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A NOTE ON FUZZY STRONGLY SEMI PRE-OPEN SETS AND FUZZY STRONG SEMI PRE-CONTINUITY

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Abstract

In this paper we will introduce a new class of fuzzy open sets which we will name as strongly semi pre-open sets. We will study their properties as well as their relation with other types of fuzzy sets that were introduced by other authors. We will also define the fuzzy strongly semi pre-continuous mappings and we will investigate their properties and their relations with other forms of fuzzy continuity.

1. Introduction

The concept of fuzzy set was introduced by Zadeh in [1]. Fuzzy topological spaces were introduced by Chang in [2]. Properties and characteristics of some other classes of open sets, semiopen sets, preopen sets as well as strongly semiopen sets were introduced by authors in [6, 7 and 9], respectively. Azad in his work [1] introduced fuzzy semiopen sets while Zhong [13] and Singal [11] introduced fuzzy

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strongly semiopen and fuzzy preopen sets, respectively. B. Krsteska in [4] introduced an independent class of fuzzy strongly preopen sets as well as the concept of fuzzy strong precontinuity, fuzzy strongly preopen and fuzzy strongly preclosed mappings. Many other authors have introduced other forms of fuzzy opened sets and other forms of fuzzy continuous mappings, see [3, 5, 8 and 10].
In our work we will introduce a new class of fuzzy sets named fuzzy strongly semi preopen sets and we will look at their relation with other similar sets introduced by other authors. We will also introduce concept of fuzzy strong semi precontinuity and investigate their relation with other forms of similar mappings introduced by other authors. The properties of fuzzy strongly semi-preopen and fuzzy strongly semi preclosed mappings will be investigated.

2. Preliminaries

Throughout this work we will use the notations (X, τ) or sometimes only by X to denote a fuzzy topological space, shortly fts, as defined by Chang in [2]. We will also denote by $intA$, clA and A^c the interior, closure and the complement of a fuzzy set A, respectively.
Definition 2.1 : A fuzzy set A of a fts X is called:

(1) Fuzzy semiopen if and only if $A \leq cl(intA)$ [1];

(2) Fuzzy preopen if and only if $A \leq int(clA)$ [11];

(3) Fuzzy strongly semiopen if and only if $A \leq int(cl(intA))$ [13].

Definition 2.2 : A fuzzy set A of a fts X is called:

(1) Fuzzy semiclosed if and only if A^c is a fuzzy semiopen set of a fts X [1];

(2) Fuzzy preclosed if and only if A^c is a fuzzy preopen set of a fts X [11];

(3) Fuzzy strongly semiclosed if and only if A^c is a fuzzy strongly semiopen set of a fts X [13].

Definition 2.3 : Let A be a fuzzy set of a fts X. Then,
$pintA = \vee\{B | B \leq A, B\, is\, a\, fuzzy\, preopen\, set\}$, is called the fuzzy preinterior of set A, [11];
$pclA = \wedge\{B | B \geq A, B\, is\, a\, fuzzy\, preclosed\, set\}$, is called the fuzzy preclosure of set A, [11].
Definition 2.4: A fuzzy set A of a fts X is called:

(a) Fuzzy strongly preopen if and only if $A \leq int(pclA)$ [4];

(b) Fuzzy strongly preclosed if and only if A^c is a fuzzy strongly preopen set of X [4].

We will also use the following Lemmas.
Lemma 2.1 [13] : Let A be a fuzzy set of a fts X. Then,

(a) $pclA^c = (pintA)^c$

(b) $pintA^c = (pclA)^c$

Lemma 2.2 [**4**] : Let $f : X \to Y$ be a mapping. For fuzzy sets A and B of X and Y respectively, the following statements hold:

(a) $ff^{-1}(B) \leq B$

(b) $f^{-1}f(A) \geq A$

(c) $f(A^c) \geq (f(A))^c$

(d) $f^{-1}(B^c) = (f^{-1}(B))^c$

(e) if f is injective, then $f^{-1}f(A) = A$;

(f) if f is surjective, then $ff^{-1}(B) = B$

(g) if f is bijective, then $f(A^c) = (f(A))^c$

Definition 2.5 : Let $f : X \to Y$ be a mapping from a fts (X, τ) to a fts (Y, δ). The mapping is called:

(1) Fuzzy continuous if $f^{-1}(B)$ is a fuzzy open set of X, for each $B \in Y$ [2];

(2) Fuzzy semicontinuous if $f^{-1}(B)$ is a fuzzy semiopen set of X, for each $B \in Y$ [1];

