International J. of Math. Sci. & Engg. Appls. (IJMSEA) ISSN 0973-9424, Vol. 11 No. III (December, 2017), pp. 29-45

THREE FINITE CLASSSES OF ORTHOGONAL LAURENT POLYNOMIALS AND THEIR APPLICATIONS IN WEIGHTED QUADRATURE RULES

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Abstract

In this work, the orthogonal Laurent polynomials on the real line for the three finite classes of classical orthogonal polynomials are discussed by finding series solution, three term recurrence relations and orthogonality relations explicitly. The two and three point weighted quadrature rules are calculated with respect to some strong weight function for the defined classes.

1. Introduction

The study of "Strong Stieltjes moment problem" was initiated by Jones, Thron and Waadeland [?] in 1980 in which orthogonal Laurent polynomial sequence (OLPS) were used. For interesting results on OLPS, we refer to [2, 8] and references therein. The strong moment problem related to the theory of OLPS is similar to the classical moment

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Key Words : Orthogonal polynomials, Laurent polynomials, Moment problems.

problem related to the theory of orthogonal polynomials sequences (OPS). Note that many results of OPS can be extended to the results of OLPS, whereas results that are true for OLPS need not be true for OPS [2].

Classical orthogonal polynomials sequences arise as a polynomial solution of the second order differential equation

$$A(x)y_n'' + B(x)y_n' - \lambda_n y_n(x) = 0,$$

where A(x) is at most quadratic polynomials and B(x) is a linear polynomials and λ_n is the eigenvalue parameter depending on $n = 0, 1, 2, \cdots$.

Hagler [6, 7] had shown the precise connection between orthogonal polynomials and orthogonal Laurent polynomials by calculating OLPS on the real line \mathbb{R} from the well known classical OPS namely Jacobi, Laguerre and Hermite.

These three classical OPS may be called as infinite class of classical OPS, since the recent investigation of Masjed-jamei [11] on three other classes of orthogonal polynomials that are due to Romanovski [16] which are finite in nature, in the sense that their parameters yielding finite number of polynomials satisfying orthogonality. These finite polynomials were initially identified by Routh [17] and by Romanovski [16] before studied extensively by Masjedjamei [11] in the recent past. For details on these polynomials see [11, 12, 14] and references therein. Basic information of these six classes of classical OPS are tabulated in Table 1 for immediate reference.

Note that these polynomials are less known in the literature but in recent years several problems ranging from super symmetric quantum mechanics over soliton physics to field theory have been solved in terms of these finite orthogonal polynomials [1, 3]. In this work, we find the orthogonal Laurent polynomials for the three finite classical OPS.

COPS	Properties: Classification, weight function and orthogonality interval				
	Polynomial	A(x)	B(x)	Weight function	Interval
	Jacobi	$1 - x^2$	$-(\alpha+\beta+2)x +$	$(1-x)^{\alpha}(1+x)^{\beta} \alpha,\beta >$	[-1, 1]
			$(\beta - \alpha)$	-1;	
Infinite	Laguerre	x	$\alpha + 1 - x$	$x^{\alpha}e^{-x}; \ \alpha > -1$	$[0,\infty)$
	Hermite	1	-2x	e^{-x^2}	$(-\infty,\infty)$
	R-Jacobi	$x^2 + x$	(2-p)x + (1+q)	$x^q (1+x)^{-(p+q)}$	$[0,\infty)$
Finite	R-Bessel	x^2	(2-p)x+1	$x^{-p}e^{-1/x}$	$[0,\infty)$
	R-Pseudo	$(ax+b)^2 +$	$2(1 - p)(a^2 +$	$((ax+b)^2 + (cx+d)^2)^{-p}$	$(-\infty,\infty)$
	Jacobi	$(cx+d)^2$	$c^2)x+q(ad-bc)+$		
			2(1-p)(ab+cd)		

Table 1 : Outline of all the six classical orthogonal polynomials

Definition 1.1 [2]: A Laurent polynomial (L-polynomial) is a function of the form $R(x) = \sum_{j=m}^{n} r_j x^j$ where x is non zero real variable and $m, n \in \mathbb{Z}$ with $m \leq n, r_j \in \mathbb{C}$. The set of all L-polynomials are denoted by $\mathcal{R}_{m,n}$ that are contained in the span of $\{x_j\}_{j=m}^n$. Two important classes of L-polynomials are

$$\mathcal{R}_{2n} = \{ R \in \mathcal{R}_{-n,n} : \text{the coefficient of } x^n \text{ is non zero} \}$$
$$\mathcal{R}_{2n+1} = \{ R \in \mathcal{R}_{-n-1,n} : \text{the coefficient of } x^{-n-1} \text{ is non zero} \}$$

for all $n \in \mathbb{Z}_0^+$.

Definition 1.2 [2]: A sequence of polynomials $\{R_n\}_{n=0}^{\infty}$ is said to be OLPS with respect to a strong moment distribution function in the following determinantal representation for $\forall m = 0, 1, 2, 3. \cdots$

$$R_{2m}(x) = \frac{1}{H_{2m}^{-2m}} \begin{vmatrix} \mu_{-2m} & \mu_{-2m+1} & \cdots & \mu_{-1} & x^{-m} \\ \mu_{-2m+1} & \mu_{-2m+2} & \cdots & \mu_0 & x^{-m+1} \\ \vdots & \vdots & & \vdots \\ \mu_{-1} & \mu_0 & \cdots & \mu_{2m-2} & x^{m-1} \\ \mu_0 & \mu_1 & \cdots & \mu_{2m-1} & x^m \end{vmatrix},$$

$$R_{2m+1}(x) = \frac{-1}{H_{2m+1}^{-2m}} \begin{vmatrix} \mu_{-2m} & \mu_{-2m} & \cdots & \mu_{-1} & x^{-m-1} \\ \mu_{-2m} & \mu_{-2m+1} & \cdots & \mu_0 & x^{-m} \\ \vdots & \vdots & & \vdots \\ \mu_{-1} & \mu_0 & \cdots & \mu_{2m-1} & x^{m-1} \\ \mu_0 & \mu_1 & \cdots & \mu_{2m} & x^m \end{vmatrix},$$

where Hankel determinants H_k^m satisfying $H_k^m > 0 \forall m = 0, \pm 1, \pm 2, \cdots, k = 1, 2, 3, \cdots$ associated the moment sequences $\{\mu_k\}$ are defined as

$$H_0^m = 1$$
, and $H_k^m = det(\mu_{m+i+j})_{i,j=0}^{k-1}$.

