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A COINCIDENCE THEOREM FOR WEAKLY RECIPROCALLY CONTINUOUS SYSTEMS OF SINGLE-VALUED AND MULTI-VALUED MAPS

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Abstract

In this paper we prove a coincidence theorem for systems of single-valued and multi-valued maps on finite product of metric spaces. Our result generalizes the results of Matkowski [14], Czerwik [3], Singh-Kulshreshtha [26] and others. We also gave some remarks on the paper of Gairola et al. [4] and Chauhan et al. [2].

1. Introduction

In 1973 Matkowski [14] generalized the Banach contraction principal for a system of n maps on a finite product of metric spaces. Czerwik [3] extend this result for a system of multi-valued maps. After that the result of Matkowski [op. cit.] has been extended and generalized by several authors (see, for instance Reddy- Subrahmanyam [21]-[22], Singh-Kulshrestha [26], Singh-Gairola [23]-[24], Baillon-Singh [1], Matkowski-Singh [16], Gairola et al. [9]-[10], Gairola-Jangwan [5]-[6] and others).

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The study of contractive maps, which does not force the map to be continuous at the fixed point was initiated by Pant (see [18]-[19]). In [18], Pant proved a common fixed point theorem without any continuity requirment by introducing the notation of reciprocal continuity for a pair of single-valued maps. Later on Pant et al. [20] improved the notation of reciprocal continuity by introducing weak reciprocal continuity for a pair of single-valued maps which states that if S and T are maps on a metric space (Y, d) then the pair (S, T) is weakly reciprocally continuous if and only if $\lim_{n\to\infty} STx_n = St$ or $\lim_{n\to\infty} TSx_n = Tt$, whenever$\{x_n\}$ is a sequence in Y such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some t in Y. Recently Gairola et al. [4], extend the idea of weak reciprocal continuity for a hybrid pair of single-valued and multi-valued maps (cf. Definition 2.6 below).

In this paper we proved a coincidence theorem for systems of single-valued and multi-valued maps on finite product of metric spaces and showed that the requirement of continuity is not necessary for existence of coincidence point on product of metric spaces. We do this by introducing a new class of maps - coordinatewise weakly reciprocally continuous systems of single-valued and multi-valued maps.

2. Notations and Definitions

Let (Y, d) be a metric space. We follow the following notations of Nadler [17] and Khan [13].

$CL(Y) = \{A : A \text{ is a non-empty closed subset of } Y\}$,

$C(Y) = \{A : A \text{ is a non-empty compact subset of } Y\}$.

For any non-empty subsets A, B of Y and $x \in Y$,

$D(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$,

$d(x, A) = \inf\{d(x, a) : a \in A\}$,

$H(A, B) = \max[\sup\{D(a, B) : a \in A\}, \sup\{D(A, b) : b \in B\}]$,

where H is called the generalized Hausdorff metric for $CL(Y)$ induced by metric d and $(CL(Y), H)$ is called generalized Hausdorff metric space.

Let (a_{ik}) be an $n \times n$ square matrix with non-negative entries defined in Matkowski [14]-[15] (see also [1], [3]).

$$c_{ik}^{(0)} = \begin{cases} a_{ik}, & i \neq k \\ 1 - a_{ik}, & i = k \end{cases} \quad i, k = 1, \ldots, n \tag{2.1}$$

$$c_{ik}^{(t+1)} = \begin{cases} c_{11}^{(t)} c_{i+1,k+1}^{(t)} + c_{i+1,1}^{(t)} c_{1,k+1}^{(t)}, & i \neq k \\ c_{11}^{(t)} c_{i+1,k+1}^{(t)} - c_{i+1,1}^{(t)} c_{1,k+1}^{(t)}, & i = k \end{cases} \tag{2.2}$$

$t = 1, \ldots, n-1, i, k = 1, \ldots, n-t.$

$$c_{ii}^{(t)} > 0, t = 1, \ldots, n, i = 1, \ldots, n+1-t. \tag{2.3}$$

Throughout the paper we shall assume that $(X_i, d_i), i = 1, ..., n$, are metric spaces, $(CL(X_i), H_i)$the generalized Hausdorff metric spaces induced by d_i. Further, let $X = X_1 \times \cdots \times X_n, x = (x_1, \ldots, x_n)$ and $\{x^m\} = \{(x_1^m, \ldots, x_n^m)\}, m \in \mathbb{N}$ (natural numbers) be a sequence in X. For $M = (M_1, \ldots, M_n) \subset X$, we use the notation $f(M) = (f_1 M_1, \ldots, f_n M_n)$ as in [1].