(3) Fuzzy precontinuous if $f^{-1}(B)$ is a fuzzy preopen set of X, for each $B \in Y$ [11];

(4) Fuzzy strong semicontinuous if $f^{-1}(B)$ is a fuzzy strongly semiopen set of X, for each$B \in Y$ [13];

(5) Fuzzy strong precontinuous if $f^{-1}(B)$ is a fuzzy strongly preopen set of X, for each $B \in Y$ [4];

(6) Fuzzy open (closed) if $f(A)$ is a fuzzy open (closed) set of Y, for each $A \in X$ [2];

(7) Fuzzy semiopen (semiclosed) if $f(A)$ is a fuzzy semiopen (semiclosed) set of Y, for each $A \in X$ [1];

(8) Fuzzy preopen (preclosed) if $f(A)$ is a fuzzy preopen (preclosed) set of Y, for each $A \in X$ [11];

(9) Fuzzy strongly semiopen (semiclosed) if $f(A)$ is a fuzzy strongly semiopen (semiclosed) set of Y, for each $A \in X$ [13];

(10) Fuzzy strongly preopen (preclosed) if $f(A)$ is a fuzzy strongly preopen (preclosed) set of Y, for each $A \in X$ [4;

Lemma 2.3 [4] : Let $\{A_\alpha\}_{\alpha\in I}$, be a family of fuzzy sets of a $ftsX$. Then:
$\vee_{\alpha\in I} pcl(A_\alpha) \leq pcl(\vee_{\alpha\in I} A_\alpha)$.

3. Fuzzy Strongly Semi-pre-open Sets and Fuzzy Strongly Semi-pre-closed sets

Definition 3.1 : A fuzzy set A on a fuzzy topological space X is called a fuzzy strongly semi-pre-open set (shortly $FSSPO$ set) if and only if it satisfies the following condition $A \leq int(pclA) \vee pcl(intA)$.
The family of fuzzy strongly semi-pre-open sets of a fuzzy topological space (X, τ) will be denoted $FSSPO(\tau)$.
From the above given definition we can very easily verify that:
Theorem 3.1 : If X is a fts, then the following holds:

(a) Every fuzzy open set A is a $FSSPO$ set;

(b) Every fuzzy strongly pre-open set is a $FSSPO$ set.

(c) Every fuzzy strongly semi-open set is a $FSSPO$ set.

(d) Every fuzzy semiopen set is a $FSSPO$ set.

Proof : (a) Every fuzzy open set is a $FSSPO$ set because if A is an open set then $A = intA$, so:

$$A = intA \leq pcl(intA) \leq int(pclA) \vee pcl(intA)$$

(b) Given any fuzzy strongly pre-open set A, it is obvious that

$$A \leq int(pclA) \leq int(pclA) \vee pcl(intA)$$

(c) If a set A is a fuzzy strongly semi open, then $A \leq int(cl(int(A)))$. From the other side we know that:

$$int(pclA) \geq A \vee cl(intA) \implies int(pclA) \geq int(cl(intA))$$

which means that A is a $FSSPO$ set.
(d) If a set A is a fuzzy semi open, then

$$A \leq cl(intA) = pcl(intA) \leq int(pclA) \vee pcl(intA)$$

Therefore, A is a $FSSPO$ set. □

Example 3.1 : Let $X = \{a, b, c\}$, and let $A = \{(a, 0.3); (b, 0.2); (c, 0.7)\}$, $B = \{(a, 0.81); (b, 0.8); (c, 0.42)\}$, $C = \{(a, 0.4); (b, 0.35); (c, 0.75)\}$ be fuzzy sets of X.
If we define a fuzzy topological space (X, τ), such that $\tau = \{0, A, B, A \vee B, A \wedge B, 1\}$.
It is easy to show that $C \leq int(pclC) \vee pcl(intC)$, in other words, the set C is a $FSSPO$ set.

Remark 3.1 : It is obvious that not all fuzzy preopen sets are $FSSPO$ sets and not all $FSSPO$ sets are fuzzy preopen sets. (see Example 4.1).

Definition 3.2 : A fuzzy set A on a fuzzy topological space X is called a fuzzy strongly semi-pre-closed set (shortly $FSSPC$ set) if and only if A^c is a fuzzy strongly semi-pre-open set of a fts X.
The family of all fuzzy strongly semi-pre-closed sets of a fts (X, τ) will be denoted $FSSPC(\tau)$.
In regards to the $FSSPC$ sets we can formulate and prove the following

Theorem 3.2 : Let A be a fuzzy set of a fts (X, τ), then A is a fuzzy strongly semi-pre-closed set if and only if $A \geq cl(pintA) \wedge pint(clA)$.