Using the above representation, orthogonality of OLPS can be characterized as

$$\langle R_{2m}(x), x^k \rangle = 0 \quad \text{for} \quad k = -m, -m+1, \cdots, m-1,$$

$$\langle R_{2m+1}(x), x^k \rangle = 0 \quad \text{for} \quad k = -m, -m+1, \cdots, m,$$

$$\|R_{2m}\|^2 = \langle R_{2m}(x), x^m \rangle = \frac{H_{2m+1}^{(-2m)}}{H_{2m}^{(-2m)}} > 0,$$

$$\|R_{2m+1}\|^2 = \langle R_{2m+1}(x), x^{-m-1} \rangle = \frac{H_{2m+2}^{(-2m-2)}}{H_{2m+1}^{(-2m)}} > 0.$$

In the sequel, we use the notation ψ as the moment distribution function (MDF) for the monic OPS with spectrum $\sigma(\psi)$ and $\tilde{\psi}$ as the strong moment distribution function (SMDF) for the monic OLPS with spectrum $\sigma(\tilde{\psi})$.

The contents of this work are as follows. In Section 2, corresponding OLPS for the three finite class of classical OPS are obtained. Gaussian quadrature rules with respect to the strong weight function are given in Section 3.

2. Finite Laurent Orthogonal Polynomials

In this section, we consider the three finite classes of classical OPS given in Table 1 [11, 12] and use the transformation formulae given in [6] and obtain properties of three finite orthogonal Laurent polynomials (FOLP). Further, for the class of polynomials $M_n^{(p,q)}(x)$ or $N_n^{(p)}(x)$ or $J_n^{(p,q)}(x; a, b, c, d)$, we denote the monic form, respectively, as $\hat{M}_n^{(p,q)}(x)$ or $\hat{N}_n^{(p)}(x)$ or $\hat{J}_n^{(p,q)}(x; a, b, c, d)$ and the corresponding monic Laurent polynomials as $\widetilde{M}_n^{(p,q)}(x)$ or $\widetilde{N}_n^{(p)}(x)$ or $\widetilde{N}_n^{(p)}(x)$ or $\widetilde{J}_n^{(p,q)}(x; a, b, c, d)$, respectively.

2.1 Romanovski Jacobi Polynomials

Romanovski Jacobi MDF [11] $\psi^{(p,q)}$ is given by

$$\frac{d\psi^{(p,q)}}{dx} = \begin{cases} \frac{x^q}{(1+x)^{p+q}} & \text{if } x \in [0,\infty);\\ 0 & \text{otherwise.} \end{cases}$$
(2.1)

Applying the transformation $v(x) = \frac{1}{\lambda} \left(x - \frac{\gamma}{x}\right)$ [[6], p. 24, Theorem 2.3.1] to a MDF (??) results in an SMDF for each choice of $\lambda > 0$ and $\gamma > 0$. The spectrum $\sigma(\tilde{\psi})$ can

be calculated as

$$\begin{aligned} \sigma(\widetilde{\psi}) &= v_{+}^{-1}(\sigma(\psi)) \cup v_{-}^{-1}(\sigma(\psi)) \\ &= v_{-}^{-1}([0,\infty)) \cup v_{+}^{-1}([0,\infty)) \\ &= (-\sqrt{\gamma}, 0) \cup (\sqrt{\gamma}, \infty), \end{aligned}$$

where

$$v_{\pm}^{-1}(y) = \frac{\lambda}{2} \left(y \pm \sqrt{y^2 + \frac{4\gamma}{\lambda^2}} \right).$$

To find Romanovski Jacobi SMDF $\tilde{\psi}^{(p,q)}$, we use the fact that $\frac{d\tilde{\psi}}{dx} = w(v(x))$ with $w(x) = \frac{d\psi}{dx}$, such that

$$\frac{d\widetilde{\psi}^{(p,q)}}{dx} = \begin{cases} \frac{\left[\frac{1}{\lambda}(x-\frac{\gamma}{x})\right]^q}{\left[1+\frac{1}{\lambda}(x-\frac{\gamma}{x})\right]^{p+q}} & \text{if } x \in (-\sqrt{\gamma},0) \cup (\sqrt{\gamma},\infty), \\ 0 & \text{if } x \in (-\infty,-\sqrt{\gamma}] \cup (0,\sqrt{\gamma}], \end{cases}$$

is an SMDF for each choice of the parameters λ and $\gamma.$

From the explicit representation of $M_n^{(p,q)}(x)$ given in [[11], p. 171, (2.2)], the corresponding result for $\hat{M}_n^{(p,q)}(x)$ with parameters p > 2N + 1 and q > -1, where $N = \max\{m, n\}$ can be rewritten as

$$\hat{M}_{n}^{(p,q)}(x) = (-1)^{n} {\binom{p-(n+1)}{n}}^{-1} \sum_{k=0}^{n} {\binom{p-(n+1)}{k}} {\binom{q+n}{n-k}} (-x)^{k}.$$
 (2.2)

The explicit representation for $\widetilde{M}_n^{(p,q)}(x)$ can be obtained by using the fact that $\widetilde{P}_{2n}(x) = \lambda^n P_n(\upsilon(x))$ and $\widetilde{P}_{2n+1}(x) = (\frac{\lambda}{\gamma})^n \frac{1}{x} P_n(\upsilon(x))$ together with (??) as

$$\widetilde{M}_{2n}^{(p,q)}(x) = (-\lambda)^n \binom{p-(n+1)}{n}^{-1}$$
$$\sum_{k=0}^n \frac{(-1)^k}{(\lambda)^k x^k} \binom{p-(n+1)}{k} \binom{q+n}{n-k} (x^2-\gamma)^k,$$

and

$$\widetilde{M}_{2n+1}^{(p,q)}(x) = \left(\frac{\lambda}{\gamma}\right)^n \binom{p-(n+1)}{n}^{-1}$$
$$\sum_{k=0}^n \frac{(-1)^k}{\lambda^k x^{k+1}} \binom{p-(n+1)}{k} \binom{q+n}{n-k} (x^2-\gamma)^k,$$

for $n = 0, 1, 2, 3, \cdots$.

