a sequence $\{R_n\}$ tending to infinity that

$$\begin{aligned}
\lambda_g^{[p+1,p]}(f)+\epsilon_4 &> \frac{log^{[p]}M_{g,D}^{-1}M_{f,D}(exp^{[p-1]}R_n)}{logR_n} \\
&> \frac{log^{[p]}M_{g,D}^{-1}(exp^{[p]}R_n^{\lambda_f^{[p+1,p]}-\epsilon})}{logR_n}, \quad using(2.4) \\
&= \frac{log^{[p]}M_{g,D}^{-1}(exp^{[p]}(R_n^{\frac{\lambda_f^{[p+1,p]}-\epsilon}{\rho_g^{[p+1,p]}+\epsilon}})^{\rho_g^{[p+1,p]}+\epsilon})}{logR_n} \\
&> \frac{log^{[p]}M_{g,D}^{-1}M_{g,D}(exp^{[p-1]}R_n^{\frac{\lambda_f^{[p+1,p]}-\epsilon}{\rho_g^{[p+1,p]}+\epsilon}})}{logR_n}, \quad using(2.3) \\
&= \frac{\lambda_f^{[p+1,p]}-\epsilon}{\rho_g^{[p+1,p]}+\epsilon}.
\end{aligned}$$

Since $\epsilon_4(>0)$ and $\epsilon(>0)$ are arbitrary, we get

$$\lambda_g^{[p+1,p]}(f) \geq \frac{\lambda_f^{[p+1,p]}}{\rho_g^{[p+1,p]}} \tag{2.15}$$

The theorem follows from (2.10), (2.11), (2.12), (2.13), (2.14) and (2.15).

Theorem 2.3 : Let f and g be entire functions of n complex variables such that $\rho_f^{[p+1,p]}=0$ and $0<\rho_g^{[p+1,p]}<\infty$. Then $\lambda_g^{[p+1,p]}(f)=0$.

Proof : From the definition, we have for arbitrary $\epsilon>0$ and for all large values of R that

$$M_{f,D}(exp^{[p-1]}R)<(exp^{[p]}R^{\epsilon}).$$

So,

$$\begin{aligned}
\frac{log^{[p]}M_{g,D}^{-1}M_{f,D}(exp^{[p-1]}R)}{logR} &< \frac{log^{[p]}M_{g,D}^{-1}(exp^{[p]}R^{\epsilon})}{logR} \\
&= \frac{log^{[p]}M_{g,D}^{-1}(exp^{[p]}(R^{\frac{\epsilon}{\rho_g^{[p+1,p]}-\epsilon}})^{\rho_g^{[p+1,p]}-\epsilon})}{logR}.
\end{aligned}$$

Now from (2.7) we get for a sequence $\{R_n\}$ tending to infinity that

$$\frac{log^{[p]}M_{g,D}^{-1}M_{f,D}(exp^{[p-1]}R_n)}{logR_n} < \frac{log^{[p]}M_{g,D}^{-1}M_{g,D}(exp^{[p-1]}(R_n^{\frac{\epsilon}{\rho_g^{[p+1,p]}-\epsilon}}))}{logR_n}.$$

Therefore,

$$\liminf_{R_n\to\infty} \frac{log^{[p]}M_{g,D}^{-1}M_{f,D}(exp^{[p-1]}R_n)}{logR_n} \leq \frac{\epsilon}{\rho_g^{[p+1,p]}-\epsilon}.$$

Since $\epsilon > 0$ is arbitrary it follows that

$$\lambda_g^{[p+1,p]}(f) = 0.$$

Theorem 2.4 : Let f and g be entire functions of n complex variables such that $0 < \rho_f^{[p+1,p]} < \infty$ and $\rho_g^{[p+1,p]} = 0$. Then $\rho_g^{[p+1,p]}(f) = \infty$.