Now we begin by briefly recalling some basic definitions which will be needed in the sequel. In the following definitions we assume that $T_i : X \to CL(X_i), i = 1, \ldots, n$, are multi-valued maps and $f_i : X \to X_i, i = 1, \ldots, n$, are single-valued maps.

Definition 2.1 [1] : Two systems of maps $(f_1, \ldots, f_n)$ and $(T_1, \ldots, T_n)$ are coordinatewise commuting at a point $x \in X$ if and only if

$$f_i(T_1 x, \ldots, T_n x) \subseteq T_i(f_1 x, \ldots, f_n x), i = 1, \ldots, n.$$

For $n = 1$, this definition is that of Itoh-Takahashi [11].

Definition 2.2 [1] : Two systems of maps $(f_1, \ldots, f_n)$ and $(T_1, \ldots, T_n)$ are coordinatewise weakly commuting at a point $x \in X$ if and only if

$$H_i(f_i(T_1 x, \ldots, T_n x), T_i(f_1 x, \ldots, f_n x)) \leq D_i(T_i x, f_i x), i = 1, \ldots, n.$$

For $n = 1$ this definition is due to Kaneko [12] (see, Singh et al. [25]). Two systems are coordinatewise weakly commuting on X if and only if they are coordinatewise weakly commuting at every point of X.

Definition 2.3 [9] : Two systems of maps $(f_1, \ldots, f_n)$ and $(T_1, \ldots, T_n)$ are coordinatewise asymptotically commuting (or simply asymptotically commuting) if and only if

$$H_i(f_i(T_1 x^m, \ldots, T_n x^m), T_i(f_1 x^m, \ldots, f_n x^m)) \to 0 \text{ as } m \to \infty,$$

whenever $\{x^m\}$ is a sequence in X such that $T_i x^m \to M_i \in CL(X_i)$ and $f_i x^m \to t_i \in M_i, i = 1, \ldots, n$.

An equivalent formulation of the above definition for two systems of single-valued maps appears in [10].

Remark 2.1 : Notice that coordinatewise weak commutativity of two systems of maps $(f_1, \ldots, f_n)$ and $(T_1, \ldots, T_n)$ on X implies their coordinatewise asymptotic commutativity, however converse need not be true (see [9]).

Remark 2.2 : Coordinatewise weak commutativity and asymptotic commutativity of two systems of maps $(f_1, \ldots, f_n)$ and $(T_1, \ldots, T_n)$ at a coincidence point z (that is, when $f_i z \in T_i z, i = 1, \ldots, n)$ is equivalent to their coordinatewise commutativity, however coordinatewise commutativity of systems $(f_1, \ldots, f_n)$ and $(T_1, \ldots, T_n)$ is more general than their weak commutativity and asymptotic commutativity at their coincidence point z (see Example 2.2 [8]).

Definition 2.4 [8] : Two systems of maps $(f_1, \ldots, f_n)$ and $(T_1, \ldots, T_n)$ are coordinatewise reciprocally continuous on X (resp. at $t \in X$) if and only if $f_i(T_1 x, \ldots, T_n x) \in CL(X_i)$ for each $x \in X$ (resp., $f_i(T_1 t, \ldots, T_n t) \in CL(X_i), i = 1, \ldots, n)$ and

$$\lim_{m\to\infty} f_i(T_1 x^m, \ldots, T_n x^m) = f_i M, \lim_{m\to\infty} T_i(f_1 x^m, \ldots, f_n x^m) = T_i t,$$

whenever $\{x^m\}$ is a sequence in X_i such that $\lim_{m\to\infty} T_i x^m = M_i \in CL(X_i)$, $\lim_{m\to\infty} f_i x^m = t_i \in M_i, i = 1, \ldots, n$.