From the orthogonality for $M_n^{(p,q)}(x)$ given in [[11], p. 173, (2.14)], the corresponding orthogonality relation for $\hat{M}_n^{(p,q)}(x)$ can be written as

$$\langle \hat{M}_{m}^{(p,q)}(x), \hat{M}_{n}^{(p,q)}(x) \rangle_{\psi}^{(p,q)} = {\binom{p-(n+1)}{n}}^{-1} B(q+n+1, p-2n-1)\delta_{mn}$$
(2.3)

where $B(\alpha, \beta)$ denote the Beta function.

Using [[6], Theorem 2.2.8], the orthogonality relation for $\widetilde{M}_n^{(p,q)}(x)$ can be obtained as

$$\langle \widetilde{M}_{m}^{(p,q)}(x), \widetilde{M}_{n}^{(p,q)}(x) \rangle_{\widetilde{\psi}^{(p,q)}} = K_{n} \delta_{mn},$$

where

$$K_n = \begin{cases} \lambda^{2n+1} {\binom{p-(n+1)}{n}}^{-1} B(q+n+1,p-2n-1), & \text{if } n \text{ is even,} \\ (\frac{\lambda}{\gamma})^{2n+1} {\binom{p-(n+1)}{n}}^{-1} B(q+n+1,p-2n-1), & \text{if } n \text{ is odd.} \end{cases}$$

To obtain the above result, we use [[6], Theorem 2.2.8] in (2.3) such that

$$\begin{split} \langle \widetilde{M}_{2m}^{(p,q)}(x), \widetilde{M}_{2n}^{(p,q)}(x) \rangle_{\widetilde{\psi}^{(p,q)}} &= \lambda^{m+n+1} \langle \hat{M}_m^{(p,q)}(x), \hat{M}_n^{(p,q)}(x) \rangle_{\psi}^{(p,q)} \\ &= \lambda^{2n+1} \binom{p-(n+1)}{n}^{-1} B(q+n+1, p-2n-1) \delta_{mn}. \end{split}$$

Similar result hold for odd index

$$\begin{split} \langle \widetilde{M}_{2m+1}^{(p,q)}(x), \widetilde{M}_{2n+1}^{(p,q)}(x) \rangle_{\widetilde{\psi}^{(p,q)}} &= \left(\frac{\lambda}{\mu}\right)^{m+n+1} \langle \hat{M}_m^{(p,q)}(x), \hat{M}_n^{(p,q)}(x) \rangle_{\psi}^{(p,q)} \\ &= \left(\frac{\lambda}{\mu}\right)^{2n+1} \binom{p-(n+1)}{n}^{-1} B(q+n+1, p-2n-1)\delta_{mn}. \end{split}$$

The fundamental recurrence formula related to $\hat{M}_n^{(p,q)}(x)$ is obtained by applying the interrelation between $\hat{M}_n^{(p,q)}(x)$ and $M_n^{(p,q)}(x)$ in [[11], p. 174, (2.19)]

$$\hat{M}_{n}^{(p,q)}(x) = \left(x - \frac{p(2n-1+q) - 2n(n-1)}{(2n-2-p)(2n-p)}\right) \hat{M}_{n-1}^{(p,q)}(x)
- \frac{(n-1)(q+n-1)(n-1-p-q)(n-1-p)}{(2n-3-p)(2n-2-p)^{2}(2n-1-p)} \hat{M}_{n-2}^{(p,q)}(x).$$
(2.4)

Using [[6], Theorem 3.5.2] with (2.4), we obtain that regular OLPS $\widetilde{M}_n^{(p,q)}(x)$ satisfy the three term recurrence relation

$$\widetilde{M}_{2n}^{(p,q)}(x) = \left(x - \lambda \frac{p(2n-1+q) - 2n(n-1)}{(2n-2-p)(2n-p)} - \frac{\gamma}{x}\right) \widetilde{M}_{2n-2}^{(p,q)}(x) - \lambda^2 \frac{(n-1)(q+n-1)(n-1-p-q)(n-1-p)}{(2n-3-p)(2n-2-p)^2(2n-1-p)} \widetilde{M}_{2n-4}^{(p,q)}(x),$$

$$\widetilde{M}_{2n+1}^{(p,q)}(x) = \left(\frac{1}{x} + \frac{\lambda}{\gamma} \frac{p(2n-1+q)-2n(n-1)}{(2n-2-p)(2n-p)} - \frac{1}{\gamma}x\right) \widetilde{M}_{2n-1}^{(p,q)}(x) - \left(\frac{\lambda}{\gamma}\right)^2 \frac{(n-1)(q+n-1)(n-1-p-q)(n-1-p)}{(2n-3-p)(2n-2-p)^2(2n-1-p)} \widetilde{M}_{2n-3}^{(p,q)}(x).$$

2.2 Romanovski Bessel Polynomials

Romanovski Bessel MDF [11] $\psi_N^{(p)}$ is given by

$$\frac{d\psi_N^{(p)}}{dx} = \begin{cases} x^{-p} \exp\left(-\frac{1}{x}\right) & \text{if } x \in (0,\infty);\\ 0 & \text{otherwise.} \end{cases}$$
(2.5)

Applying the transformation $v(x) = \frac{1}{\lambda} \left(x - \frac{\gamma}{x}\right)$ [[6], p. 24, Theorem 2.3.1], to a MDF (2.5) results in an SMDF for each choice of $\lambda > 0$ and $\gamma > 0$. The spectrum $\sigma(\tilde{\psi})$ can be calculated as

$$\begin{aligned} \sigma(\widetilde{\psi}) &= v_+^{-1}(\sigma(\psi)) \cup v_-^{-1}(\sigma(\psi)) \\ &= v_-^{-1}([0,\infty)) \cup v_+^{-1}([0,\infty)) \\ &= (-\sqrt{\gamma}, 0) \cup (\sqrt{\gamma}, \infty). \end{aligned}$$