Proof : It follows from the definition. □

Theorem 3.3 :

(a) Any union of $FSSPO$ sets is also a $FSSPO$ set

(b) Any intersection of $FSSPC$ sets is also a $FSSPC$ set.

Proof : (a)Given any family of $FSSPO$ sets $\{A_i\}_{i \in I}$, we can conclude that:

$$\vee_{i\in I}A_i \leq \vee_{i\in I}[int(pclA_i) \vee pcl(intA_i)] \leq [int(pcl(\vee_{i\in I}A_i) \vee pcl(int \vee_{i\in I} A_i)]$$

(b) Similar to (a). □

Given any family of $FSSPO$ sets $\{A_i\}_{i\in I}$, it is obvious that its intersection may not always be a $FSSPO$ set.

Let us illustrate this with an example.

Example 3.2 : Let $X = \{a, b, c\}$, and let $D = \{(a, 0.0); (b, 0.0), (c, 0.25)\}$, $E = \{(a, 0.85); (b, 0.7); (c, 0.0)\}$, $F = \{(a, 0.1); (b, 0.3), (c, 0.25)\}$ and $G = \{(a, 0.9); (b, 0.8); (c, 0.1)\}$ be fuzzy sets of X.

If we define a fuzzy topological space (X, δ), such that $\delta = \{0, D, E, D \vee E, 1\}$.

It is easy to show that fuzzy sets F and G are $FSSPO$ sets, but their intersection $F \wedge G$ is not a $FSSPO$ set.

Definition 3.3 : If A is a fuzzy set of a fuzzy topological space X , then,

(1) The union of all fuzzy strongly semi-pre-open sets contained in a set A is called a fuzzy strong semi-pre-interior of set A and is denoted as $sspintA$.

(2) The intersection of all fuzzy strongly semi-pre-closed sets containing set A is called the strong semi-pre-closure and is denoted as $sspclA$.

We can formulate the following:

Theorem 3.4 sets A and B are fuzzy sets of a fuzzy topological space X, then the following stand:

(a) $intA \leq sspintA \leq A$	$A \leq sspclA \leq clA$
(b) $sspintA \in FSSPO(\tau)$	$sspclA \in FSSPC(\tau)$
(c) $A \in FSSPO(\tau) \iff A = sspintA$	$A \in FSSPC(\tau) \iff A = sspclA$
(d) $A \leq B \implies sspintA \leq sspintB$	$A \leq B \implies sspclA \leq sspclB$
(e) $sspint(sspintA) = sspintA$	$sspcl(sspclA) = sspclA$
(f) $sspintA \wedge sspintB \geq sspint(A \wedge B)$	$sspcl \vee sspintB \leq sspcla(A \vee B)$
(g) $sspintA \vee sspintB \leq sspint(A \vee B)$	$sspcl \wedge sspintB \geq sspcla(A \wedge B)$
(h) $sspintX = X$,$sspint\emptyset = \emptyset$	$sspclX = X$, $sspcl\emptyset = \emptyset$

Proof : The proof follows from the definitions and theorems given above. □

Theorem 3.5 :

(a) Given a fts X, a fuzzy set B is a $FSSPO$ set if and only if there exists a fuzzy set A of X such that $A \leq B \leq int(pclA) \vee pcl(intA)$

(b) Given a fts X, a fuzzy set B is a $FSSPC$ set if and only if there exists a fuzzy set A of X such that $cl(pintA) \wedge pint(clA) \leq B \leq A$

Proof : (a)If there is a set A which satisfies such conditions, then it is obvious that from $A \leq B$ and from the given condition we will get

$$B \leq int(pclA) \vee pcl(intA) \leq int(pclB) \vee pcl(intB)$$

meaning that B is a $FSSPO$ set.
Conversely, if B is a fuzzy strongly semi-pre-open set, then the result will follow if we take $A = B$.
(b) similary to case (a).
We can formulate the following theorem which gives the relation between fuzzy strong semi-pre-interior and fuzzy strong semi-pre-closure. □

Theorem 3.6 : If A is a fuzzy set of a fuzzy topological space X, then:

(1) $sspclA^c = (sspintA)^c$

(2) $sspintA^c = (sspclA)^c$.