For $n = 1$, this definition is due to Singh-Mishra [27]. An equivalent formulation of the above definition for two systems of single-valued maps appears in [7].

If two systems $(f_1, \ldots, f_n)$ and $(T_1, \ldots, T_n)$ both are continuous then they are obviously coordinatewise reciprocally continuous but converse need not be true (see [8], [27]).

Definition 2.5 : Two systems of maps $(f_1, \ldots, f_n)$ and $(T_1, \ldots, T_n)$ are coordinatewise weakly reciprocally continuous on X (resp. at $t \in X$) if and only if $f_i(T_1 x, \ldots, T_n x) \in CL(X_i)$ for each $x \in X$ (resp., $f_i(T_1 t, \ldots, T_n t) \in CL(X_i), i = 1, \ldots, n)$ and

$$\lim_{m\to\infty} f_i(T_1 x^m, \ldots, T_n x^m) = f_i M, \text{ or } \lim_{m\to\infty} T_i(f_1 x^m, \ldots, f_n x^m) = T_i t,$$

whenever $\{x^m\}$ is a sequence in X_i such that $\lim_{m\to\infty} T_i x^m = M_i \in CL(X_i)$ and $\lim_{m\to\infty} f_i x^m = t_i \in M_i, i = 1, \ldots, n$.

As a special case of the above definition for $n = 1$, we have the following definition introduced in [4].

Definition 2.6 : The mapping $f_1 : X_1 \to X_1$ and $T_1 : X_1 \to CL(X_1)$ are weakly reciprocally continuous on X_1 (resp. at $t \in X_1$) if and only if $f_1 T_1 x \in CL(X_1)$ for each

$x \in X_1$ (resp., $f_1T_1 \in CL(X_1)$) and

$$\lim_{n\to\infty} f_1T_1x_n = f_1M_1 \text{ or } \lim_{n\to\infty} T_1f_1x_n = T_1t,$$

whenever $\{x_n\}$ is a sequence in X_1 such that $\lim_{n\to\infty} T_1x_n = M_1 \in CL(X_1)$ and $\lim_{n\to\infty} f_1x_n = t \in M_1$.

If the map T_1 in Definition 2.6 is single-valued then M_1 has just a single element t, and we get the definition of weak reciprocal continuity for single-valued self maps introduced by Pant et al. [20].

If systems of maps $(f_1, \ldots, f_n)$ and $(T_1, \ldots, T_n)$ are coordinatewise reciprocally continuous then they are obviously coordinatewise weakly reciprocally continuous but converse need not be true. The following example shows the coordinatewise weak reciprocal continuity of two systems of maps and illustrates that the coordinatewise weak reciprocal continuity of two systems of maps does not imply their reciprocal continuity.

Example 2.1 : Let $X_1 = X_2 = [0, \infty)$ be usual metric spaces and $T_i : X_1 \times X_2 \to CL(X_i), f_i : X_1 \times X_2 \to X_i, i = 1, 2$, be such that

$$T_1(x_1, x_2) = \begin{cases} [0, x_1] & \text{if } x_1 \leq 3 \\ [4, 2 + x_1] & \text{if } x_1 > 3 \end{cases}, \quad f_1(x_1, x_2) = \begin{cases} x_1 & \text{if } x_1 < 3 \\ 5 & \text{if } x_1 \geq 3 \end{cases},$$

$$T_2(x_1, x_2) = \begin{cases} [0, x_2] & \text{if } x_2 \leq 3 \\ [4, 2 + x_2] & \text{if } x_2 > 3 \end{cases}, \quad f_2(x_1, x_2) = \begin{cases} x_2 & \text{if } x_2 < 3 \\ 5 & \text{if } x_2 \geq 3 \end{cases}.$$