Proof : From the definition of $(p+1,p)^{th}$ relative Gol'dberg order, we get for arbitrary $\epsilon_1 > 0$ and for all large values of R that

$$\rho_g^{[p+1,p]}(f) + \epsilon_1 > \frac{log^{[p]}M_{g,D}^{-1}M_{f,D}(exp^{[p-1]}R))}{logR}.$$

Now from (2.6) we get for a sequence $\{R_n\}$ tending to infinity that,

$$\begin{aligned}
\rho_g^{[p+1,p]}(f) + \epsilon_1 &> \frac{log^{[p]}M_{g,D}^{-1}(exp^{[p]}R_n^{\rho_f^{[p+1,p]}-\epsilon})}{logR_n} \\
&= \frac{log^{[p]}M_{g,D}^{-1}(exp^{[p]}(R_n^{\frac{\rho_f^{[p+1,p]}-\epsilon}{\epsilon}})^{\epsilon})}{logR_n} \\
&> \frac{log^{[p]}M_{g,D}^{-1}M_{g,D}(exp^{[p-1]}R_n^{\frac{\rho_f^{[p+1,p]}-\epsilon}{\epsilon}})}{logR_n}, \quad using (2.3)\ and\ \rho_g^{[p+1,p]} = 0 \\
&= \frac{\rho_f^{[p+1,p]}-\epsilon}{\epsilon}.
\end{aligned}$$

Since $\epsilon_1(>0)$ and $\epsilon(>0)$ are arbitrary it follows that

$$\rho_g^{[p+1,p]}(f) = \infty.$$

Theorem 2.5 : Let f and g be two entire functions and $\rho_f^{[p+1,p]}$ and $\rho_g^{[p+1,p]}$ be the $(p+1,p)^{th}$ Gol'dberg order of f and g respectively.Then the $(p+1,p)^{th}$ relative Gol'dberg order $\rho_g^{[p+1,p]}(f)$ of $f(z)$ with respect to $g(z)$ satisfies

$$(i) \qquad \rho_g^{[p+1,p]}(f) \geq \frac{\rho_f^{[p+1,p]}}{\rho_g^{[p+1,p]}}$$

(ii) If g is of $(p+1,p)^{th}$ regular growth then

$$\rho_g^{[p+1,p]}(f) = \frac{\rho_f^{[p+1,p]}}{\rho_g^{[p+1,p]}}.$$

Proof : From the definition of $(p+1,p)^{th}$ Gol'dberg order we have for arbitrary $\epsilon > 0$ and for all large values of R

$$M_{f,D}(exp^{[p-1]}R) < exp^{[p]}R^{\rho_f^{[p+1,p]}+\epsilon} \tag{2.16}$$

and

$$M_{g,D}(exp^{[p-1]}R) < exp^{[p]}R^{\rho_g^{[p+1,p]}+\epsilon} \tag{2.17}$$

Also for a sequence $\{R_n\}$ tending to infinity, we get that

$$M_{f,D}(exp^{[p-1]}R_n) > exp^{[p]}R_n^{\rho_f^{[p+1,p]}-\epsilon}. \tag{2.18}$$

Now,

$$\begin{aligned}
\rho_g^{[p+1,p]}(f) &= \limsup_{R\to\infty} \frac{log^{[p]}M_{g,D}^{-1}(M_{f,D}(exp^{[p-1]}R))}{logR} \\
&\geq \limsup_{R_n\to\infty} \frac{log^{[p]}M_{g,D}^{-1}(exp^{[p-1]}R_n^{\rho_f^{[p+1,p]}-\epsilon})}{logR_n} \quad by \quad (2.18) \\
&= \limsup_{R_n\to\infty} \frac{log^{[p]}M_{g,D}^{-1}(exp^{[p-1]}(R_n^{\frac{\rho_f^{[p+1,p]}-\epsilon}{\rho_g^{[p+1,p]}+\epsilon}})^{\rho_g^{[p+1,p]}+\epsilon})}{logR_n} \\
&\geq \limsup_{R_n\to\infty} \frac{log^{[p]}M_{g,D}^{-1}M_{g,D}(exp^{[p-1]}R_n^{\frac{\rho_f^{[p+1,p]}-\epsilon}{\rho_g^{[p+1,p]}+\epsilon}})}{logR_n}, \quad using \quad (2.17) \\
&= \frac{\rho_f^{[p+1,p]}-\epsilon}{\rho_g^{[p+1,p]}+\epsilon}.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary,

$$\rho_g^{[p+1,p]}(f) \geq \frac{\rho_f^{[p+1,p]}}{\rho_g^{[p+1,p]}} \tag{2.19}$$

This proves (i).