Proof : (1) According to the definition, we have:

$$(sspintA)^c = (\vee\{C|C \leq A, C \in FSSPO(\tau)\})^c = \wedge\{C^c|C^c \geq A^c, C^c \in FSSPC(\tau)\} = sspclA^c$$

(2) similarly as in (1) we can show that:

$$(sspclA)^c = (sspcl(A^c)^c)^c = ((sspintA^c)^c)^c = sspintA^c$$

□

We can also formulate the following theorem which gives the relation between strong semi-pre-interior and strong semi-pre-closure.

Theorem 3.7 : Let A be a fuzzy set of a fuzzy topological space X, then:

(1) $sspintA \leq A \wedge (int(pclA) \vee pcl(intA))$

(2) $sspclA \geq A \vee (pint(clA) \wedge cl(pintA))$.

Proof : (1) Based on the fact that $sspintA$ is also a $FSSPO$ set, as a union of $FSSPO$ sets, we can conclude that:

$$sspintA \leq (int(pcl(sspintA)) \vee pcl(int(sspintA)) \leq (int(pclA) \vee pcl(intA))$$

The last relation as well as the fact that $sspintA \leq A$ gives the conclusion given in (1).
(2) If we find the compliment of (1) we will get

$$(sspintA)^c \geq (A \wedge (int(pclA) \vee pcl(intA)))^c = A^c \vee (cl(pintA^c) \wedge pint(clA^c))$$

Which gives the desired result. □

Definition 3.4 : Fuzzy topological space X is called an *SSPO-extremely disconnected* if and only if the $sspclA$ is a $FSSPO$ for every $FSSPO$ set A of X.
We can formulate the following theorem which gives some interesting characteristics of the mentioned spaces.

Theorem 3.8 : Let X be a fuzzy topological space, the following statements are equivalent:

(i) X is an *SSPO-extremely disconnected.*

(ii) $sspintA$ is a strongly semi-pre-closed set for each strongly semi-pre-closed set A of the fuzzy topological space X.

(iii) $sspcl(sspclA)^c = (sspclA)^c$, where A is a fuzzy strongly semi-pre-open set of X.

(iv) If $B = (sspclA)^c$, then $sspclB = (sspclA)^c$ for any pair of fuzzy strongly-semi-pre-open sets A, B of X.

Proof : $(i) \implies (ii)$ Given any fuzzy strongly semi-pre-closed set A, its compliment A^c is a $FSSPO$ set, which means that $sspclA^c$ is a $FSSPO$ set, and then according to Theorem **??** we can conclude that $sspclA^c = (sspintA)^c$ meaning that $(sspintA)^c$ is a $FSSPO$ set, and consequently $sspintA$ is a $FSSPC$ set.
$(ii) \implies (iii)$ Let us suppose that A is fuzzy strongly semi-pre-open set, then $sspcl(sspclA)^c = sspcl(sspintA^c)$, and since according to (ii) $sspintA^c$ is a $FSSPC$ set, which means that $sspcl(sspintA^c) = sspintA^c$, and this gives us the desired conclusion $sspcl(sspclA)^c = (sspclA)^c$.
$(iii) \implies (iv)$ Let us suppose that sets A, B are any pair of fuzzy strongly semi-pre-open sets of X such that $B = (sspclA)^c$ then $sspclB = sspcl(sspclA)^c = (sspclA)^c$.
$(iv) \implies (i)$ Let us suppose that A is a $FSSPO$ set of X. If we consider a set $B = (sspclA)^c$, then from the assumption we have that $sspclB = (sspclA)^c$ which means that $(sspclA)^c$ is a $FSSPC$ set and its compliment $sspclA$ is a $FSSPO$ set. Thus, if A is a fuzzy strongly semi-pre-open set of X, then $sspclA$ is also a fuzzy strongly semi-pre-open set, that is X is a *SSPO-extremely disconnected.* □

4. Fuzzy Strongly Semi-pre-continuity

Definition 4.1 : A mapping $f : (X, \tau_1) \to (Y, \tau_2)$, which maps a fuzzy topological space X into a fuzzy topological space Y is called a fuzzy strongly semi-pre-continuous if $f^{-1}(B) \in FSSPO(\tau_1)$ for every $B \in \tau_2$.
It can be very easily verified that any fuzzy continuous mapping is also a fuzzy strongly semi-pre-continuous mapping. Also any fuzzy strongly semi-continuous mapping and fuzzy strongly pre-continuous mappings are fuzzy strongly semi-pre-continuous mappings.

