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EFFECT OF EDGE REMOVAL ON SOME PARAMETERS OF HYPERGRAPH

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Abstract

In this paper we have studied the effect of removing an edge on some parameters of hypergraph mainly on edge h-domination number, edge covering number and strong edge covering number. We have proved a necessary and sufficient condition under which the edge h-domination number decreases when an edge is removed from the hypergraph. We have also proved that edge covering number of a hypergraph increases or remains same when an edge is removed from the hypergraph. However the strong edge covering number decreases or remains same when an edge is removed from the hypergraph.

1. Introduction

The concept of edge domination in hypergraphs was studied in [7, 8, 9]. This concept

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was defined using the adjacency relation among the edges of a hypergraph. We introduce a new concept called edge h-domination in hypergraphs [10]. This concept is stronger than the edge domination in hypergraphs. In this paper we observe the effect of removing an edge on the edge h-domination number of a hypergraph.

The concept of edge cover is well known for graphs. An edge cover of a graph is a set of edges such that every vertex of the graph is an end vertex of some member of this set. We introduce the concept of an edge cover for hypergraphs [11]. An edge cover of a hypergraph is a set of edges which covers all the vertices of the hypergraph. In this paper we consider the operation of removing an edge from the hypergraph and observe its effect on the edge covering number of a hypergraph. We also introduce a new concept called strong edge cover of a hypergraph. If a strong edge cover contains all the isolated edges then it is also an edge h- dominating set. In a hypergraph with minimum edge degree ≥ 2 every strong edge cover is an edge cover. For strong edge cover also we have considered the operation of edge removal.

2. Preliminaries

Definition 2.1 (Hypergraph) [4] : A hypergraph G is an ordered pair $(V(G), E(G))$ where $V(G)$ is a non-empty finite set and $E(G)$ is a family of non-empty subsets of $V(G) \ni$ their union $= V(G)$. The elements of $V(G)$ are called vertices and the members of $E(G)$ are called *edges of the hypergraph* G.

We make the following assumption about the hypergraph.

(1) Any two distinct edges intersect in at most one vertex.

(2) If e_1 and e_2 are distinct edges with $|e_1|, |e_2| > 1$ then $e_1 \not\subseteq e_2$ and $e_2 \not\subseteq e_1$.

Definition 2.2 (Edge Degree) [4] : Let G be a hypergraph amd $v \in V(G)$ then the *edge degree* of $v = d_E(v) =$ the number of edges containing the vertex v. The minimum edge degree among all the vertices of G is denoted as $\delta_E(G)$ and the maximum edge degree is denoted as $\Delta_E(G)$.

Definition 2.3 (Dominating Set in Hypergraph) [1] : Let G be a hypergraph and $S \subseteq V(G)$ then S is said to be a *dominating set* of G if for every $v \in V(G) - S$ there is $u \in S \ni u$ and v are adjacent vertices.

A dominating set with minimum cardinality is called minimum dominating set and cardinality of such a set is called *domination number* of G and it is denoted as $\gamma(G)$.

Definition 2.4 (Edge Dominating Set) [7] : Let G be a hypergraph and $S \subseteq E(G)$ then S is said to be an *edge dominating set* of G if for every $e \in E(G) - S$ there is some f in $S \ni e$ and f are adjacent edges.

An edge dominating set with minimum cardinality is called a *minimum edge dominating set* and cardinality of such a set is called *edge domination number* of G and it is denoted as $\gamma_E(G)$.

Definition 2.5 (Sub Hypergraph and Partial Sub Hypergraph) [3] : Let G be a hypergraph and $v \in V(G)$. Consider the subset $V(G) - \{v\}$ of $V(G)$. This set will induce two types of hypergraphs from G.

(1) First type of hypergraph: Here the vertex set $= V(G) - \{v\}$ and the edge set $= \{e'/e' = e - \{v\}\ \textit{for some}\ e \in E(G)\}$. This hypergraph is called the *sub hypergraph of* G and it is denoted as $G - \{v\}$.

