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PARTIAL PERMUTATION GENERATION METHOD : REVISITED

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Abstract

This paper deals with a new method of forming a partial random permutation of n integers which is based on coin tossing experiment and discusses the properties of random variable associated with them.

1. Introduction

Permutations and Combinations have always played a significant role in many aspects of Discrete Mathematics and Statistics. The foundation of probability theory rests on these two important features. The study of random variables associated with randomly constructed permutations continues to attract the attention of many researchers worldwide.

In this article we propose a new method of forming a partial random permutation of n integers which is based on coin tossing experiment and study the properties of random variable associated with them. We describe the method in the next section.

Key Words and Phrases : *Partial random permutation, Coin tossing experiment.*

2. Description of Method

We denote the permutation of n integers $(1, 2, \cdots, n)$ by $\pi = (\pi_1, \pi_2, \cdots, \pi_n)$.
The n positions corresponding to the n integers $(1, 2, \cdots, n)$ can be represented as follows.

Position	1	2	$\cdots$	i	$i+1$	$i+2$	$\cdots$	n
Integer	$*$	$*$	$\cdots$	$*$	$*$	$*$	$\cdots$	$*$

The method of permutation generation depends upon the outcomes of an unbiased coin which is tossed $(n-1)$ times.
If the first toss results in a head (H), we write 1 at the n^{th} position whereas, if the first toss results in a tail (T), we write 1 at the first position. The next vacant positions are successively filled from the right or left end with the appropriate successive numbers depending upon the outcomes at the successive trials i.e, whenever the outcome is head (H), the number corresponding to that trial is written from the right end at the next available vacant position whereas the vacant position from the left end is filled with the appropriate successive number if the outcome is tail (T).
This process is followed for $(n-1)$ tosses. Finally, the last integer n is placed at the last empty place.
Illustration : Suppose $n = 4$ i.e we are interested in obtaining a random permutation of the integers (1, 2, 3, 4). Here an unbiased coin is tossed three times. The possible outcomes of the first three trials and the resulting permutation are presented below.

Sr. no.	Outcome Trial no.			Positions filled				Resultiing permutation
	1	2	3	1	2	3	4	
1	H	H	H	4	3	2	1	(4 3 2 1)
2	H	H	T	3	4	2	1	(3 4 2 1)
3	H	T	H	2	4	3	1	(2 4 2 1)
4	H	T	T	2	3	4	1	(2 3 4 1)
5	T	T	T	1	2	3	4	(1 2 3 4)
6	T	T	H	1	2	4	3	(1 2 4 3)
7	T	H	H	1	4	3	2	(1 4 3 2)
8	T	H	T	1	3	4	2	(1 3 4 2)

In this case, the eight distinct permutations are as follows $(4\ 3\ 2\ 1), (3\ 4\ 2\ 1), (2\ 4\ 3\ 1), (2\ 3\ 4\ 1), (1\ 2\ 3\ 4), (1\ 2\ 4\ 3), (1\ 4\ 3\ 2)$ and $(1\ 3\ 4\ 2)$.

Remark : It may be noted that for $n = 4$, we obtain $2^3 = 8$ permutations. In general for n integers $(1, 2 \cdots, n)$, the number of distinct permutations possible are 2^{n-1}. Also, the permutations obtained are equiprobable.

In the next sections we define the random variable associated with the permutations and obtain its probability distribution and other properties.

3. Probability Distribution of the random Variable Y

Let the random variable Y denotes the absolute difference between the first and n^{th} integers of the random permutation generated using above method. If π_1 denotes the integer at the first position and π_n denotes the integer at the n^{th} position of the random permutation then, $Y = |\pi_n - \pi_1|$. It may be noted that Y can take values from $1, 2, 3, \cdots, n-1$.

Theorem 1 : The probability distribution of the random variable Y is given by

$$P(Y = i) = \begin{cases} \frac{1}{2^i}, & i = 1, 2, 3, \cdots, n-2 \\ \\ \frac{1}{2^{n-2}}, & i = n-1 \end{cases} \tag{3.1}$$

Proof : We discuss the following two cases.

Case I : $Y = |\pi_n - \pi_1| = (n-1)$.