We can give an example to show that not all fuzzy pre-continuous mappings are fuzzy strongly semi-pre-continuous.

Example 4.1 : Let $X = \{a, b, c\}$, and let us define fuzzy topological spaces (X, τ) and (X, τ_1) such that $\tau = \{A, B, A \vee B, A \wedge B, 1\}$; $\tau_1 = \{0, C, 1\}$, where sets A, B and C are defined the same as in Example 3.1. Let $f = id : (X, \tau) \to (X, \tau_1)$, it

is obvious that f is a fuzzy strongly semi-pre-continuous mapping but it is not a fuzzy continuous mapping.
If we consider the fuzzy topological space $\tau_2 = \{0, A, 1\}$ and $\tau_3 = \{0, B, 1\}$, then the mapping $f = id : (X, \tau_2) \to (X, \tau_3)$ is a fuzzy pre-continuous but it is not a fuzzy strongly semi-precontinuous.
If we consider the fuzzy topological space $\delta_1 = \{0, G, 1\}$ where $G = \{(a, 0.3); (b, 0.20); (c, 0.4)\}$ and the fuzzy topological space $\delta_2 = \{0, V, 1\}$ such that $V = \{(a, 0.5); (b, 0.6); (c, 0.57)\}$, the mapping $f = id : (X, \delta_1) \to (X, \delta_2)$ is a fuzzy strongly semi-precontinuous mapping which is not a fuzzy pre-continuous mapping.
Theorem 4.1 : Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a mapping from a fuzzy topological space X into a fuzzy topological space Y then the following statements are equivalent:

(a) f is a fuzzy strongly semi-pre-continuous mapping;

(b) $f^{-1}(B)$ is a fuzzy strongly semi-pre-closed set of X, for each fuzzy closed set B of Y;

(c) $f(sspclA) \leq clf(A)$for each fuzzy set A of X;

(d) $sspclf^{-1}(B) \leq f^{-1}(clB)$, for each fuzzy set B of Y;

(e) $f^{-1}(intB) \leq sspint(f^{-1}(B))$, for each fuzzy set B of Y;

(f) There is a base β for τ_2 such that $f^{-1}(B)$ is a fuzzy strongly semi-pre-open set of X for each $B \in \beta$;

(g) There is a base β for τ_2 such that $f^{-1}(B)$ is a fuzzy strongly semi-pre-closed set of X for each $B^c \in \beta$;

Proof : The proof is standard and is therefore omitted. □
Theorem 4.2 : Let $f : X \to Y$ be a bijective mapping from a fuzzy topological space X into a fuzzy topological space Y. The mapping f is fuzzy strongly semi-pre-continuous if and only if $intf(A) \leq f(sspintA)$, for each fuzzy set A of X.
Proof : Let us suppose that the given function $f : X \to Y$ is a fuzzy strongly semi-pre-continuous. Given any fuzzy set A of X then $f^{-1}(intf(A))$ is a fuzzy strongly semi preopen set, and from the Theorem 4.1, as well as from the fact that $f : X \to Y$ is bijective, we have:

$$f^{-1}(intf(A)) \leq sspint(f^{-1}(intf(A))) \leq sspint(f^{-1}(f(A))) = sspintA$$

From the fact that f is surjective, we can conclude that:

$$intf(A) = f(f^{-1}(intf(A))) \leq f(sspintA)$$

Conversely, let us suppose that B is a fuzzy open set of Y. It is obvious that $intB = B$, and according to the assumption we have:

$$f(sspintf^{-1}(B)) \geq intf(f^{-1}(B)) = intB = B$$

The last implies that

$$f^{-1}(f(sspintf^{-1}(B))) \geq f^{-1}(B)$$

And since f is injective, we get the following:

$$f^{-1}(f(sspintf^{-1}(B))) = sspintf^{-1}(B) \geq f^{-1}(B).$$

On the other side, it is obvious that $sspintf^{-1}(B) \leq f^{-1}(B)$, which gives us the conclusion that $sspintf^{-1}(B) = f^{-1}(B)$, which means that for each fuzzy opened set B of Y,$f^{-1}(B)$ is a $FSSPO$ set, in other words, the mapping f is fuzzy strongly semi pre-continuous. □

Theorem 4.3 : Let $f : X \to Y$ and $g : y \to Z$ be mappings and let X, Y, Z be fuzzy topological spaces. If a mapping f is a fuzzy strongly semi-pre-continuous and g is a fuzzy continuous mapping, then the product gf is a fuzzy strongly semi-pre-continuous mapping.