(2) Second type of hypergraph: Here also the vertex set $= V(G) - \{v\}$ and edges in this hypergraph are those edges of G which do not contain the vertex v.This hypergraph is called the *partial sub hypergraph* of G.

Definition 2.6 (Edge h-Dominating Set) [10] : Let G be a hypergraph. A collection F of edges of G is called an edge *h-dominating set* of G if

(1) All isolated edges of G are in F.

(2) If f is not an isolated edge and $f \notin F$ then there is a vertex x in $f \ni$ edge degree of $x \geq 2$ and all the edges containing x except f are in F.

An edge h-dominating set with minimum cardinality is called a *minimum edge h-dominating set* of G and its cardinality is called edge h-domination number of G and it is denoted as $\gamma'_h(G)$.

Definition 2.7 (Edge Cover in Hypergraph) [11] : Let G be a hypergraph and F be a set of edges of G then F is said to be an edge cover of G if for every vertex x there is an edge e in $F \ni x \in e$.

Definition 2.8 (Minimal Edge Cover in Hypergraph) [11] : Let G be a hypergraph and F be an edge cover of G then F is said to be a minimal edge cover of G if no proper subset of F is an edge cover of G. Equivalently for every e in F, $F - \{e\}$ is not an edge cover of G.

Definition 2.9 (Minimum Edge Cover in Hypergraph) [11] : An edge cover with minimum cardinality is called a minimum edge cover of G.

Definition 2.10 (Edge Covering Number) [11] : Let G be a hypergraph. The cardinality of a minimum edge cover is called the *edge covering number* of the hypergraph G and it is denoted as $\alpha_1(G)$.

3. Main Results

We assume that all the hypergraphs considered here are linear. This means for any two distinct vertices u and v there is at most one edge which contains both u and v.

We consider the operation of removing an edge from the hypergraph. We will prove a necessary and sufficient condition under which the edge h-domination number decreases when an edge is removed from the hypergraph.

Theorem 3.1 : Let G be a hypergraph and e be an isolated edge of G then $\gamma'_h(G-e) < \gamma'_h(G)$ iff there is a minimum edge h-dominating set F of $G \ni e \in F$.

Proof : Suppose $\gamma'_h(G-e) < \gamma'_h(G)$.

Let F_1 be a minimum edge h-dominating set of $G - e$.

Let $F = F_1 \cup \{e\}$. Let h be any edge of G different from e. If h is an isolated edge of G then h is also an isolated edge of $G - e$ and therefore $h \in F_1$ which is a subset of F. Thus, $h \in F$.

Suppose h is not an isolated edge of G and $h \notin F$ then $h \notin F_1$ and h is not an isolated edge of $G - e$. Therefore, there is a vertex x in $h \ni$ edge degree of $x \geq 2$ and all the edges of $G - e$ containing x and different from h are in F_1. These are the edges of G also and they are in F. Note that none of them is e. Thus, all the edges of G containing x (except h) are in F.

We have proved that F is an edge h-dominating set of G containing e. Since $|F| = |F_1|+1$ and $\gamma'_h(G-e) < \gamma'_h(G)$, F must be a minimum edge h- dominating set of G. Thus, the condition is satisfied.

Conversely suppose there is a minimum edge h-dominating set F of $G \ni e \in F$.

Let $F_1 = F - \{e\}$. Let h be any edge of $G - e$. If h is an isolated edge of $G - e$ then $h \in F_1$ because h is also an isolated edge of G.

Suppose h is not an isolated edge of $G - e$ and $h \notin F_1$ then h is not an isolated edge of G and $h \notin F$. Since, F is an edge h- dominating set of G there is a vertex x in h with edge degree of $x \geq 2$ and all the edges of G containing x (except h) are in F. Note that none of them is e because e is an isolated edge in G. Thus, all the edges of $G - e$ containing x (except h) are in F_1.

Thus, We have proved that F_1 is an edge h-dominating set of $G - e$.

$\therefore \quad \gamma'_h(G - e) \leq |F_1| < |F| = \gamma'_h(G)$.