Consider the following two possibilities of outcomes for this case. The $(n-1)$ tosses either result in all heads or all tails i.e

$$H\ H\ H\ H \cdots \quad \text{or} \quad T\ T\ T\ T \cdots$$

If all the $(n-1)$ tosses result in heads, the integers $1, 2, \cdots, n-1$ will be placed at successive positions from the right end with the last integer n automatically getting placed at the first position after the $(n-1)^{th}$ toss. The resulting permutation will be $(n, n-1, n-2, \cdots, 3, 2, 1)$.

The probability of the above event happening is given by

$$P(Y = n-1) = \frac{1}{2^{n-1}}. \tag{3.2}$$

Similarly, if all $(n-1)$ tosses result in tails (T) the resulting permutation will be $(1, 2, 3, \cdots, n-1, n)$ as the integers are now placed successively from the left end.

Again, the last integer n gets automatically placed at the last position after the $(n-1)^{th}$ toss. As in the previous situation, the probability of the above event happening is given by,

$$P(Y = n-1) = \frac{1}{2^{n-1}}. \tag{3.3}$$

On combining (3.2) and (3.3), we get

$$\begin{aligned} P(Y = n-1) &= \frac{1}{2^{n-1}} + \frac{1}{2^{n-1}} \\ &= \frac{2}{2^{n-1}} \\ &= \frac{1}{2^{n-2}}. \end{aligned}$$

Therefore

$$P(Y = i) = \frac{1}{2^{n-2}}, \quad i = (n-1). \tag{3.4}$$

Case II : $Y = |\pi_n - \pi_1| = i, \quad i = 1, 2, 3, \cdots, n-2.$

In this case also, there are again two possibilities of outcomes. In $n-1$ tosses, either i consecutive heads (H) occur before the first tail (T) or i consecutive tails (T) occur before the first head (H)

i.e either $\underbrace{H\ H\ H \cdots H}_{i}$ or $\underbrace{T\ T\ T \cdots T}_{i} H \cdots$ happens.

In the first string, after i consecutive heads (H), at the $(i+1)^{th}$ toss, tail (T) occurs. The integers $1, 2, \cdots, i, i+1$ occupy the positions shown below.

Position	1	2	$\cdots$	$(i-1)$	i	$(i+1)$	$\cdots$	$(n-1)$	n
Outcome	T	$*$	$\cdots$	$*$	H	H	$\cdots$	H	H
Integer	$(i+1)$	$*$	$\cdots$	$*$	i	$*$	$\cdots$	2	1

The remaining $(n-i-1)$ tosses may consist of heads or tails.

Thus here $\pi_n = 1$ and $\pi_1 = i+1$.

Therefore, $Y = |\pi_n - \pi_1| = i$.

The probability associated with this string of outcomes is thus given by

$$P(Y = i) = \frac{1}{2^{i+1}}, \quad i = 1, 2, \cdots, (n-2). \tag{3.5}$$

Similarly, in the second string, after i consecutive tails (T), at the $(i+1)^{th}$ toss, head (H) occurs. The integers $1, 2, \cdots, i+1$ now occupy the following positions.

Position	1	2	$\cdots$	$(i-1)$	i	$(i+1)$	$\cdots$	$(n-1)$	n
Outcome	T	T	$\cdots$	T	T	$*$	$\cdots$	$*$	H
Integer	1	2	$\cdots$	$*$	i	$*$	$\cdots$	$*$	$(i+1)$

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As before, the remaining $(n-i-1)$ tosses may consist of heads or tails.

Thus here $\pi_1 = 1$ and $\pi_n = i+1$.

Therefore, $Y = |\pi_n - \pi_1| = i$.

The probability associated with this string of outcomes is thus given by,

$$P(Y = i) = \frac{1}{2^{i+1}}, \quad i = 1, 2, \cdots, (n-2) \tag{3.6}$$

On combining (3.5) and (3.6), we get

$$\begin{aligned} PY = i) &= \frac{1}{2^{i+1}} + \frac{1}{2^{i+1}} \\ &= \frac{1}{2^i}. \end{aligned}$$

Therefore,

$$P(Y = i) = \frac{1}{2^i}, \quad i = 1, 2, \cdots, (n-2). \tag{3.7}$$

Hence, from Case I and Case II, the probability distribution of the random variable Y is given by

$$P(Y = i) = \begin{cases} \frac{1}{2^i}, & i = 1, 2, 3, \cdots, n-2 \\ \\ \frac{1}{2^{n-2}}, & i = n-1 \end{cases}.$$

This proves Theorem (1).

