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ESTIMATION OF PARAMETERS OF α-LAPLACE DISTRIBUTION

S. SATHEESH
School of Data Analytics,
M. G. University,
Kottayam - 686560, India

Abstract

In this note we derive the point estimators of the parameters of α-Laplace distribution. The asymptotic distribution of the estimators are found. Based on this large sample confidence interval for the estimators are obtained.

Keywords: Characteristic function, Estimator, Consistency, Asymptotic normality

1 Introduction

A random variable (*r.v.*) X has α-Laplace distribution ($X \sim \alpha L(\alpha, \lambda)$) if its characteristic function (CF) is $\phi(t) = 1/(1 + \lambda|t|^{\alpha})$; $t \in \mathbf{R}$, $\alpha \in (0, 2]$, $\lambda > 0$. This distribution is also known as Linnik distribution, see Kotz *et al.* (2001) for more details on this. This monograph also discusses its applications to Communications, Economics, Engineering and Finance. Some other relevant references on this topic are: Pillai and George (1984), Pillai and Kalyanaraman (1984), Pillai (1985), George and Pillai (1988) and George (1990).

This infinitely divisible distribution has been studied in some detail by Pillai and Kalyanaraman (1984) obtaining a characterisation of it. Pillai and George (1984) proved that it is in class-L and George (1990) showed that it is geometrically infinitely divisible. These divisibility properties make this a candidate suitable for describing Lévy processes, Pillai (1985), random time changed Lévy processes and AR(1) processes.

In this note we derive the point estimators of the parameters of $\alpha L(\alpha, \lambda)$ distribution, their asymptotic distribution and large sample confidence intervals. Since the density of α-Laplace law is not in a closed form unless $\alpha = 2$ (Laplace), we adapt the method based on empirical CFs, given in Press (1972).

2 Estimation of Parameters

Since the distribution of $X \sim \alpha L(\alpha, \lambda)$ is symmetric about zero, the CF of X is $\phi(t) = E(\cos tX)$. Again, since α-Laplace distribution is infinitely divisible, its CF has no real zeros. Now, choose two non-zero values $t_1 \neq t_2$ and write $[\phi(t_i)]^{-1} = 1 + \lambda|t_1|^\alpha;\ i = 1, 2$. Solving these two simultaneous equations for α and λ we get

$$\alpha = \frac{\log\left[\frac{1-\phi(t_1)}{\phi(t_1)}\right] - \log\left[\frac{1-\phi(t_2)}{\phi(t_2)}\right]}{\log(|t_1/t_2|)} \tag{1}$$

$$and \quad \log\lambda = \frac{\log(|t_2|)\log\left[\frac{1-\phi(t_1)}{\phi(t_1)}\right] - \log(|t_1|)\log\left[\frac{1-\phi(t_2)}{\phi(t_2)}\right]}{\log(|t_1/t_2|)} \tag{2}$$

Let $x_1, x_2, ..., x_n$ be a simple random sample from $X \sim \alpha L(\alpha, \lambda)$. Define

$$X_n(t) = \frac{1}{n}\sum_{i=1}^{n} \cos tx_i \tag{3}$$

the empirical CF corresponding to $\phi(t)$. Replacing $\phi(t_i)$ by $X_n(t_i),\ i = 1, 2$ in (1) and (2) we get estimators of α and $\log\lambda$. Since $|X_n(t)| \leq 1$, all the moments of $X_n(t)$ are finite and further, $X_n(t)$ is an unbiased and consistent estimator of $\phi(t)$. We now derive certain asymptotic properties of these estimators.