To find Romanovski Bessel SMDF $\tilde{\psi}_N^{(p)}$, we use the fact that $\frac{d\tilde{\psi}}{dx} = w(v(x))$ with $w(x) = \frac{d\psi}{dx}$, so that

$$\frac{d\widetilde{\psi}_{N}^{(p)}}{dx} = \begin{cases} \exp\left(-\frac{\lambda x}{x^{2}-\gamma}\right) \left[\frac{1}{\lambda}(x-\frac{\gamma}{x})\right]^{-p} & \text{if} \quad x \in (-\sqrt{\gamma},0) \cup (\sqrt{\gamma},\infty), \\ 0 & \text{if} \quad x \in (-\infty,-\sqrt{\gamma}] \cup (0,\sqrt{\gamma}], \end{cases}$$

is an SMDF for each choice of the parameters λ and γ .

From the explicit representation of $N_n^{(p)}(x)$ given in [[11], p. 180, (4.3)], the corresponding result for $\hat{N}_n^{(p)}(x)$ with parameters p > 2N + 1, where $N = \max\{m, n\}$ can be rewritten as

$$\hat{N}_{n}^{(p)}(x) = \frac{(-1)^{n}}{n!} {\binom{p-(n+1)}{n}}^{-1} \sum_{k=0}^{n} k! {\binom{p-(n+1)}{k}} {\binom{n}{n-k}} (-x)^{k}.$$
 (2.6)

The explicit representation for $\widetilde{N}_n^{(p)}(x)$ can be obtained by using the fact that $\widetilde{P}_{2n}(x) = \lambda^n P_n(v(x))$ and $\widetilde{P}_{2n+1}(x) = (\frac{\lambda}{\gamma})^n \frac{1}{x} P_n(v(x))$ together with (??) as

$$\widetilde{N}_{2n}^{(p)}(x) = \frac{(-\lambda)^n}{n!} {\binom{p - (n+1)}{n}}^{-1} \\ \sum_{k=0}^n \frac{(-1)^k k!}{(\lambda)^k x^k} {\binom{p - (n+1)}{k}} {\binom{n}{n-k}} (x^2 - \gamma)^k$$

and

$$\widetilde{N}_{2n+1}^{(p)}(x) = \left(\frac{\lambda}{\gamma}\right)^n \frac{1}{n!} \binom{p-(n+1)}{n}^{-1}$$
$$\sum_{k=0}^n \frac{(-1)^k k!}{\lambda^k x^{k+1}} \binom{p-(n+1)}{k} \binom{n}{n-k} (x^2-\gamma)^k$$

for $n = 0, 1, 2, 3, \cdots$.

From the orthogonality for $N_n^{(p)}(x)$ given in [?, p. 182, (4.17)], the corresponding orthogonality relation for $\hat{N}_n^{(p)}(x)$ can be written as

$$\langle \hat{N}_m^{(p)}(x), \hat{N}_n^{(p)}(x) \rangle_{\psi_N^{(p)}} = {\binom{p - (n+1)}{n}}^{-1} (p - (2n+2))! \delta_{mn}.$$
 (2.7)

Using [[6], Theorem 2.2.8], the orthogonality relation for $\widetilde{N}_n^{(p)}(x)$ is given by

$$\langle \widetilde{N}_m^{(p)}(x), \widetilde{N}_n^{(p)}(x) \rangle_{\widetilde{\psi}_N^{(p)}} = K_n \delta_{mn},$$

where

$$K_n = \begin{cases} \lambda^{2n+1} {\binom{p-(n+1)}{n}}^{-1} (p-(2n+2))!, & \text{if } n \text{ is even,} \\ \left(\frac{\lambda}{\mu}\right)^{2n+1} {\binom{p-(n+1)}{n}}^{-1} (p-(2n+2))!, & \text{if } n \text{ is odd.} \end{cases}$$

To obtain the above result, we apply [[6], Theorem 2.2.8] to (2.7) so that

$$\langle \tilde{N}_{2m}^{(p)}(x), \tilde{N}_{2n}^{(p)}(x) \rangle_{\tilde{\psi}_{N}^{(p)}} = \lambda^{m+n+1} \langle \hat{N}_{m}^{(p)}(x), \hat{N}_{n}^{(p)}(x) \rangle_{\psi_{N}^{(p)}}$$

$$= \lambda^{2n+1} \binom{p-(n+1)}{n}^{-1} (p-(2n+2))! \delta_{mn}$$

Similar result hold for odd index

$$\langle \tilde{N}_{2m+1}^{(p)}(x), \tilde{N}_{2n+1}^{(p)}(x) \rangle_{\tilde{\psi}_N^{(p)}} = \langle \hat{N}_m^{(p)}(x), \hat{N}_n^{(p)}(x) \rangle_{\psi_N^{(p)}}$$

= $\left(\frac{\lambda}{\mu}\right)^{2n+1} \binom{p-(n+1)}{n}^{-1} (p-(2n+2))! \delta_{mn}.$

The fundamental recurrence formula for $\hat{N}_n^{(p)}(x)$ is obtained by applying the interrelationship between $\hat{N}_n^{(p)}(x)$ and $N_n^{(p)}(x)$ in [[11], p. 182, (4.19)]

$$\hat{N}_{n}^{(p)}(x) = \left(x - \frac{p}{(2n-2-p)(2n-p)}\right) \hat{N}_{n-1}^{(p)}(x)
- \frac{(n-1)(p+1-n)}{(2n-3-p)(2n-2-p)^{2}(2n-1-p)} \hat{N}_{n-2}^{(p)}(x).$$
(2.8)