$\therefore \quad \gamma'_h(G - e) < \gamma'_h(G)$. □

Definition 3.2 (Weakly h-isolated edge) : Let G be a hypergraph. F be a set of edges and $e \in F$ then e is said to be a weakly h-isolated edge of F if any one of the following two conditions is satisfied by e.

(1) e is an isolated edge of G.

(2) If e is not an isolated edge of G then $\forall$ x in e with edge degree of $x \geq 2$ there is an edge h_x containing x and different from $e \ni h_x \notin F$.

Note that if $F = E(G)$ and $e \in F$ then e is weakly h-isolated in F iff h is an isolated edge of G.

Definition 3.3 (h-adjacency of two edges) : Let G be a hypergraph. F be a set of edges and $e \in F$ and f be any edge of G then we say that f is h-adjacent to e with respect to F if the following conditions are satisfied.

(i) There is a vertex x in $e \cap f \ni$ all the edges containing x (except possibly f) are in F.

(ii) $\forall$ y in f with $y \neq x$ and with edge degree of $y \geq 2$ there is an edge hy containing $y \ni h_y \neq f$ and $h_y \notin F$.

Definition 3.4 (Private edge h-neighbourhood) : Let G be a hypergraph. F be a set of edges and $e \in F$.

(1) $e \in$ private edge h-neighbourhood of e w.r.t. F if e is a weakly h-isolated edge of F.

(2) If $h \notin F$ then $h \in$ private edge h-neighbourhood of e w.r.t. F if

(i) There is a vertex x in $e \cap h \ni$ all the edges containing x (except h) are in F.

(ii) $\forall y$ in h with $y \neq x$ and with edge degree of $y \geq 2$ there is an edge h_y containing $y \ni h_y \neq h$ and $h_y \notin F$.

It is denoted by $Prn_{eh}[e, F]$.

We can rewrite the definition of private edge h-neighbourhood in the following manner.

Definition 3.5 (Private edge h-neighbourhood) : Let G be a hypergraph. F be a set of edges and $e \in F$.

(1) $e \in$ private edge h-neighbourhood of e w.r.t. F if e is a weakly h-isolated edge of F.

(2) If $h \notin F$ then h is h-adjacent to e w.r.t F.

Theorem 3.6 : Let G be a hypergraph with minimum edge degree ≥ 3 and let e be any edge of G then $\gamma'_h(G - e) < \gamma'_h(G)$ iff there is a minimum edge h-dominating set F of $G \ni e \in F$.

Proof : Suppose $\gamma'_h(G - e) < \gamma'_h(G)$.

Let F_1 be any minimum edge h-dominating set of $G - e$ then F_1 cannot be an edge h-dominating set of G.

Let $F = F_1 \cup \{e\}$. Let h be any edge of $G \ni h \notin F$ then h is an edge of $G - e \ni h \notin F_1$. Since F_1 is an edge h-dominating set of $G - e$ there is some x in $h \ni$ all the edges containing x except h are in F_1. It follows that all the edges in G containing x except h are in F. Thus, F is an edge h-dominating set containing e. Since $|F| = |F_1| + 1$, F is a minimum edge h-dominating set of G containing e.

Conversely suppose there is a minimum edge h-dominating set F of G containing e.

Let $F_1 = F - \{e\}$. Let h be any edge of $G - e \ni h \notin F_1$ then $h \notin F$. Since, F is an edge h-dominating set of G there is some x in $h \ni$ all the edges of G containing x except h are in F. It follows that all the edges of $G - e$ containing x except h are in F_1. Thus, F_1 is an edge h-dominating set of $G - e$.

$\therefore \quad \gamma'_h(G - e) \leq |F_1| < |F| = \gamma'_h(G)$.

$\therefore \quad \gamma'_h(G - e) < \gamma'_h(G)$. □

Now, we consider the operation of removing an edge from a hypergraph and observe its effect on the edge covering number of a hypergraph.

When we remove an edge e from the hypergraph, we will assume that every vertex in e has edge degree ≥ 2.