Suppose $\{x^m\}$ be a sequence in $X_1 \times X_2$ such that $T_ix^m \to M_i \in CL(X_i)$ and $f_ix^m \to t_i$, for some $t_i \in M_i, i = 1, 2$, as $m \to \infty$. Then for $t = (3, 3)$ and $\{x^m\} = \{(3-\epsilon_m, 3-\epsilon_{m'})\}$ where $m, m' \in \mathbb{N}$ and $\epsilon_m, \epsilon_{m'} \to 0$ as $m, m' \to \infty$ resp.. We have $T_ix^m \to [0, 3] = M_i, f_ix^m \to 3 = t_i \in M_i$ and $T_i(f_1x^m, f_2x^m) \to [0, 3] = T_it, f_i(T_1x^m, T_2x^m) \to [0, 3] \neq f_i(M_1, M_2), i = 1, 2$, as $m \to \infty$. Hence the systems of maps $\{f_1, f_2\}$ and $\{T_1, T_2\}$ are coordinatewise weakly reciprocally continuous but not coordinatewise reciprocally continuous at $t = (3, 3)$. However it is easy to see that each system of maps $\{f_1, f_2\}$ and $\{T_1, T_2\}$ is discontinuous at $t = (3, 3)$.

Since at $t = (4, 4)$, systems of maps $\{f_1, f_2\}$ and $\{T_1, T_2\}$ both are continuous, hence they are obviously weakly reciprocally continuous at this point. However there does not exist any sequence $\{x^m\} \in X_1 \times X_2$ such that $T_ix^m \to M_i \in CL(X_i)$ and $f_ix^m \to t_i$, for some $t_i \in M_i, i = 1, 2$.

Remark 2.3 : The coordinatewise weak reciprocal continuity of two systems of maps $(T_1, \ldots, T_n)$ and $(f_1, \ldots, f_n)$ at a point $t \in X$ may be verified by considering all se-

quences $\{x^m\} \in X$ such that $T_i x^m = M_i \in CL(X_i)$ and $f_i x^m = t_i \in M_i, i = 1, \ldots, n$. If there does not exist such a sequence then the definition of coordinatewise weak reciprocal continuity holds vacuously. The same observation applies for coordinatewise reciprocally continuous maps and asymptotically commuting maps.

3. Coincidence Theorem

Now we state our main result.

Theorem 3.1 : Let $(X_i, d_i), i = 1, \ldots, n$, be complete metric spaces and $T_i : X \to CL(X_i), f_i : X \to X_i$, be such that

$$T_i(X) \subset f_i(X), i = 1, \ldots, n. \tag{3.1}$$

The systems of maps $(T_1, \ldots, T_n)$ and $(f_1, \ldots, f_n)$ are coordinatewise weakly reciprocally continuous and coordinatewise asymptotically commuting on X. (3.2)

If there exist non-negative numbers $b < 1$ and $a_{ik}, i, k = 1, \ldots, n$, defined in (2.1) and (2.2) such that (2.3) and the following hold:

$$H_i(T_i x, T_i y) \leq \max_i \left\{ \sum_{k=1}^{n} a_{ik} d_k(f_k x, f_k y), b \max \left\{ \begin{array}{c} D_i(f_i x, T_i x), D_i(f_i y, T_i y), \\ \frac{D_i(f_i x, T_i y) + D_i(f_i y, T_i x)}{2} \end{array} \right\} \right\} \tag{3.3}$$

for all $x, y \in X$. Then there exists a point $v \in X$ such that

$$f_i v \in T_i v, i = 1, \ldots, n. \tag{3.4}$$

Proof : First we note that the system (2.3) and

$$\sum_{k=1}^{n} a_{ik} r_k < r_i, i = 1, \ldots, n,$$

are equivalent for some positive numbers $r_1, \ldots, r_n$. Further if we put

$$h = \max \left\{ r_i^{-1} \sum_{k=1}^{n} a_{ik} r_k \right\}$$

then $h \in (0, 1)$ and we may choose positive numbers $r_1, \ldots, r_n$ such that