Using [[6], Theorem 3.5.2] with (2.8), we obtain that regular OLPS $\widetilde{N}_n^{(p)}(x)$ satisfy the following three term recurrence relation

$$\begin{split} \widetilde{N}_{2n}^{(p)}(x) &= \left(x - \lambda \frac{p}{(2n-2-p)(2n-p)} - \frac{\gamma}{x}\right) \widetilde{N}_{2n-2}^{(p)}(x) \\ &- \lambda^2 \frac{(n-1)(p+1-n)}{(2n-3-p)(2n-2-p)^2(2n-1-p)} \widetilde{N}_{2n-4}^{(p)}(x), \\ \widetilde{N}_{2n+1}^{(p)}(x) &= \left(\frac{1}{x} + \frac{p}{(2n-2-p)(2n-p)} - \frac{1}{\gamma}x\right) \widetilde{N}_{2n-1}^{(p)}(x) \\ &- (\frac{\lambda}{\gamma})^2 \frac{(n-1)(p+1-n)}{(2n-3-p)(2n-2-p)^2(2n-1-p)} \widetilde{N}_{2n-3}^{(p)}(x). \end{split}$$

2.3 Generalized Romanovski Hermite type Polynomials

Note that these polynomials are closely related to Jacobi polynomials and hence they are particular case of Romanovski pseudo Jacobi polynomials. This class is introduced in [12] and studied extensively in [14]. We call this class as generalized Romanovski Hermite class because this class reduces to Romanovski class given in [11] as a particular case of parameters $\hat{J}_n^{(p-\frac{1}{2},0)}(x;1,0,0,1)$, which is related to the Hermite polynomials. Generalized Romanovski Hermite MDF [12] $\psi_I^{(p)}$ is given by

$$\frac{d\psi_J^{(p,q)}}{dx} = ((ax+b)^2 + (cx+d)^2)^{-p} \exp\left(q \arctan\frac{ax+b}{cx+d}\right), \quad x \in (-\infty,\infty).$$
(2.9)

Applying the transformation $v(x) = \frac{1}{\lambda} \left(x - \frac{\gamma}{x}\right)$ [[6], p. 24, Theorem 2.3.1] to a MDF (2.9) results in an SMDF for each choice of $\lambda > 0$ and $\gamma > 0$. The spectrum $\sigma(\tilde{\psi})$ can be calculated as

$$\begin{aligned} \sigma(\widetilde{\psi}) &= v_{+}^{-1}(\sigma(\psi)) \cup v_{-}^{-1}(\sigma(\psi)) \\ &= v_{-}^{-1}([-\infty,\infty)) \cup v_{+}^{-1}([-\infty,\infty)) \\ &= (-\infty,0) \cup (0,\infty). \end{aligned}$$

To find generalized Romanovski Hermite SMDF $\tilde{\psi}_J^{(p,q)}$, we use the fact that $\frac{d\tilde{\psi}}{dx} = w(v(x))$ with $w(x) = \frac{d\psi}{dx}$, so that

$$\frac{d\widetilde{\psi}_{J}^{(p,q)}}{dx} = \left[\left(\frac{a(x^2 - \gamma)}{\lambda x} + b \right)^2 + \left(\frac{c(x^2 - \gamma)}{\lambda x} + d \right)^2 \right]^{-p} \\ \times \exp\left(q \arctan \frac{a(x^2 - \gamma) + b\lambda x}{c(x^2 - \gamma) + d\lambda x} \right), \quad x \in (-\infty, 0) \cup (0, \infty)$$

is an SMDF for each choice of the parameters λ and $\gamma.$

From the explicit representation of $J_n^{(p,q)}(x; a, b, c, d)$ given in [?, p. 139, (6)], the corresponding result for $\hat{J}_n^{(p,q)}(x; a, b, c, d)$ with parameters $p > N + \frac{1}{2}$ and $ad - bc \neq 0$, where $N = \max\{m, n\}$ can be rewritten as

$$\hat{J}_{n}^{(p,q)}(x;a,b,c,d) = (-1)^{n}(a^{2}+c^{2})^{n}(n+1-2p)_{n}\sum_{k=0}^{n} \binom{n}{k} \left(\frac{a^{2}+c^{2}}{(ab+cd)+i(ad-bc)}\right)^{k} \times {}_{2}F_{1}\left(k-n,p-n-iq/2;2p-2n;\frac{2(ad-bc)}{(ab+cd)+i(ad-bc)}\right)x^{k},$$
(2.10)

in which $i = \sqrt{-1}$ and ${}_2F_1(.)$ is the well known Gaussian hypergeometric function. The explicit representation for $\widetilde{J}_n^{(p,q)}(x; a, b, c, d)$ can be obtained by using the fact that $\widetilde{P}_{2n}(x) = \lambda^n P_n(\upsilon(x))$ and $\widetilde{P}_{2n+1}(x) = (\frac{\lambda}{\gamma})^n \frac{1}{x} P_n(\upsilon(x))$ together with (??) as

$$\begin{aligned} \widetilde{J}_{2n}^{(p,q)}(x;a,b,c,d) &= (-1)^n (a^2 + c^2)^n (n+1-2p)_n \sum_{k=0}^n \binom{n}{k} \left(\frac{a^2 + c^2}{(ab+cd) + i(ad-bc)} \right)^k \\ &\times {}_2F_1 \left(k - n, p - n - iq/2; 2p - 2n; \frac{2(ad-bc)}{(ab+cd) + i(ad-bc)} \right) \lambda^{n-k} \times (x - \frac{\gamma}{x})^k, \end{aligned}$$

and

$$\widetilde{\hat{J}}_{2n+1}^{(p,q)}(x;a,b,c,d) = \frac{(-1)^n}{\gamma^n x} (a^2 + c^2)^n (n+1-2p)_n \sum_{k=0}^n \binom{n}{k} \left(\frac{a^2 + c^2}{(ab+cd) + i(ad-bc)} \right)^k \\ \times {}_2F_1 \left(k - n, p - n - iq/2; 2p - 2n; \frac{2(ad-bc)}{(ab+cd) + i(ad-bc)} \right) \lambda^{n-k} (x - \frac{\gamma}{x})^k,$$
for $n = 0, 1, 2, 3, \cdots$.