When g is of $(p+1,p)^{th}$ regular growth, we have for $\epsilon > 0$ and for all $R > R_0$

$$M_{g,D}(exp^{[p-1]}R) > exp^{[p]}R^{\rho_g^{[p+1,p]}-\epsilon}. \tag{2.20}$$

Now,

$$\rho_g^{[p+1,p]}(f) = \limsup_{R\to\infty} \frac{log^{[p]} M_{g,D}^{-1}(M_{f,D}(exp^{[p-1]}R))}{logR}$$

$$\leq \limsup_{R\to\infty} \frac{log^{[p]} M_{g,D}^{-1}(exp^{[p]} R^{\rho_f^{[p+1,p]}+\epsilon})}{logR} \quad from \quad (2.16)$$

$$= \limsup_{R\to\infty} \frac{log^{[p]} M_{g,D}^{-1}(exp^{[p]}(R^{\frac{\rho_f^{[p+1,p]}+\epsilon}{\rho_g^{[p+1,p]}-\epsilon}})^{\rho_g^{[p+1,p]}-\epsilon})}{logR}$$

$$\leq \limsup_{R\to\infty} \frac{log^{[p]} M_{g,D}^{-1}(exp^{[p-1]} R^{\frac{\rho_f^{[p+1,p]}+\epsilon}{\rho_g^{[p+1,p]}-\epsilon}})}{logR} \quad from \quad (2.20)$$

$$= \frac{\rho_f^{[p+1,p]}+\epsilon}{\rho_g^{[p+1,p]}-\epsilon}.$$

Since $\epsilon > 0$ is arbitrary, so

$$\rho_g^{[p+1,p]}(f) \leq \frac{\rho_f^{[p+1,p]}}{\rho_g^{[p+1,p]}}. \tag{2.21}$$

Hence from (2.19) and (2.21) we have

$$\rho_g^{[p+1,p]}(f) = \frac{\rho_f^{[p+1,p]}}{\rho_g^{[p+1,p]}}.$$

3. Sum and Product Theorems

Theorem 3.1 : Let f_1, f_2 and g be three transcendental entire functions. If $\rho_g^{[p+1,p]}(f_1)$ and $\rho_g^{[p+1,p]}(f_2)$ be the $(p+1,p)^{th}$ relative Gol'dberg order of f_1 and f_2 respectively with respect to g and
$\rho_g^{[p+1,p]}(f_1) \neq \rho_g^{[p+1,p]}(f_2)$, then the $(p+1,p)^{th}$ relative Gol'dgerg order of f_1+f_2 is given by
$\rho_g^{[p+1,p]}(f_1+f_2) = max\{\rho_g^{[p+1,p]}(f_1), \rho_g^{[p+1,p]}(f_2\}$.

Proof : We have,

$$\rho_g^{[p+1,p]}(f_1) = \limsup_{R\to\infty} \frac{log^{[p]} M_{g,D}^{-1}(M_{f_1,D}(exp^{[p-1]}R))}{logR},$$

$$\rho_g^{[p+1,p]}(f_2) = \limsup_{R\to\infty} \frac{log^{[p]} M_{g,D}^{-1}(M_{f_2,D}(exp^{[p-1]}R))}{logR}$$

and

$$\rho_g^{[p+1,p]}(f_1+f_2) = \limsup_{R\to\infty} \frac{log^{[p]} M_{g,D}^{-1}(M_{f_1+f_2,D}(exp^{[p-1]}R))}{logR}.$$

Without loss of generality, we assume that $\rho_g^{[p+1,p]}(f_1) > \rho_g^{[p+1,p]}(f_2)$.
Now,

$$\begin{aligned} M_{f_1+f_2,D}(exp^{[p-1]}R) &\le M_{f_1,D}(exp^{[p-1]}R) + M_{f_2,D}(exp^{[p-1]}R) \\ &< M_{g,D}(exp^{[p-1]}R^{\rho_g^{[p+1,p]}(f_1)+\epsilon}) + M_{g,D}(exp^{[p-1]}R^{\rho_g^{[p+1,p]}(f_2)+\epsilon}) \ for\ large\ R \\ &< 2M_{g,D}(exp^{[p-1]}R^{\rho_g^{[p+1,p]}(f_1)+\epsilon}) \\ &< M_{g,D}(exp^{[p-1]}R^{\rho_g^{[p+1,p]}(f_1)+2\epsilon}),\ for\ sufficiently\ large\ R. \end{aligned}$$