$$\sum_{k=1}^{n} a_{ik} r_k \leq h r_i, i = 1, \ldots, n.$$

Pick x_i^0 in $X_i, i = 1, \ldots, n$. Since (3.1) holds, we can find a point $x^1 \in X$ such that $f_i x^1 \in T_i x^0, i = 1, \ldots, n$. For a suitable $x^2 \in X$ we can have a point $f_i x^2 \in T_i x^1, i = 1, \ldots, n$, such that

$$d_i(f_i x^1, f_i x^2) \leq c^{-1/2} H_i(T_i x^0, T_i x^1), i = 1, \ldots, n,$$

where $c = \max\{h, b\}$ and $c^{-1/2} > 1$. In general, we choose a sequence $\{x^m\}$ in X such that $f_i x^{m+1} \in T_i x^m$ and

$$d_i(f_i x^{m+1}, f_i x^{m+2}) \leq c^{-1/2} H_i(T_i x^m, T_i x^{m+1}), i = 1, \ldots, n; m = 0, 1, \ldots$$

If at any stage $f_i x^{m+1} = f_i x^{m+2}$ then $f_i x^{m+1} \in T_i x^{m+1}$ that is, x^{m+1} is a coincidence point of f_i and T_i and the proof is complete. So we assume that $f_i x^{m+1} \neq f_i x^{m+2}, m = 0, 1, 2, \ldots$. Without loss of generality, we may assume that

$$d_i(f_i x^1, f_i x^2) \leq r_i, i = 1, \ldots, n.$$

Then by (3.3), we have

$$\begin{aligned}
d_i(f_i x^2, f_i x^3) &\leq c^{-1/2} H_i(T_i x^1, T_i x^2) \\
&\leq c^{-1/2} \max \left\{ \sum_{k=1}^{n} a_{ik} d_k(f_k x^1, f_k x^2), b \max \left\{ \begin{array}{c} D_i(f_i x^1, T_i x^1), D_i(f_i x^2, T_i x^2), \\ \frac{D_i(f_i x^1, T_i x^2) + D_i(f_i x^2, T_i x^1)}{2} \end{array} \right\} \right\} \\
&\leq c^{-1/2} \max \left\{ \sum_{k=1}^{n} a_{ik} d_k(f_k x^1, f_k x^2), b \max \left\{ \begin{array}{c} d_i(f_i x^1, f_i x^2), d_i(f_i x^2, f_i x^3), \\ \frac{d_i(f_i x^1, f_i x^3)}{2} \end{array} \right\} \right\} \\
&\leq c^{-1/2} \max \left\{ \sum_{k=1}^{n} a_{ik} d_k(f_k x^1, f_k x^2), b \max \left\{ \; d_i(f_i x^1, f_i x^2), d_i(f_i x^2, f_i x^3) \; \right\} \right\}.
\end{aligned}$$

If $d_i(f_i x^2, f_i x^3) > d_i(f_i x^1, f_i x^2)$ then

$$\begin{aligned}
d_i(f_i x^2, f_i x^3) &\leq c^{-1/2} \max \left\{ \sum_{k=1}^{n} a_{ik} d_k(f_k x^1, f_k x^2), b d_i(f_i x^2, f_i x^3) \right\} \\
&\leq c^{-1/2} h r_i \leq c^{1/2} r_i,
\end{aligned}$$

since otherewise we get a contradiction. On the other hand if $d_i(f_i x^2, f_i x^3) \leq d_i(f_i x^1, f_i x^2)$ then

$$\begin{aligned}
d_i(f_i x^2, f_i x^3) &\leq c^{-1/2} \max \left\{ \sum_{k=1}^{n} a_{ik} d_k(f_k x^1, f_k x^2), b d_i(f_i x^1, f_i x^2) \right\} \\
&\leq c^{-1/2} \max\{h r_i, b r_i\} = c^{1/2} r_i, \text{ wherein } c = \max\{h, b\}.
\end{aligned}$$