From the orthogonality of $J_n^{(p,q)}(x; a, b, c, d)$ given in [[12], p. 142, (23)], the corresponding orthogonality relation for $\hat{J}_n^{(p,q)}(x; a, b, c, d)$ can be written as

$$\begin{aligned} \langle \hat{J}_{n}^{(p,q)}(x;a,b,c,d), \hat{J}_{m}^{(p,q)}(x;a,b,c,d) \rangle_{\psi_{J}^{(p,q)}} &= \\ \int_{-\infty}^{\infty} \hat{J}_{n}^{(p,q)}(x;a,b,c,d) \hat{J}_{m}^{(p,q)}(x;a,b,c,d) ((ax+b)^{2} + (cx+d)^{2})^{-p} \exp\left(q \arctan \frac{ax+b}{cx+d}\right) dx \\ &= \left(\frac{2^{2n+1-2p}(ad-bc)^{2n-2p+1} \exp(-q \arctan(c/a))}{(2p-2n-1)(a^{2}+c^{2})^{-p+1}} \frac{n!\Gamma(2p-n)}{\Gamma(p-n+iq/2)\Gamma(p-n-iq/2)}\right) \\ &\times \left(\frac{(ab+cd)+i(ad-bc)}{(a^{2}+c^{2})}\right)^{2n} \delta_{mn}. \end{aligned}$$

$$(2.11)$$

Apply [[6], Theorem 2.2.8] to (??), the orthogonality relation for $\widetilde{J}_n^{(p,q)}(x; a, b, c, d)$ is given by

$$\langle \tilde{J}_n^{(p,q)}(x;a,b,c,d), \tilde{J}_m^{(p,q)}(x;a,b,c,d) \rangle_{\tilde{\psi}_J^{(p,q)}} = K_n \delta_{mn},$$

where

$$K_n = \begin{cases} \lambda^{2n+1} \ \overline{K}_n & \text{if } n \text{ is even,} \\ (\frac{\lambda}{\gamma})^{2n+1} \overline{K}_n & \text{if } n \text{ is odd,} \end{cases}$$

with
$$\overline{K}_{n} = \langle \hat{J}_{n}^{(p,q)}(x; a, b, c, d), \hat{J}_{m}^{(p,q)}(x; a, b, c, d) \rangle_{\psi_{J}^{(p,q)}}$$
 explicitly given as

$$\overline{K}_{n} = \frac{2^{2n+1-2p}(ad-bc)^{2n-2p+1}\exp(-q\arctan(c/a))}{(2p-2n-1)(a^{2}+c^{2})^{-p+1}} \times \frac{n!\Gamma(2p-n)}{\Gamma(p-n+iq/2)\Gamma(p-n-iq/2)} \left(\frac{(ab+cd)+i(ad-bc)}{(a^{2}+c^{2})}\right)^{2n} \delta_{nm}.$$
(2.12)

3. Strong Gaussian Quadrature rules

Let ψ be a moment distribution function with spectrum $\sigma(\psi)$. Let $\{Q_n(x)\}_{n=0}^{\infty}$ denote an orthogonal polynomial sequence with respect to ψ , let n be any positive index and let $x_{n,1}, x_{n,2}, x_{n,3} \cdots, x_{n,n}$ denote the zeros of $Q_n(x)$. There exist positive numbers $A_{n,1}, A_{n,2}, A_{n,3}, \cdots, A_{n,n}$ such that for every polynomials f(x) of degree at most 2n-1

$$\int_{\sigma(\psi)} f(x)w(x)dx = \sum_{k=1}^n f(x_{n,k})A_{(n,k)}.$$

We will refer to the positive number $A_{n,1}, A_{n,2}, A_{n,3}, \cdots, A_{n,n}$ as the quadrature weights and the zeros $x_{n,1}, x_{n,2}, x_{n,3}, \cdots, x_{n,n}$ as the quadrature nodes.

In general, the above quadrature rule has the highest degree of precision 2n - 1 if and only if $\{x_j\}_{j=1}^n$ are the roots of the orthogonal polynomials of degree n with respect to the weight function w(x), see [10].

The following theorem summarizes, how the quadrature rules based on the OPS can be extended to OLPS via the relationship between the quadrature nodes and quadrature weights for the OPS and OLPS.

Theorem 3.1 [5] (Zeros): Let n be a positive integer and suppose $\{x_{n,k}\}_{k=1}^n$ are the zeros of $P_n(x)$ such that $x_{n,1}, x_{n,2}, x_{n,3} \cdots, x_{n,n}$. then the zeros of $\tilde{P}_{2n}(x)$ and $\tilde{P}_{2n+1}(x)$ are $x_{n,j}^* = v_*^{-1}(x_{n,j})$, for $* \in \{+, -\}$ with 0 be a root for odd degree polynomials. and have the ordering $x_{n,1}^- < x_{n,2}^- < \cdots < x_{n,n}^- < 0 < x_{n,1}^+ < x_{n,2}^+ < \cdots < x_{n,n}^+$.

Theorem 3.2 [5] (Weights) : Let $\{x_{n,j}\}$, $\{x_{n,j}^*\}$ be the quadrature nodes given in Theorem 3.1 and let $\{A_{n,k}\}$ and $\{A_{n,k}^*\}$, $k = 1, 2, 3, \dots, n$ denote the corresponding gauss quadrature weights respectively. Then $A_{n,k} = v'(x_{n,k}^*)A_{n,k}^*$, $k = 1, 2, 3, \dots, n$ for $* \in \{+, -\}$. Moreover, to obtain the coefficients $\{A_j\}_{j=1}^n$, one can use the following formulae

$$\frac{1}{A_j} = \sum_{i=0}^{n-1} Q_i^{*2}, \qquad j = 1, 2, 3, \cdots, n,$$

where Q_i^* are orthonormal polynomials of $Q_i(x)$ defined by

$$Q_i^* = \frac{Q_i}{\langle Q_i(x) | Q_i(x) \rangle^{\frac{1}{2}}}$$

Now, we focus on Gaussian quadrature rules [4, 5] associated with the strong moment distribution functions for the three finite class of classical orthogonal polynomials. We call these quadrature formulae as L-Quadrature formulae. Further, for the class of polynomials $M_n^{(p,q)}(x)$ or $N_n^{(p)}(x)$ or $J_n^{(p,q)}(x; a, b, c, d)$, we denote the orthonormal form, respectively, as $\hat{M}_n^{*(p,q)}(x)$ or $\hat{N}_n^{*(p)}(x)$ or $\hat{J}_n^{*(p,q)}(x; a, b, c, d)$.