Therefore,

$$\limsup_{R\to\infty} \frac{log^{[p]} M_{g,D}^{-1}(M_{f_1+f_2,D}(exp^{[p-1]}R))}{logR} \le \rho_g^{[p+1,p]}(f_1) + 2\epsilon.$$

So,

$$\rho_g^{[p+1,p]}(f_1+f_2) \le \rho_g^{[p+1,p]}(f_1). \tag{3.1}$$

On the other hand, there exists a sequence $\{R_n\}$ of value of R, tending to infinity, such that,

$$M_{f_1,D}(exp^{[p-1]}R_n) > M_{g,D}(exp^{[p-1]}R_n^{\rho_g^{[p+1,p]}(f_1)-\epsilon}). \tag{3.2}$$

We have

$$M_{f_1+f_2,D}(exp^{[p-1]}R_n) \ge M_{f_1,D}(exp^{[p-1]}R_n) - M_{f_2,D}(exp^{[p-1]}R_n).$$

Let $\epsilon > 0$ such that $\rho_g^{[p+1,p]}(f_1) - \epsilon > \rho_g^{[p+1,p]}(f_2) + \epsilon$.
Then

$$M_{f_2,D}(exp^{[p-1]}R_n) < M_{g,D}(exp^{[p-1]}R_n^{\rho_g^{[p+1,p]}(f_2)+\epsilon}), \tag{3.3}$$

for sufficiently lagre n.
So from (3.2) and (3.3) we have,

$$\begin{aligned} M_{f_1+f_2,D}(exp^{[p-1]}R_n) &> M_{g,D}(exp^{[p-1]}R_n^{\rho_g^{[p+1,p]}(f_1)-\epsilon}) - M_{g,D}(exp^{[p-1]}R_n^{\rho_g^{[p+1,p]}(f_2)+\epsilon}) \\ &= M_{g,D}(exp^{[p-1]}R_n^{\rho_g^{[p+1,p]}(f_1)-\epsilon})[1 - \frac{M_{g,D}(exp^{[p-1]}R_n^{\rho_g^{[p+1,p]}(f_2)+\epsilon})}{M_{g,D}(exp^{[p-1]}R_n^{\rho_g^{[p+1,p]}(f_1)-\epsilon})}] \\ &> \frac{1}{2} M_{g,D}(exp^{[p-1]}R_n^{\rho_g^{[p+1,p]}(f_1)-\epsilon}) \ for\ sufficiently\ large\ n \\ &> M_{g,D}(exp^{[p-1]}R_n^{\rho_g^{[p+1,p]}(f_1)-2\epsilon}) \ for\ large\ n \end{aligned}$$

or,

$$\limsup_{R\to\infty} \frac{log^{[p]} M_{g,D}^{-1}(M_{f_1+f_2,D}(exp^{[p-1]}R))}{logR} \geq \rho_g^{[p+1,p]}(f_1) - 2\epsilon.$$

So,

$$\rho_g^{[p+1,p]}(f_1+f_2) \geq \rho_g^{[p+1,p]}(f_1). \tag{3.4}$$

Hence from (3.1) and (3.4) we get $\rho_g^{[p+1,p]}(f_1+f_2) = max\{\rho_g^{[p+1,p]}(f_1), \rho_g^{[p+1,p]}(f_2\}$. This proves the theorem.

Theorem 3.2 : Let f_1 and f_2 be two entire functions of $(p+1,p)^{th}$ relative Gol'dberg order $\rho_g^{[p+1,p]}(f_1)$ and $\rho_g^{[p+1,p]}(f_2)$ respectively. If g has the property (A), then the $(p+1,p)^{th}$ relative Gol'dberg order of $f_1.f_2$ is
$\rho_g^{[p+1,p]}(f_1.f_2) \leq max\{\rho_g^{[p+1,p]}(f_1), \rho_g^{[p+1,p]}(f_2)\}$.