Again from (3.3), we have

$$d_i(f_ix^3, f_ix^4) \leq c^{-1/2} \max \left\{ \sum_{k=1}^{n} a_{ik} d_k(f_kx^2, f_kx^3), b \max \{ \ d_i(f_ix^2, f_ix^3), d_i(f_ix^3, f_ix^4) \ \} \right\}$$
$$\leq c^{-1/2} \max \left\{ \sum_{k=1}^{n} a_{ik} c^{1/2} r_k, b \max \{ \ c^{1/2} r_i, d_i(f_ix^3, f_ix^4) \ \} \right\},$$

and arguing same as before this implies

$$d_i(f_ix^3, f_ix^4) \leq c^{2/2} r_i.$$

Inductively

$$d_i(f_ix^{m+1}, f_ix^{m+2}) \leq c^{m/2} r_i.$$

So each $\{f_ix^m\}$ is a Cauchy sequence in $X_i, i = 1, \ldots, n$ and X_i is a complete metric space. Therefore there exist a point t_i (say) in X_i such that the sequence $\{f_ix^m\}$ converges to t_i.

Since $f_ix^{m+1} \in T_ix^m$, it follows that the sequence $\{T_ix^m\}$ is also Cauchy in $CL(X_i), i = 1, \ldots, n$. So there exists M_i in $CL(X_i)$ such that $\{T_ix^m\}$ converges to M_i for each $i = 1, \ldots, n$. Thus

$$D_i(t_i, M_i) \leq d_i(t_i, f_ix^{m+1}) + D_i(f_ix^{m+1}, M_i)$$
$$< d_i(t_i, f_ix^{m+1}) + H_i(T_ix^m, M_i) \to 0,$$

as $m \to \infty$. This gives $t_i \in M_i, i = 1, \ldots, n$.

If systems $(T_1, \ldots, T_n)$ and $(f_1, \ldots, f_n)$ are coordinatewise weakly reciprocally continuous then $f_i(T_1x, \ldots, T_nx) \in CL(X_i)$ for each $x \in X$ and $\lim_{m\to\infty} f_i(T_1x^m, \ldots, T_nx^m) = f_iM$ or $\lim_{m\to\infty} T_i(f_1x^m, \ldots, f_nx^m) = T_it, i = 1, \ldots, n,$.

Case (I): Let us suppose that $\lim_{m\to\infty} f_i(T_1x^m, \ldots, T_nx^m) = f_iM, i = 1, \ldots, n$ then coordinatewise asymptotic commutativity of systems of maps $(T_1, \ldots, T_n)$ and $(f_1, \ldots, f_n)$ gives

$$H_i(T_i(f_1x^m, \ldots, f_nx^m), f_i(T_1x^m, \ldots, T_nx^m)) \to 0, \text{ as } m \to \infty,$$

that is

$$\lim_{m\to\infty} T_i(f_1x^m, \ldots, f_nx^m) = \lim_{m\to\infty} f_i(T_1x^m, \ldots, T_nx^m) = f_iM. \tag{3.5}$$

Since

$$f_i x^{m+1} \in T_i x^m$$

therefore

$$f_i(f_1 x^{m+1}, \ldots, f_n x^{m+1}) \in f_i(T_1 x^m, \ldots, T_n x^m)$$

and

$$\lim_{m\to\infty} f_i(f_1 x^{m+1}, \ldots, f_n x^{m+1}) = z_i \text{ (say) } \in f_i M. \tag{3.6}$$

Now as $z_i \in f_i M$, there exists a point $v \in X$ such that

$$f_i v = z_i, i = 1, \ldots, n. \tag{3.7}$$

From (3.3), with $fx^{m+1} := (f_1 x^{m+1}, ..., f_n x^{m+1})$,

$$H_i(T_i f x^{m+1}, T_i v) \leq \max \left\{ \begin{array}{l} \sum\limits_{k=1}^{n} a_{ik} d_k(f_k f x^{m+1}, f_k v), \\ b \max \left\{ \begin{array}{c} D_i(f_i f x^{m+1}, T_i f x^{m+1}), D_i(f_i v, T_i v), \\ \frac{D_i(f_i f x^{m+1}, T_i v) + D_i(f_i v, T_i f x^{m+1})}{2} \end{array} \right\} \end{array} \right\}. \tag{3.8}$$