Proof : Due to the conditions of the theorem, and from the fact that $(gf)^{-1}(B) = f^{-1}(g^{-1}(B))$, it is obvious that for any fuzzy open set B of Z, $g^{-1}(B)$ is fuzzy open set and therefore $f^{-1}(g^{-1}(B))$ is a fuzzy strongly semi-pre-open set thus gf is fuzzy strongly semi-pre-continuous. □

Corollary 4.3.1 : If X, X_1, X_2 are fuzzy topological spaces and let $p_i : X_1 \times X_2 \to X_i$
$(i = 1, 2)$, be projections of $X_1 \times X_2$ onto X_i. If $f : X \to X_1 \times X_2$ is a fuzzy strongly semi-pre-continuous then the compositions $p_i f$ are also strongly semi-pre-continuous.

Proof : Since projections $p_i : X_1 \times X_2 \to X_i, (i = 1, 2)$ are continuous mappings, the statement follows from Theorem 4.3.

5. Fuzzy Strongly Semi-pre-open and Fuzzy Strongly Semi-pre-closed Mappings

Definition 5.1 : A mapping $f : (X, \tau_1) \to (Y, \tau_2)$ is called:

(a) fuzzy strongly semi-pre-open if $f(A) \in FSSPO(\tau_2)$ for each $A \in \tau_1$.

(b) fuzzy strongly semi-pre-closed if $f(A) \in FSSPC(\tau_2)$ for each $A^c \in \tau_1$.

Since any open (closed) sets are $FSSPO$ $(FSSPC)$ sets then we can conclude that any fuzzy open (closed) mapping is always a $FSSPO$ $(FSSPC)$ mapping. For similar reasons any fuzzy strongly preopen (preclosed) mapping is a $FSSPO$ $(FSSPC)$ mapping.

Theorem 5.1 : Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a mapping from a fts X to a fts Y. The following statements are equivalent:

(1) f is a fuzzy strongly semi-pre-open mapping;

(2) $f(intA) \leq sspintf(A)$, for each fuzzy set A of X;

(3) $intf^{-1}(B) \leq f^{-1}(sspintB)$, for each fuzzy set B of Y;

(4) $f^{-1}(sspclB) \leq cl(f^{-1}(B))$, for each fuzzy set B of Y;

(5) There is a base α for τ_1 such that $f(A)$ is a fuzzy strongly semi-pre-open set of Y for each $A \in \alpha$.

Proof : Proof of this theorem is standard and therefore will be omitted. □

Theorem 5.2 : Let $f : X \to Y$ be a mapping from a fts X to a fts Y. The mapping f is a fuzzy strongly semi-pre-closed mapping if and only if $sspclf(A) \leq f(clA)$, for each fuzzy set A of X.

Proof : If f is a $fsspc$ mapping then $f(clA)$ is a $FSSPC$ set of Y which contains $f(A)$ and since $sspclf(A)$ is the intersection of all $FSSPC$ sets that contain $f(A)$, it is obvious that $sspclf(A) \leq f(clA)$.

Conversely, let B be a closed set of X, then from the condition of theorem we have $sspclf(B) \leq f(clB) = f(B)$, but from the other side we also have that $f(B) \leq sspclf(B)$, for any B of X, which therefore yields $sspclf(B) = f(B)$.

In other words, for any closed set B, the image of it under f is a $FSSPC$ set, therefore f is a $FSSPC$ mapping. □

Theorem 5.3 : Let $f : X \to Y$ be a bijective mapping from a fts X to a fts Y. The mapping f is a fuzzy strongly semi-pre-closed mapping if and only if it is a fuzzy strongly semi-pre-open mapping.

Proof : This is obvious, it follows immediately from Lemma 2.2.

Corollary 5.3.1 : Let $f : X \to Y$ be a bijective mapping from a fts X to a fts Y. The mapping f is a fuzzy strongly semi-pre-open if and only if $sspclf(A) \leq f(clA)$, for each fuzzy set A of X.

Proof : It is obvious, it follows from Theorem 5.3, the mapping is a fuzzy strongly semi-pre-open if and only if f is also a $FSSPC$ and then from Theorem 5.2 it follows that f is $FSSPC$ mapping if and only if $sspclf(A) \leq f(clA)$. □

Corollary 5.3.2 : Let $f : X \to Y$ be a bijective mapping from a fts (X, τ_1) to a fts (Y, τ_2). The following statements are equivalent:

(a) f is a fuzzy strongly semi-pre-closed mapping;

(b) $f(intA) \leq sspintf(A)$, for each fuzzy set A of X;

(c) $intf^{-1}(B) \leq f^{-1}(sspintB)$, for each fuzzy set B of Y;

(d) $f^{-1}(sspclB) \leq cl(f^{-1}(B))$, for each fuzzy set B of Y;

(e) There is a base α for τ_1 such that $f(A)$ is a fuzzy strongly semi-pre-closed set of Y for each $A^c \in \alpha$.