3.1 L-Quadrature formulae for Romanovski Jacobi polynomials $\hat{M}_n^{(p,q)}(x)$ The two term quadrature formula for $\hat{M}_n^{(p,q)}(x)$ with parameter $p = \frac{11}{2}$ and $q = \frac{1}{2}$ is given in [11] as,

$$\int_0^\infty \frac{\sqrt{x}}{(1+x)^6} f(x) dx \cong 0.0828349625 f(0.333333333) + 0.0030679615 f(3),$$

where [15] nodes are taken as the zeros of $M_2^{\left(\frac{11}{2},\frac{1}{2}\right)}(x)$ and weights are calculated by $\frac{1}{A_j} = \sum_{k=0}^{1} \left(M_k^{*\left(\frac{15}{2},\frac{1}{2}\right)}(x_j) \right)^2$.

The respective L-quadrature formula is calculated for $\lambda = \gamma = 1$ as $\int_{\sigma(\tilde{\psi})} \frac{(x - \frac{1}{x})^{\frac{1}{2}}}{(1 + x - \frac{1}{x})^6} f(x) dx$ $\cong 0.000257631 f(-0.302775637) + 0.034608489 f(-0.847127088)$ + 0.048226473 f(1.180460421) + 0.0028103299 f(3.302775637),

where nodes and weights are calculated by using Theorem 3.1 and Theorem 3.2 respectively.

Table 2 : Two point Gaussian quadrature approximation with respect to strong Romanovski Jacobi distribution for $p = \frac{11}{2}$, $q = \frac{1}{2}$.

f(x)	$\int_{\sigma(\widetilde{\psi})} f(x) d\widetilde{\psi}(x)$	L-quadrature	Rel.error
e^{-x}	0.0972432644	0.0960036004	1.2×10^{-2}
$\frac{1}{\sqrt{1+2x^2}}$	0.0473686842	0.0477847678	0.9×10^{-2}
$\cos x$	0.0368308620	0.0387377612	$5.1{\times}10^{-2}$

The three term quadrature formula for $\hat{M}_n^{(p,q)}(x)$ with parameter $p = \frac{15}{2}$ and $q = \frac{1}{2}$ is given by

$$\int_0^\infty \frac{\sqrt{x}}{(1+x)^8} f(x) dx$$

$$\cong 0.04447882987 f(0.1715728753) + 0.00613592315 f(1.00000000)$$

$$+ 0.0000066129 f(5.828427125),$$

where nodes are taken as the zeros of $M_3^{(\frac{15}{2},\frac{1}{2})}(x)$ and weights are calculated by $\frac{1}{A_j} = \sum_{k=0}^2 \left(M_k^{*(\frac{15}{2},\frac{1}{2})}(x_j) \right)^2$. The respective L-quadrature formula is calculated for $\lambda = \gamma = 1$ as $\int_{\sigma(\tilde{\psi})} \frac{(x-\frac{1}{x})^{\frac{1}{2}}}{(1+x-\frac{1}{x})^8} f(x) dx$ $\approx 0.0203385564f(-0.9178864736) + 0.00169592744f(-0.6180339887)$ + 0.000001790f(-0.1667993696) + 0.0241402734f(1.0894593489)+ 0.0044399957f(1.6180339887) + 0.0000064339f(5.9952264943),

where nodes and weights are calculated by using Theorem 3.1 and Theorem 3.2 respectively.

Table 3 : Three point Gaussian quadrature approximation with respect to strong Romanovski Jacobi distribution for $p = \frac{15}{2}$, $q = \frac{1}{2}$.

f(x)	$\int_{\sigma(\widetilde{\psi})} f(x) d\widetilde{\psi}(x)$	L-quadrature	Rel.error
e^{-x}	0.0630917762	0.06307538213	2.6×10^{-4}
$\frac{1}{\sqrt{1+2r^2}}$	0.0286171341	0.0286104949	$2.3{\times}10^{-4}$
$\cos x$	0.0247564237	0.0247106749	$1.8{ imes}10^{-3}$

3.2 L-Quadrature formulae for Romanovski Bessel polynomials $\hat{N}_n^{(p)}(x)$ The two term quadrature formula for $\hat{N}_n^{(p)}(x)$ with parameter p = 6 is given in [11] as, $\int_0^\infty x^{-6} e^{-\frac{1}{x}} f(x) dx$

 $\cong 1.6076951545f(0.78867513459) + 22.3923048454f(0.2113248654),$

where nodes are taken as the zeros of $N_2^{(6)}(x)$ and weights are calculated using [15].

The respective L-quadrature formula is calculated for
$$\lambda = \gamma = 1$$
 as

$$\int_{\sigma(\tilde{\psi})} (x - \frac{1}{x})^{-6} \exp\left(-\frac{x}{x^2 - 1}\right) f(x) dx$$

$$\cong 10.0196888351 f(-0.8999043476) + 0.5089699932 f(-0.6806052762)$$

$$+ 1.0987351613 f(1.4692804108) + 12.3726160102 f(1.1112292130),$$

where nodes and weights are calculated by using Theorem 3.1 and Theorem 3.2 respectively.