Making $m \to \infty$ and using (3.5), (3.6) and (3.7), we have

$$\begin{aligned} H_i(f_i M, T_i v) &\leq \max \left\{ \sum_{k=1}^{n} a_{ik} d_k(f_k v, f_k v), b \max \left\{ \begin{array}{c} D_i(f_i v, f_i M), D_i(f_i v, T_i v), \\ \frac{D_i(f_i v, T_i v) + D_i(f_i v, f_i M)}{2} \end{array} \right\} \right\} \\ &= b D_i(f_i v, T_i v) \leq b H_i(f_i M, T_i v), \end{aligned}$$

implies that

$$H_i(f_i M, T_i v) = 0.$$

This gives $f_i M = T_i v, i = 1, \ldots, n$. As $f_i v \in f_i M$ then

$$f_i v \in T_i v, i = 1, \ldots, n.$$

Thus the system (3.4) has a solution $v = (v_1, \ldots, v_n)$ in X.

Case (II) : Let us assume that $\lim\limits_{m\to\infty} T_i(f_1 x^m, \ldots, f_n x^m) = T_i t, i = 1, \ldots, n$, then coordinatewise asymptotic commutativity of systems of maps $(T_1, \ldots, T_n)$ and $(f_1, \ldots, f_n)$ yields

$$\lim_{m\to\infty} T_i(f_1 x^m, \ldots, f_n x^m) = \lim_{m\to\infty} f_i(T_1 x^m, \ldots, T_n x^m) = T_i t. \tag{3.9}$$

Since

$$f_i x^{m+1} \in T_i x^m$$

then

$$f_i(f_1 x^{m+1}, \ldots, f_n x^{m+1}) \in f_i(T_1 x^m, \ldots, T_n x^m)$$

and

$$\lim_{m\to\infty} f_i(f_1 x^{m+1}, \ldots, f_n x^{m+1}) = z_i \text{ (say) } \in T_i t. \tag{3.10}$$

From (3.1) and (3.10), there exists a point $v \in X$ such that

$$f_i v = z_i, i = 1, \ldots, n. \tag{3.11}$$

By (3.3), with $fx^{m+1} := (f_1 x^{m+1}, \ldots, f_n x^{m+1})$,

$$H_i(T_i f x^{m+1}, T_i v) \leq \max \left\{ \begin{array}{l} \sum\limits_{k=1}^{n} a_{ik} d_k(f_k f x^{m+1}, f_k v), \\ b \max \left\{ \begin{array}{c} D_i(f_i f x^{m+1}, T_i f x^{m+1}), D_i(f_i v, T_i v), \\ \frac{D_i(f_i f x^{m+1}, T_i v) + D_i(f_i v, T_i f x^{m+1})}{2} \end{array} \right\} \end{array} \right\}. \tag{3.12}$$

Making $m \to \infty$ and using (3.9), (3.10) and (3.11), we have

$$\begin{aligned} H_i(T_i t, T_i v) &\leq \max \left\{ \sum_{k=1}^{n} a_{ik} d_k(f_k v, f_k v), b \max \left\{ \begin{array}{c} D_i(f_i v, T_i t), D_i(f_i v, T_i v), \\ \frac{D_i(f_i v, T_i v) + D_i(f_i v, T_i t)}{2} \end{array} \right\} \right\} \\ &= b D_i(f_i v, T_i v) \leq b H_i(T_i t, T_i v). \end{aligned}$$

This gives

$$H_i(T_i t, T_i v) = 0,$$

which implies

$$T_i t = T_i v.$$

As $f_i v \in T_i t, i = 1, \ldots, n$. Then by the above we have

$$f_i v \in T_i v, i = 1, \ldots, n.$$

This proves that the system (3.4) has a solution $v = (v_1, \ldots, v_n)$ in X. □

If we take $f_i x = x_i, i = 1, \ldots, n$, in Theorem 3.1 then the following Corollary is an immediate consequence from Theorem 3.1.