Theorem 5.4 : Let $f : X \to Y$ be a mapping from a fts X to a fts Y. The mapping f is a fuzzy strongly semi-pre-open if and only if $f(intA) \leq int(pcl(f(A)) \vee pcl(int(f(A))$, for each fuzzy set A of X.
Proof : Let f be a fuzzy strongly semi-pre-open mapping and let A be any fuzzy set of X. Since $intA$ is an opened fuzzy set then $f(intA)$ is a fuzzy strongly semi-pre-open set of Y, that is:

$$f(intA) \leq int(pcl(f(intA))) \vee pcl(int(f(intA))) \leq int(pcl(f(A)) \vee pcl(int(f(A))$$

Conversely, if A is a fuzzy open set of X, then $A = intA$ and

$$f(A) = f(intA) \leq int(pcl(f(intA))) \vee pcl(int(f(intA)))$$

which means that $f(A)$ is always a $FSSPO$ set, in other words the mapping f is a fuzzy strongly semi-pre-open. □
Theorem 5.5 : Let $f : X \to Y$ be a mapping from a fts X to a fts Y. The mapping f is a fuzzy strongly semi-pre-closed if and only if $cl(pintf(A)) \wedge pint(clf(A)) \leq f(clA)$, for each fuzzy set A of X.
Proof : Similarly to the Theorem 5.4 and also by using the Theorem 3.5. *Box*
Theorem 5.6 : Let $f : X \to Y$ be a mapping from a fts X to a fts Y.

(a) If $f(int(pclA) \vee pcl(intA)) \leq int(pclf(A)) \vee pcl(intf(A))$, for each fuzzy set A of X, then f is a fuzzy strongly semi-pre-open mapping.

(b) If $f(cl(pintA) \wedge pint(clA)) \geq cl(pintf(A)) \wedge pint(clf(A))$, for each fuzzy set A of X, then f is a fuzzy strongly semi-pre-closed mapping.

Proof : (a) If A is a fuzzy open set of X, then $A \leq int(pclA) \vee pcl(intA)$ and subsequently

$$f(A) \leq f(int(pclA) \vee pcl(intA))$$

Now, according to the assumption, we have:

$$f(A) \leq f(int(pclA) \vee pcl(intA)) \leq int(pclf(A)) \vee pcl(intf(A))$$

Which means that $f(A)$ is a fuzzy strongly semi-pre-open set and accordingly f is a fuzzy strongly semi-pre-open mapping.
(b) If A is a fuzzy closed set of X, then $A \geq (cl(pintA) \wedge pint(clA))$ and subsequently $f(A) \geq f((cl(pintA) \wedge pint(clA)))$. Now, according to the assumption, we have:

$$f(A) \geq f((cl(pintA) \wedge pint(clA))) \geq (cl(pintf(A)) \wedge pint(clf(A)))$$

Which according to Theorem **??** means that $f(A)$ is a fuzzy strongly semi-pre-closed set and accordingly f is a fuzzy strongly semi-pre-closed mapping. $\square$

Theorem 5.7 : Let $f : X \to Y$ be a mapping from a fts X to a fts Y. The mapping f is a fuzzy strongly semi-pre-open mapping if and only if for each fuzzy set B of Y and each fuzzy closed set A of X such that $f^{-1}(B) \leq A$ there exists a fuzzy strongly semi-pre-closed set C of Y such that $B \leq C$ and $f^{-1}(C) \leq A$.

Proof : Let B be any fuzzy set of a fts Y and let A be a fuzzy closed set of a fts X such that $f^{-1}(B) \leq A$. Then $(f^{-1}(B))^c \geq A^c$ and $(f^{-1}(B^c)) \geq A^c$ or $(f(A^c)) \leq f(f^{-1}(B^c)) \leq B^c$.