Table 4 :Two point Gaussian quadrature approximation with respect to strong Romanovski Bessel distribution for p = 6.

f(x)	$\int_{\sigma(\widetilde{\psi})} f(x) d\widetilde{\psi}(x)$	L-quadrature	Rel.error
e^{-x}	29.97824013	29.97264020	1.9×10^{-4}
$\frac{1}{\sqrt{1+2r^2}}$	13.6759544785	13.6760512536	$7.1{\times}10^{-6}$
$\cos x$	12.2277659222	12.2239983818	$3.1{ imes}10^{-4}$

The three term quadrature formula for $\hat{N}_n^{(p)}(x)$ with parameter p = 8 is given in [4] as, $\int_0^\infty x^{-8} e^{-\frac{1}{x}} f(x) dx$

 $\cong 565.2150607824f(0.1288864005) + 154.3624193336f(0.3025345782)$

+ 0.4225198843f(1.0685790213),

where nodes are taken as the zeros of $N_3^{(8)}(x)$ and weights are calculated by $\frac{1}{A_j} = \sum_{k=0}^2 \left(N_k^{*(8)}(x_j)\right)^2$. The respective L-quadrature formula is calculated for $\lambda = \gamma = 1$ as $\int_{\sigma(\widetilde{\psi})} (x - \frac{1}{x})^{-8} \exp\left(-\frac{x}{x^2 - 1}\right) f(x) dx$ $\cong 264.4330961301 f(-0.9376311113) + 65.63754025 f(-0.8601088984)$ + 0.1117048155 f(-0.5994941023) + 300.7819645987 f(1.0665175118) + 88.7248790707 f(1.1626434766) + 0.3108150687 f(1.6680731236),where nodes and weights are calculated by using Theorem 3.1 and Theorem 3.2 respectively a statement of the second s

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strong itomanovski besser distribution for $p = 6$.				
f(x)	$\int_{\sigma(\widetilde{\psi})} f(x) d\widetilde{\psi}(x)$	L-quadrature	Rel.error	
e^{-x}	962.0045358940	962.0044006861	1.4×10^{-7}	
$\frac{1}{\sqrt{1+2r^2}}$	413.421100013	413.4209368305	3.9×10^{-7}	
$\cos x$	379.8926804937	379.8927829756	2.7×10^{-7}	

Table 5 : Three point Gaussian quadrature approximation with respect to strong Romanovski Bessel distribution for p = 8.

3.3 L-Quadrature formulae for Generalized Romanovski Hermite polynomials $\hat{J}_n^{(p,q)}(x;a,b,c,d)$

Consider the two term quadrature formula for $\hat{J}_n^{(p,q)}(x; a, b, c, d)$ with parameters p = 4, q = 1, a = d = 1, b = c = 0 is given in [?] as, $\int_{-\infty}^{\infty} \frac{\exp\left(\arctan x\right)}{(1+x^2)^4} f(x) dx$

 $\cong 0.6220884910f(-0.2109772229) + 0.4316063958f(0.7109772229),$

where nodes are taken as the zeros of $J_2^{(4,1)}(x;1,0,0,1)$ and weights are calculated by $\frac{1}{A_j} = \sum_{k=0}^1 \left(J_k^{*(4,1)}(x;1,0,0,1) \right)^2.$

The respective L-quadrature formula is calculated for $\lambda = \gamma = 1$ as $\int_{\sigma(\tilde{\psi})} \frac{\exp\left(\arctan\left(x - \frac{1}{x}\right)\right)}{(1 + (x - \frac{1}{x})^2)^4} f(x) dx$ $\cong 0.3436748193 f(-1.1110371419) + 0.1435191249 f(-0.7058182022)$ + 0.2784136716 f(0.9000599190) + 0.2880872708 f(1.4167954251),

where nodes and weights are calculated by using Theorem 3.1 and Theorem 3.2 respectively.

Table 6 : Two point Gaussian quadrature approximation with respect togeneralized strong Romanovski Hermite distribution for

f(x)	$\int_{\sigma(\widetilde{\psi})} f(x) d\widetilde{\psi}(x)$	L-quadrature	Rel.error
$\frac{1}{1+e^x}$	0.4910992188	0.4913335249	4.8×10^{-4}
$\frac{1}{\sqrt{1+2r^2}}$	0.5884790363	0.58674796350	2.9×10^{-3}
$\cos x$	0.48410953800	0.47897149960	$1.1{ imes}10^{-2}$

p = 4, q = 1, a = d = 1, b = c = 0

The three term quadrature formula for $\hat{J}_{n}^{(p,q)}(x; a, b, c, d)$ with parameter p = 4, q = 1, a = d = 1, b = c = 0 is given by $\int_{-\infty}^{\infty} \frac{\exp(\arctan x)}{(1+x^{2})^{4}} f(x) dx$ $\cong 0.25322550273 f(-0.5229034027) + 0.7679319861 f(0.3293582536)$ + 0.0325373985 f(1.693545149),

where nodes are taken as the zeros of $J_3^{(4,1)}(x;1,0,0,1)$ and weights are calculated by $\frac{1}{A_j} = \sum_{k=0}^{2} \left(J_k^{*(4,1)}(x;1,0,0,1) \right)^2.$

The respective L-quadrature formula is calculated for $\lambda = \gamma = 1$ as

$$\begin{split} &\int_{\sigma(\widetilde{\psi})} \frac{\exp\left(\arctan\left(x - \frac{1}{x}\right)\right)}{(1 + (x - \frac{1}{x})^2)^4} f(x) dx \\ &\cong 0.158639342846 f(-1.2950652616) + 0.3215751447 f(-0.8487897749) \\ &\quad + 0.0057555832 f(-0.4635799707) + 0.0945861597 f(0.7721618589) \\ &\quad + 0.4463568385 f(1.1781480285) + 0.0267818152 f(2.1571251197), \end{split}$$

where nodes and weights are calculated by using Theorem 3.1 and Theorem 3.2 respectively.

Table 7 : Three point Gaussian quadrature approximation with respect togeneralized strong Romanovski Hermite distribution for

f(x)	$\int_{\sigma(\widetilde{\psi})} f(x) d\widetilde{\psi}(x)$	L-quadrature	Rel.error
$\frac{1}{1+e^x}$	0.4910992188	0.4910014528	1.9×10^{-4}
$\frac{1}{\sqrt{1+2r^2}}$	0.5884790363	0.5915109163	5.2×10^{-3}
$\cos x$	0.48410953800	0.4845999436	1.1×10^{-3}

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p = 4, q = 1, a = d = 1, b = c = 0
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