4. Properties of the Random Variable Y

In this section, we obtain the expectation and variance of Y.

4.1 Expectation of Y

Theorem 2 : The expected value of Y is given by,

$$E[Y] = 2 - \frac{1}{2^{n-2}}. \tag{4.1.1}$$

Proof : We define

$$\begin{aligned} E[Y] &= \sum_{i=1}^{n-1} ip(i) \\ &= \sum_{i=1}^{n-2} \frac{i}{2^i} + \frac{n-1}{2^{n-2}} \qquad \text{(using (3.1))} \end{aligned}$$

Let

$$S = \sum_{i=1}^{n-2} \frac{i}{2^i}.$$

It can be easily shown that,

$$S = 2\left[1 - \frac{1}{2^{n-2}}\right] - \frac{n-2}{2^{n-2}}.$$

Therefore,

$$\begin{aligned} E[Y] &= \sum_{i=1}^{n-2} \frac{i}{2^i} + \frac{n-1}{2^{n-2}} \\ &= 2\left[1 - \frac{1}{2^{n-2}}\right] - \frac{n-2}{2^{n-2}} + \frac{n-1}{2^{n-2}} \end{aligned}$$

which upon simplification gives

$$E[Y] = 2 - \frac{1}{2^{n-2}}.$$

This proves Theorem (2).

We note that for large n, $E[Y] \to 2$.

4.2 Variance of Y :

Theorem 3 : Variance of Y is given by

$$V[Y] = 2 - \frac{1}{2^{n-2}}\left[2n - 3 + \frac{1}{2^{n-2}}\right]. \tag{4.2.1}$$

Proof : We first evaluate $E[Y^2]$ using the probability distribution given in Theorem 1. We define,

$$\begin{aligned} E[Y^2] &= \sum_{i=1}^{n-1} i^2 p(i) \\ &= \sum_{i=1}^{n-2} \frac{i^2}{2^i} + \frac{(n-1)^2}{2^{n-1}}. \end{aligned}$$

Let

$$S = \sum_{i=1}^{n-2} \frac{i^2}{2^i}.$$

It can be easily shown that,

$$S = 6 - \frac{n^2+2}{2^{n-2}}.$$

Therefore,

$$\begin{aligned} E[Y^2] &= \sum_{i=1}^{n-2} \frac{i^2}{2^i} + \frac{(n-1)^2}{2^{n-2}} \\ &= 6 - \frac{(n^2+2)}{2^{n-2}} + \frac{(n-1)^2}{2^{n-2}} \end{aligned}$$

which upon simplification gives,

$$E[Y^2] = 6 - \frac{(2n+1)}{2^{n-2}}. \tag{4.2.2}$$

Now,

$$V[Y] = E[Y^2] - [E[Y]]^2.$$

Using equations (4.1.1) and (4.2.2), we get

$$V[Y] = 6 - \frac{(2n+1)}{2^{n-2}} - \left[2 - \frac{1}{2^{n-2}}\right]^2$$

which upon simplification gives

$$V[Y] = 2 - \frac{1}{2^{n-2}}\left[2n - 3 + \frac{1}{2^{n-2}}\right].$$

This proves Theorem (3).

5. Numerical results

Using (4.1.1) and (4.2.1), we have evaluated the numerical values of expectation and variance of Y for different values of n. These are presented in the following table.

Table 1: Expectation and Variance of Y

n	$E[Y]$	$V[Y]$	n	$E[Y]$	$V[Y]$
3	1,5000	0.2500	7	1.9688	1.6553
4	1.7500	0.6875	8	1.9844	1.7966
5	1.8750	1.1094	9	1.9922	1.8828
6	1.9375	1.4336	10	1.9961	1.9336

It can be observe that both expectation and variance of Y approach 2 with an increase in n as proved theoretically.

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