2.1 Asymptotic distribution of the estimators

Defining $Z_n = (X_n(t_1), X_n(t_2))'$ we have $E(Z_n) = \Theta = (\phi(t_1), \phi(t_2))' = (\theta_1, \theta_2)'$. Now we evaluate the variance-covariance matrix $\Sigma = [\sigma_{ij}] = n\ V(Z_n)$.

$$\begin{aligned}
Cov\,(X_n(t_1), X_n(t_2)) &= Cov\left(\frac{1}{n}\sum_k \cos t_1 x_k,\ \frac{1}{n}\sum_l \cos t_2 x_l\right) \\
&= \frac{1}{n^2}\ \sum_k\sum_l Cov\,(\cos t_1 x_k,\ \ \cos t_2 x_l) \\
&= \frac{1}{n}\ Cov\,(\cos t_1 X,\ \ \cos t_2 X)\,.
\end{aligned}$$

Using the trigonometric identity, $2\cos A\cos B = \cos(A+B)+\cos(A-B)$, we get, $\sigma_{ij} = n\ Cov\,(X_n(t_1), X_n(t_2)) = \frac{1}{2}\,[\phi(t_1+t_2)+\phi(t_1-t_2)-2\phi(t_1)\phi(t_2)]$. Hence,

$$\begin{aligned}
\sigma_{12} &= \frac{1}{2}\,\{\phi(t_1+t_2)+\phi(t_1-t_2)-2\phi(t_1)\phi(t_2)\} \\
\sigma_{11} &= \frac{1}{2}\,\left\{1+\phi(2t_1)-2\phi^2(t_1)\right\}\ \ and \\
\sigma_{22} &= \frac{1}{2}\,\left\{1+\phi(2t_2)-2\phi^2(t_2)\right\}.
\end{aligned}$$

Thus we have Z_n possessing the first and second order moments, $E(Z_n)=\Theta$ and $nV(Z_n)=\Sigma$. Hence by the central limit theorem, for fixed t_1 and t_2,

$$\sqrt{n}(Z_n-\Theta) \xrightarrow{\text{d}} Z \sim N_2(0,\Sigma). \tag{4}$$

Next, define the function

$$g(Z_n) = a_1\log\left[\frac{1-X_n(t_1)}{X_n(t_1)}\right] + a_2\log\left[\frac{1-X_n(t_2)}{X_n(t_2)}\right].$$

When $a_1 = [\log|t_1/t_2|]^{-1} \equiv a_{11},\ a_2 = -a_{11} \equiv a_{21}$ we get $g(Z_n)=\hat{\alpha}$ and when $a_1 = \frac{\log|t_2|}{\log|t_1/t_2|} \equiv a_{12},\ a_2 = \frac{-\log|t_1|}{\log|t_1/t_2|} \equiv a_{22}$, we get $g(Z_n) = \log\hat{\lambda}$ which are the estimators of α and $\log\lambda$. Setting $g(\Theta) = a_1\log\left[\frac{1-\theta_1}{\theta_1}\right] + a_2\log\left[\frac{1-\theta_2}{\theta_2}\right]$ we have

$$\begin{aligned}
\frac{\partial g}{\partial\theta_1} &= \frac{a_1}{\theta_1(\theta_1-1)}\ \ and \\
\frac{\partial g}{\partial\theta_2} &= \frac{a_2}{\theta_2(\theta_2-1)}.
\end{aligned}$$

Observing that g is a totally differentiable function, we can use a result in large sample theory, Rao (1973, p.387) to get

$$\sqrt{n}(\hat{\alpha}-\alpha) \xrightarrow{\text{d}} Z_1 \sim N(0,\sigma_1^2)\ and \tag{5}$$

$$\sqrt{n}(\log\hat{\lambda}-\log\lambda) \xrightarrow{\text{d}} Z_2 \sim N(0,\sigma_2^2). \tag{6}$$

Here $\sigma_k^2 = \sigma^2(a_{1k}, a_{2k}) = \sum_{i=1}^{2}\sum_{j=1}^{2}\sigma_{ij}\ \frac{\partial g}{\partial\theta_i}\ \frac{\partial g}{\partial\theta_j}; k = 1, 2$. Thus;