Proof : Without loss of generality, let us assume that $\rho_g^{[p+1,p]}(f_1) \geq \rho_g^{[p+1,p]}(f_2)$.
For $\epsilon > 0$ and for all large R, we have

$$M_{f_1,D}(exp^{[p-1]}R) < M_{g,D}(exp^{[p-1]}R^{\rho_g^{[p+1,p]}(f_1)+\epsilon})$$

and

$$M_{f_2,D}(exp^{[p-1]}R) < M_{g,D}(exp^{[p-1]}R^{\rho_g^{[p+1,p]}(f_2)+\epsilon}).$$

Now we have

$$\begin{aligned} M_{f_1.f_2,D}(exp^{[p-1]}R) &\leq M_{f_1,D}(exp^{[p-1]}R).M_{f_2,D}(exp^{[p-1]}R) \\ &< M_{g,D}(exp^{[p-1]}R^{\rho_g^{[p+1,p]}(f_1)+\epsilon}).M_{g,D}(exp^{[p-1]}R^{\rho_g^{[p+1,p]}(f_2)+\epsilon}) \\ &< [M_{g,D}(exp^{[p-1]}R^{\rho_g^{[p+1,p]}(f_1)+\epsilon})]^2 \\ &< M_{g,D}(exp^{[p-1]}R^{\alpha(\rho_g^{[p+1,p]}(f_1)+\epsilon)}), \alpha > 1 from\ property(A). \end{aligned}$$

Hence,

$log^{[p-1]} M_{g,D}^{-1} M_{f_1.f_2,D}(exp^{[p-1]}R) < R^{\alpha(\rho_g^{[p+1,p]}(f_1)+\epsilon)}$

or, $\limsup_{R\to\infty} \frac{log^{[p]} M_{g,D}^{-1} M_{f_1.f_2,D}(exp^{[p-1]}R)}{logR} \leq \alpha(\rho_g^{[p+1,p]}(f_1)+\epsilon)$.

Letting $\alpha \to 1^+$ and since $\epsilon > 0$ is arbitrary, so we have
$\rho_g^{[p+1,p]}(f_1.f_2) \leq \rho_g^{[p+1,p]}(f_1)$.
Hence the theorem.

Theorem 3.3 : Let f be an entire function of $(p+1,p)^{th}$ relative Gol'dberg order $\rho_g^{[p+1,p]}(f) \neq 0$ and $P(z)$ be a polynomial. If g has the property (A), then the $(p+1,p)^{th}$ relative Gol'dberg order $\rho_g^{[p+1,p]}(f.P)$ of $f(z).P(z)$ is same as $\rho_g^{[p+1,p]}(f)$.

Proof : From Theorem 3.2 we have,

$$\rho_g^{[p+1,p]}(f.P) \leq \rho_g^{[p+1,p]}(f) \tag{3.5}$$

since the $(p+1,p)^{th}$ relative Gol'dberg order of $P(z)$ with respect to g is zero.
Since $M_{P,D}(exp^{[p-1]}R) \geq 1$ for all sufficiently large R,
$\sup|f(z)P(z)| \geq \sup|f(z)|$,where $z \in D_{exp^{[p-1]}R}$ and for all sufficiently large R
i.e.,$M_{f.P,D}(exp^{[p-1]}R) \geq M_{f,D}(exp^{[p-1]}R)$.
So, $\limsup_{R\to\infty} \frac{log^{[p]}M_{g,D}^{-1}(M_{f.P,D}(exp^{[p-1]}R))}{logR} \geq \limsup_{R\to\infty} \frac{log^{[p]}M_{g,D}^{-1}(M_{f,D}(exp^{[p-1]}R))}{logR}$

$$i.e., \rho_g^{[p+1,p]}(f.P) \geq \rho_g^{[p+1,p]}(f). \tag{3.6}$$

Hence from (3.5) and (3.6) we get, $\rho_g^{[p+1,p]}(f.P) = \rho_g^{[p+1,p]}(f)$.

4. Asymptotic Behaviour

Definition 4.1 : Two entire functions g_1 and g_2 are said to be asymptotically equivalent if
$\frac{M_{g_1,D}(exp^{[p-1]}R)}{M_{g_2,D}(exp^{[p-1]}R)} \to 1$ as $R \to \infty$ and in this case we write $g_1 \sim g_2$.
Theorem 4.1 : Let g_1, g_2 and f be three entire functions and $\rho_{g_1}^{[p+1,p]}(f)$ and $\rho_{g_2}^{[p+1,p]}(f)$ be the $(p+1,p)^{th}$ relative Gol'dberg order of f with respect to g_1 and g_2 respectively. If $g_1 \sim g_2$ then $\rho_{g_1}^{[p+1,p]}(f) = \rho_{g_2}^{[p+1,p]}(f)$.
Proof : Since $g_1 \sim g_2$, we have for $\epsilon > 0$ and for all large R,

$$M_{g_1,D}(exp^{[p-1]}R) < (1+\epsilon)M_{g_2,D}(exp^{[p-1]}R) < M_{g_2,D}(exp^{[p-1]}R^{(1+\epsilon)}).$$