We begin with the following proposition.

Proposition 3.7 : Let G be a hypergraph and e be an edge of G then $\alpha_1(G{-}e) \geq \alpha_1(G)$.

Proof : Let F be any minimum edge covering set of $G - e$ then obviously F is an edge covering set of G also. Therefore, $\alpha_1(G) \leq |F| \leq \alpha_1(G - e)$. □

Example 3.8 : Consider the hypergraph G whose vertex set $V(G) = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6\}$.

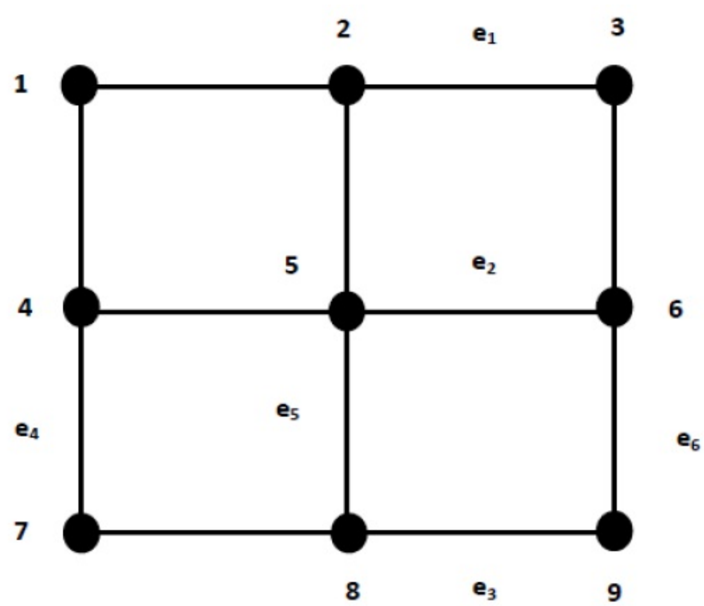

Figure 1

Here, $\alpha_1(G) = 3$ and $\alpha_1(G - e_5) = 3$.

$\therefore \quad \alpha_1(G - e) = \alpha_(G)$.

Example 3.9 : Consider the hypergraph G whose vertex set $V(G) = \{1, 2, 3, 4, 5, 6, 7\}$ and $E(G) = \{e_1, e_2, e_3, e_4\}$.

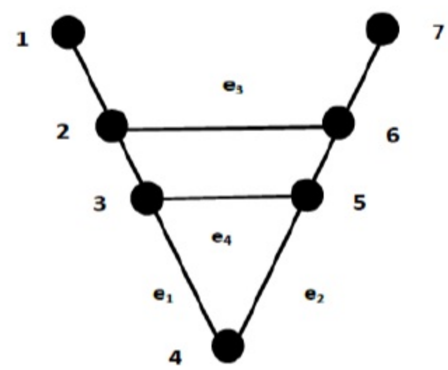

Figure 2

Here, $\alpha_1(G) = 2$ and $\alpha_1(G - e_1) = 3 \therefore \quad \alpha_1(G - e) > \alpha_1(G)$.

We stat and prove a necessary and sufficient condition under which the edge covering number remains same when an edge is removed from the hypergraph.

Theorem 3.10 : Let G be a hypergraph and e be an edge of G then $\alpha_1(G-e) = \alpha_1(G)$ iff there is a minimum edge cover F of $G \ni e \notin F$.

Proof : First suppose that there is a minimum edge cover F of $G \ni e \notin F$. Then obviously F is an edge cover of $G - e$ also.

$\therefore \quad \alpha_1(G - e) \leq |F| = \alpha_1(G) \leq \alpha_1(G - e)$.

$\therefore \quad \alpha_1(G - e) = \alpha_1(G)$.

Conversely suppose $\alpha_1(G - e) = \alpha_1(G)$. Let F be any minimum edge cover of $G - e$ then F is also a minimum edge cover of G. Since F is a set of edges of $G - e$, $e \notin F$. Thus the condition is satisfied. □

Corollary 3.11 : Let G be a hypergraph and e be an edge of G then $\alpha_1(G-e) > \alpha_1(G)$ iff for every minimum edge cover F of $G, e \in F$.