Corollary 3.1 : Let $(X_i, d_i), i = 1, \ldots, n$, be complete metric spaces. If $T_i : X \to CL(X_i), i = 1, \ldots, n$ satisfy (2.1), (2.2) (2.3) and

$$H_i(T_i x, T_i y) \leq \max_i \left\{ \sum_{k=1}^{n} a_{ik} d_k(x_k, y_k), b \max \left\{ \begin{array}{c} D_i(x_i, T_i x), D_i(y_i, T_i y), \\ \frac{D_i(x_i, T_i y) + D_i(y_i, T_i x)}{2} \end{array} \right\} \right\}$$

for all $x, y \in X$. Then there exists a point $v \in X$ such that $v_i \in T_i v, i = 1, \ldots, n$.
Here it is remarkable that if $T_i, i = 1, \ldots, n$ are single-valued maps in Corollary 3.1 then v is necessarily unique. Result of Matkowski [14] may be obtained as a special case from Corollary 3.1.

Corollary 3.2 : Let $T : Y \to CL(Y)$ and $f : Y \to Y$ are weakly reciprocally continuous and asymptotically commuting (or compatible) maps in a complete metric space (Y, d) such that $T(Y) \subset f(Y)$ and satisfying

$$H(Tx, Ty) \leq k \max \left\{ d(fx, fy), D(fx, Tx), D(fy, Ty), \frac{D(fx, Ty) + D(fy, Tx)}{2} \right\}$$

for all $x, y \in Y$. Then there exists a point $v \in Y$ such that $fv \in Tv$.

Proof : Proof may be completed by putting $(Y, d) = (X_i, d_i), T = T_i, f = f_i, i = 1, \ldots, n$ and $n = 1, k = \max\{a_{11}, b\}$ in the proof of Theorem 3.1. □

The following is the statement of Theorem 1 of Gairola et al. [4].

Theorem 3.2 : Let $T : Y \to C(Y)$ and $f : Y \to Y$ are weakly reciprocally continuous and non-vacuously compatible maps of a metric space (Y, d) satisfying condition $T(Y) \subset f(Y)$ and

$$\begin{aligned} H(Tx, Ty) \leq & a(x, y) d(fx, fy) + b(x, y) \max\{d(fx, Tx), d(fy, Ty)\} \\ & + c(x, y) \left[d(fx, Ty) + d(fy, Tx) \right] \end{aligned}$$

where a, b, c are non-negative function from $Y \times Y \to [0, 1)$ such that $\beta = \inf_{x,y \in Y} b(x, y) > 0, \gamma = \inf_{x,y \in Y} c(x, y) > 0$, and

$$sup_{x,y \in Y} \left[a(x, y) + b(x, y) + c(x, y) \right] = 1.$$

Then T and f have a coincidence point. Further, if $fft = ft$ for some $t \in C(T, f)$ then f and T have a common fixed point.

Remark 3.1 :

(i) In the proof of the above theorem authors assume that if $fx_n \to t$ then $ffx_n \to ft$ as $n \to \infty$ (see [4], page 708, line 12 from below). However it is not always true in case, when maps f and T are weakly reciprocally continuous.

(ii) In paper [2], authors used the same technique to prove a fixed point theorem for a hybrid pair of weak reciprocal continuous maps by employing an implicit relation and compatibility (see [2], page 78, line 12 from below). The following example illustrates this concept.

Example 3.1 : Let $Y = [0, \infty)$ be usual metric space and $T : Y \to C(Y), f : Y \to Y$ be such that

$$T(x) = \begin{cases} [0, x] & \text{if } x \leq 1 \\ [2, 2 + x] & \text{if } x > 1 \end{cases}, \quad f(x) = \begin{cases} x & \text{if } x < 1 \\ 3 & \text{if } x \geq 1 \end{cases}.$$

For $x = 1$, there exist a sequence $x_n = \{1 - \epsilon_n\} \in Y$ such that $fx_n \to 1, Tx_n \to [0, 1]$ as $n \to \infty$ and $1 \in [0, 1]$. We observe that $fTx_n \to [0, 1] \neq f[0, 1]$ and $Tfx_n \to [0, 1] = T1$ as $n \to \infty$. Hence pair of maps (T, f) is weakly reciprocally continuous at $x = 1$ but $ffx_n \to 1 \neq f1$.

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