$$\sigma_1^2 = \frac{1+\phi(2t_1)-2\phi^2(t_1)}{2\left[\phi(t_1)(\phi(t_1)-1)\log\left|\frac{t_1}{t_2}\right|\right]^2} + \frac{1+\phi(2t_2)-2\phi^2(t_2)}{2\left[\phi(t_2)(\phi(t_2)-1)\log\left|\frac{t_1}{t_2}\right|\right]^2} - \frac{\phi(t_1+t_2)+\phi(t_1-t_2)-2\phi(t_1)\phi(t_2)}{\phi(t_1)(\phi(t_1)-1)\phi(t_2)(\phi(t_2)-1)\left[\log\left|\frac{t_1}{t_2}\right|\right]^2} \tag{7}$$

$${\sigma_2}^2 = \frac{1+\phi(2t_1)-2\phi^2(t_1)[\log|t_2|]^2}{2\left[\phi(t_1)(\phi(t_1)-1)\log\left|\frac{t_1}{t_2}\right|\right]^2} + \frac{1+\phi(2t_2)-2\phi^2(t_2)[\log|t_1|]^2}{2\left[\phi(t_2)(\phi(t_2)-1)\log\left|\frac{t_1}{t_2}\right|\right]^2} - \frac{\phi(t_1+t_2)+\phi(t_1-t_2)-2\phi(t_1)\phi(t_2)[\log|t_1|][\log|t_2|]}{\phi(t_1)(\phi(t_1)-1)\phi(t_2)(\phi(t_2)-1)\left[\log\left|\frac{t_1}{t_2}\right|\right]^2} \tag{8}$$

Replacing $\phi(t_i)$ by $X_n(t_i)$ in (7) and (8) we get estimates $\hat{\sigma}_1^2$ and $\hat{\sigma}_2^2$ of σ_1^2 and σ_2^2. Equations (5) to (8) thus provide consistent and asymptotically normal (CAN) estimators of $\hat{\alpha}$ and $\log\hat{\lambda}$. Now we will obtain large sample confidence intervals.

2.2 Large sample confidence intervals

From the above we can also get the large sample confidence intervals. Let $z_{\epsilon/2},\ \epsilon > 0$ be such that $P\{-z_{\epsilon/2} < Z < z_{\epsilon/2}\} = 1-\epsilon$, where $Z \sim N(0,1)$. Then in large samples we have;

$$\hat{\alpha} - \frac{z_{\epsilon/2}\hat{\sigma_1}}{\sqrt{n}} < \alpha < \hat{\alpha} + \frac{z_{\epsilon/2}\hat{\sigma_1}}{\sqrt{n}}\ \textit{and}$$
$$\hat{\lambda}\exp\left\{\frac{-z_{\epsilon/2}\hat{\sigma_2}}{\sqrt{n}}\right\} < \lambda < \hat{\lambda}\exp\left\{\frac{z_{\epsilon/2}\hat{\sigma_2}}{\sqrt{n}}\right\}.$$

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References

[1] George S.,Studies on Certain Distributions Normally Attracted to Stable, Unpublished Ph.D. Thesis, University of Kerala, (1990), India.

[2] George S. and Pillai R. N., Multivariate α-Laplace distribution by using special functions, Proceedings of the National Symposium on Special Functions and Problem Oriented Research, Centre for Mathematical Sciences, Vazhuthacaud, Trivandrum, India (1988).

[3] Kotz S., Kozubowski T. J. and Podgórski K., The Laplace Distribution and Generalizations: A Revisit with Applications to Communications, Economics, Engineering and Finance, Birkhäuser, Boston, (2001).

[4] Pillai R. N., α-Laplace stochastic process, J. Orissa Math. Soc., 4 (1985), 123-128.

[5] Pillai R. N. and George S., A certain class of distributions under normal attraction, Proc. of the VI annual conference of ISPS, (1984), 107-112.

[6] Pillai R. N. and Kalyanaraman K., Characterisation of a law that is normally attracted to a stable law, Proc. of the VI annual conference of ISPS, (1984), 9-18.

[7] Press S. J., Estimation in univariate and multivariate stable distributions., J. Amer. Statist. Assoc., 67 (1972), 842-846.

[8] Rao C. R., Linear Statistical Inference and Its Applications, 2^{nd} Edition, (1973), Wiley, New York.