Proof : Obvious.

Example 3.12 : Consider the finite projective plane with r^2-r+1 vertices and r^2-r+1 edges $(r \geq 2)$. Let e be any edge of this hypergraph. Let x be any vertex which is not in e. Let $F = \{f \in E(G) \ni x \in f\}$ then F contains r edges and it is a minimum edge cover of G. Also $e \notin F$. Therefore, by the above theorem $\alpha_1(G - e) = \alpha_1(G)$.

Definition 3.13 (Strong Edge Cover in Hypergraph) : Let G be a hypergraph and F be a set of edges of G. Then F is said to be a *strong edge cover* of G if whenever

e and f are adjacent edges of G then $e \in F$ or $f \in F$.

Definition 3.14 (Strong Edge Covering Number) : Let G be a hypergraph. A strong edge cover with minimum cardinality is called a *minimum strong edge cover* and its cardinality is called strong edge covering number of the hypergraph G. It is denoted by $\alpha_s^1(G)$.

Example 3.15 : Consider the hypergraph G whose vertex set $V(G) = \{1, 2, 3, 4, 5, 6\}$ and $E(G) = \{e_1, e_2, e_3\}$.

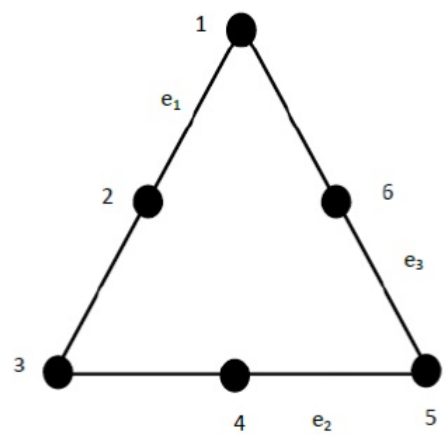

Figure 3

$F = \{e_1, e_2\}$ is a strong edge cover of this hypergraph.

Example 3.16 : Consider the finite projective plane G with $r^2 - r + 1$ vertices and $r^2 - r + 1$ edges ($r \geq 3$).

In this hypergraph every edge contains exactly r vertices and every vertex contained in exactly r edges. Also any two edges have a non-empty intersection and for any two vertices u and v there is exactly one edge which contains u and v.

Let v be a fixed vertex of G. There are exactly r edges say $e_1, e_2, \cdots, e_r$ which contains the vertex v. Let $F = \{e_1, e_2, \cdots, e_r\}$ then F is an edge cover of G. Let H be any set of edges which contain $r^2 - r - 1$ or less number of edges. Then $E(G) - H$ contains at least two edges.

Let $f, g \in E(G) - H$. Now, f and g are adjacent edges but $f \notin H$ and $g \notin H$. Thus, H is not a strong edge cover of G. Thus, any set of edges whose cardinality $\leq r^2 - r - 1$ cannot be a strong edge cover of G.

$\therefore$ The set F mentioned above cannot be a strong edge cover of G.

First we give a Characterization of Strong Edge Cover of a hypergraph.

Theorem 3.17 : Let G be a hypergraph and F be a set of edges of G. Then F is a strong edge cover of G iff $\forall\ \ v \in V(G)$ with edge degree of $v \geq 2$ there is at most one edge containing v which is not in F.

Proof : Suppose F is a strong edge cover of G. Let $v \in V(G) \ni$ edge degree of $v \geq 2$. If all the edges containing v are in F then the statement is proved.

Suppose e is an edge $\ni v \in e$ and $e \notin F$. Let f be any edge $\ni v \in f$ and $f \neq e$. Now, f and e are adjacent edges of G because $v \in e \cap f$. Since F is a strong edge cover of G and $e \notin F \Rightarrow f \in F$.

Thus, all the edges containing v except e are in F.