Hence

$$R < log^{[p-1]}M_{g_1,D}^{-1}\{M_{g_2,D}(exp^{[p-1]}R^{(1+\epsilon)})\} \tag{4.1}$$

for all large R.
Let $M_{g_2,D}(exp^{[p-1]}R^{(1+\epsilon)}) = R_1$.
Then $R = \{log^{[p-1]}M_{g_2,D}^{-1}(R_1)\}^{\frac{1}{(1+\epsilon)}}$.
Now from (1.1) we get,
$log^{[p-1]}M_{g_2,D}^{-1}(R_1) < (log^{[p-1]}M_{g_1,D}^{-1}(R_1))^{1+\epsilon}$ for large R_1.

Now,

$$\begin{aligned}
\rho_{g_2}^{[p+1,p]}(f) &= \limsup_{R_1\to\infty} \frac{log^{[p]}M_{g_2,D}^{-1}(M_{f,D}(exp^{[p-1]}R_1))}{logR_1} \\
&= \limsup_{R_1\to\infty} \frac{log[log^{[p-1]}M_{g_2,D}^{-1}(M_{f,D}(exp^{[p-1]}R_1))]}{logR_1} \\
&\le \limsup_{R_1\to\infty} \frac{log[log^{[p-1]}M_{g_1,D}^{-1}(M_{f,D}(exp^{[p-1]}R_1))]^{1+\epsilon}}{logR_1} \\
&= (1+\epsilon)\limsup_{R_1\to\infty} \frac{log^{[p]}M_{g_1,D}^{-1}(M_{f,D}(exp^{[p-1]}R_1))]}{logR_1} \\
&= (1+\epsilon)\rho_{g_1}^{[p+1,p]}(f).
\end{aligned}$$

So,

$$\rho_{g_2}^{[p+1,p]}(f) \le \rho_{g_1}^{[p+1,p]}(f).$$

Also if $g_2 \sim g_1$ so

$$\rho_{g_1}^{[p+1,p]}(f) \le \rho_{g_2}^{[p+1,p]}(f).$$

Hence, $\rho_{g_2}^{[p+1,p]}(f) = \rho_{g_1}^{[p+1,p]}(f)$.

Theorem 4.2 : Let f_1, f_2 and g be three transcendental entire functions. If $\rho_g^{[p+1,p]}(f_1)$ and $\rho_g^{[p+1,p]}(f_2)$ be the $(p+1,p)^{th}$ relative Gol'dberg order of f_1 and f_2 respectively with respect to g. If $f_1 \sim f_2$ then $\rho_g^{[p+1,p]}(f_1) = \rho_g^{[p+1,p]}(f_2)$.

Proof : Since $f_2 \sim f_1$, we have for $\epsilon > 0$ and for all large R,

$$M_{f_2,D}(exp^{[p-1]}R) < (1+\epsilon)M_{f_1,D}(exp^{[p-1]}R).$$

Now,

$$\begin{aligned}
\rho_g^{[p+1,p]}(f_2) &= \limsup_{R\to\infty} \frac{log^{[p]}M_{g,D}^{-1}(M_{f_2,D}(exp^{[p-1]}R))}{logR} \\
&\le \limsup_{R\to\infty} \frac{log^{[p]}M_{g,D}^{-1}((1+\epsilon)M_{f_1,D}(exp^{[p-1]}R))}{logR} \\
&\le \limsup_{R\to\infty} \frac{log^{[p]}M_{g,D}^{-1}(M_{f_1,D}(exp^{[p-1]}R^{(1+\epsilon)}))}{logR^{(1+\epsilon)}}.(1+\epsilon) \\
&= (1+\epsilon)\rho_g^{[p+1,p]}(f_1).
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary so,

$$\rho_g^{[p+1,p]}(f_2) \le \rho_g^{[p+1,p]}(f_1).$$

Also if $f_1 \sim f_2$ so

$$\rho_g^{[p+1,p]}(f_1) \leq \rho_g^{[p+1,p]}(f_2).$$

Hence, $\rho_g^{[p+1,p]}(f_1) = \rho_g^{[p+1,p]}(f_2)$.

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