Since the mapping f is a fuzzy strongly semi-pre-open, then $f(A^c)$ is a $FSSPO$ set which means that $f(A^c) \leq sspintB^c$. From the last we will get:

$$f^{-1}(f(A^c)) \leq f^{-1}(sspintB^c) \implies A^c \leq f^{-1}(f(A^c)) \leq f^{-1}(sspintB^c)$$

and hence

$$A \geq f^{-1}((sspintB^c)^c) = f^{-1}(ssplB). \text{ If we take } C = ssplB$$

then the following conditions are met, C is a $FSSPC$ set of Y, $B \leq C$ and $f^{-1}(C) \leq A$. Conversely, let us suppose that V is a fuzzy open set of X. We have to show that $f(V)$ is a $FSSPO$ set.

If we start from the fact that $f^{-1}(f(V)) \geq V \implies f^{-1}(f(V)^c) \leq V^c$ and then if we substitute $f(V)^c = B$, a fuzzy set of Y, and $V^c = A$, a fuzzy closed set of X, then from the assumption of the theorem, there is a $FSSPC$ set C of Y such that $B = f(V)^c \leq C$ and $f^{-1}(C) \leq A = V^c$.

From $B = f(V)^c \leq C$ we conclude that $sspcl(f(V)^c) \leq sspclC = C$ and subsequently

$$C^c \leq (sspcl(f(V)^c))^c \implies C^c \leq sspint(f(V))$$

From $f^{-1}(C) \leq A = V^c$ we obtain

$$\begin{gathered}(f^{-1}(C))^c \geq (V^c)^c \implies f^{-1}(C^c) \geq V \text{ and} \\ f(V) \leq f(f^{-1}(C^c)) \leq C^c \leq sspintf(V)\end{gathered}$$

It is also obvious that $sspinf(V) \leq f(V)$, so from the last two expressions we will get that $f(V) = sspintf(V)$, which means that $f(V)$ is a $FSSPO$ set, in other words the mapping f is a fuzzy strongly semi-pre-open mapping. $\square$

Corollary 5.7.1 : Let $f : X \to Y$ be a fuzzy strongly semi-pre-open mapping from a fts X to a fts Y, then:

(1) $f^{-1}(cl(pintB)) \leq cl(f^{-1}(B))$, for each fuzzy set B of Y;

(2) $f^{-1}(cl(B)) \leq cl(f^{-1}(B))$, for each fuzzy set B of Y.

Proof : (1) Let B be a fuzzy set of Y, then $cl(f^{-1}(B))$ is a fuzzy closed set of X which contains $f^{-1}(B)$. According to Theorem **??**, there exists a $FSSPC$ set C of Y such that $B \leq C$ and $f^{-1}(C) \leq cl(f^{-1}(B))$.
Thus

$$f^{-1}(cl(pintB)) \leq f^{-1}(cl(pintC)) \leq f^{-1}(C) \leq cl(f^{-1}(B))$$

(2) In a similar manner, let B be a fuzzy set of Y, then $cl(f^{-1}(B))$ is a fuzzy closed set of X which contains $f^{-1}(B)$. According to Theorem **??**, there exists a $FSSPC$ set C of Y such that $B \leq C$ and $f^{-1}(C) \leq cl(f^{-1}(B))$. Thus

$$f^{-1}(clB) \leq f^{-1}(clC) = f^{-1}(C) \leq cl(f^{-1}(B))$$

□

Theorem 5.8 : Let $f : X \to Y$ be a mapping from a fts X to a fts Y. The mapping f is a fuzzy strongly semi-pre-closed mapping if and only if for each fuzzy set B of Y and each fuzzy open set A of X such that when $f^{-1}(B) \leq A$ there exists a fuzzy strongly semi-pre-open set C of Y such $B \leq C$ and $f^{-1}(C) \leq A$.

Proof : It is similar to the proof of the Theorem 5.7. □

Theorem 5.9 : Let $f : X \to Y$ and $g : Y \to Z$ be mappings and let X, Y, Z be fuzzy topological spaces. If a mapping g is a fuzzy strongly semi-pre-open (semi-pre-closed) and f is a fuzzy open (closed) mapping, then the product gf is fuzzy strongly semi-pre-open (semi-pre-closed) mapping.

Proof : If a set V is a fuzzy open (closed) set of X, then according to the conditions of the theorem, $f(V)$ is also a fuzzy opened (closed) set of Y, and then $g(f(V))$ is a $FSSPO$ ($FSSPC$) set of Z.

6. Conclusion

In this work we have introduced a concept of fuzzy strongly semi-pre-open sets and fuzzy strongly semi-pre-continuous mappings. We have shown that this class is an independent class and we have investigated their properties and their connections with other forms of fuzzy sets and fuzzy continuous mappings.

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