Conversely suppose the condition is satisfied. Let f and e be adjacent edges of G and suppose $f \cap e = \{v\}$. Then edge degree of $v \geq 2$

$\therefore$ $e \in F$ or $f \in F$.

$\therefore$ F is a strong edge cover of G. □

Corollary 3.18 : Let G be a hypergraph and let F be a strong edge cover of G which contains all the isolated edges of G then F is an edge h-dominating set of G.

Proof : From the above theorem the result follows. □

Proposition 3.19 : Let G be a hypergraph with minimum edge degree of $G \geq 2$ then every strong edge cover of G is an edge cover of G.

Proof : Suppose F is a strong edge cover of G. Let $v \in V(G)$. Since minimum edge degree of $G \geq 2$ there are two edges e and $f \ni v \in e \cap f$. If $e \notin F \Rightarrow f \in F$. Thus, f is an edge of $F \ni v \in f$. Thus, F is an edge cover of G. □

Theorem 3.20 : Let G be a hypergraph and e be any edge of $G \ni$ for every x in e, edge degree of $x \geq 2$ then $\alpha_s^1(G - e) \leq \alpha_s^1(G)$.

Proof : Let F be a minimum strong edge cover of G.

Case 1 : $e \notin F$.

Then F is a set of edges of $G - e$. Suppose g and h are two edges of $G - e$ such that g and h are adjacent in $G - e$. Then g and h are also adjacent in e. Then $g \in F$ or $h \in F$.

Case 2 : $e \in F$.

Let $F_1 = F - \{e\}$. Then F_1 is a set of edges of $G - e$. It can be easily seen that F_1 is a strong edge cover of $G - e$.

From both the cases it follows that there is a set of edges, whose cardinality is $\leq \alpha_s^1(G)$ and it is a strong edge cover of $G - e$.

$\therefore \quad \alpha_s^1(G-e) \leq \alpha_s^1(G)$. □

Theorem 3.21 : Let G be a hypergraph and e be any edge of $G \ni$ for every x in e, edge degree of $x \geq 2$ then $\alpha_s^1(G-e) < \alpha_s^1(G)$ iff there is a minimum strong edge cover F of $G \ni e \in F$.

Proof : First suppose that $\alpha_s^1(G-e) < \alpha_s^1(G)$. Let F_1 be a minimum strong edge cover of $G - e$. Then F_1 cannot be a strong edge cover of G. Therefore, there are two edges g and h of G which are adjacent but $g \notin F_1$ and $h \notin F_1$.

$\therefore$ If $g \neq e$ and $h \neq e$ then g and h are edges of $G - e$ and they are also adjacent. Since F_1 is a strong edge cover of $G - e$, $g \in F_1$ or $h \in F_1$. This contradicts the above statement. Thus, $g = e$ or $h = e$.

Now, let $F = F_1 \cup \{e\}$. Then F is a strong edge cover of G with $|F| = |F_1| + 1$. Since $\alpha_s^1(G) > \alpha_s^1(G-e)$, F is a minimum strong edge cover of G and $e \in F$.

Conversely suppose there is a minimum strong edge cover F of $G \ni e \in F$. Let $F_1 = F - \{e\}$. Then F_1 is a strong edge cover of $G - e$.

$\therefore \quad \alpha_s^1(G-e) \leq |F_1| < |F| = \alpha_s^1(G)$.

$\therefore \quad \alpha_s^1(G-e) < \alpha_s^1(G)$. □

Example 3.22 : Consider the finite projective plane G mentioned in above example. Let e be any edge of G. Let f be any edge of $G \ni f \neq e$. Now, let $F = E(G) - \{f\}$. Then F is a minimum strong edge cover of G and $e \in F$.

By the above theorem $\alpha_s^1(G-e) < \alpha_s^1(G)$.

Thus, for every edge e of G $\alpha_s^1(G-e) < \alpha_s^1(G)$.

Note that $\alpha_s^1(G-e) = r^2 - r - 1 \;\; \forall \, e \in G$.

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