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FUNNELS IN BCL^+ ALGEBRAS

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Abstract

The two concepts of filtrations and deductive systems were introduced in BCL^+ algebras by the author. In this article, we introduce the concept of funnels, is fairly fundamental concept in BCL^+ algebras. We show that the new version of the definition is equivalent to the filtration and it more general case. We studies unions and intersections of the funnels. We discuss the funnels coming into the deductive systems of BCL^+ algebras and how to generate funnels by subalgebra. An interesting outcome from our result is that we shall prove that for BCL^+ algebras there exists a filtration H such that H is funnel to a deductive system.

1. Introduction

In 2011 the author [1] first studied that the BCL-algebras. Later on, more papers have appeared in which BCL-algebras and their related concepts were developed by D. Al-Kadi and R. Hosny [2], and soft BCL-algebras were treated by D. Al-Kadi [3]. The author [4] in 2012 was introduced BCL^+ algebras. For the general development of BCL^+ algebras, the author [5-8] gives a characterization of a partial order and a

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topology in BCL^+ algebras; discuss some distributions of BCL^+ algebras; introduce filtrations and deductive systems in BCL^+ algebras.

In this paper, we will introduce a new concept called funnels in BCL^+ algebras. We discuss their interesting properties and relationships relevant to subalgebra, deductive systems and filtrations.

2. Preliminary (1)

In this section, let's first review relevant concept, as follows.

Definition 1 ([4]) : A BCL^+ algebra is a triple $(Y;*,1)$, where Y is a nonempty set, $'*'$ is a binary operation on Y and $1 \in Y$ is an element such that the following three axioms hold for any $x, y, z \in Y$:

$(BCL^+\ 1)$ $\quad x * x = 1$.

$(BCL^+\ 2)$ $\quad x * y = 1$ and $y * x = 1$ imply $x = y$.

$(BCL^+\ 3)$ $\quad ((x * y) * z) * ((x * z) * y) = (z * y) * x$.

Theorem 2 ([4]) : Assume that $(Y;*,1)$ is a BCL^+ algebra. Then the following hold for any $x, y, z \in Y$.

(i) $(x * (x * y)) * y = 1$.

(ii) $x * 1 = x$ implies $x = 1$.

(iii) $((x * y) * (x * z)) * (z * y) = 1$.

(iv) $(BCL^+\ 2)$ $\quad x * y = 1$ and $y * x = 1$ imply $x = y$.

3. Preliminary (2)

The concept of filtration has been intensively studied in [8] for BCL^+ algebras. It has become an important tool in BCL^+ algebras, as follows.

Definition 3 ([8]) : If D be a nonempty subset of a BCL^+ algebra $(Y;*,1)$. Then we say that D be a deductive system if

(D1) $1 \in D$.

(D2) $x \in D$ and $x * y \in D$ imply $y \in D$.

Lemma 4 ([8]) : Let D be a deductive system of a BCL^+ algebra $(Y;*,1)$ and suppose $a \leq x$ whenever $a \in D$. Then $x \in D$.
Lemma 5 ([8]) : Let Y be commutative and let for all $x, y, z \in Y$. Then the following equalities are satisfied:

(Y1) $x * (y * z) = y * (x * z)$.

(Y2) $(x * y) * z = (x * z) * y$.

Theorem 6 ([8]) : Suppose that B is subalgebra of Y. Then B is a filtration and $x \in B$ if and only if $1 * x \in B$.
Theorem 7 ([8]) : Let H be a nonempty subset of a BCL^+ algebra $(Y;*,1)$. Then H is a filtration if and only if it is a deductive system.
Theorem 8 ([8]) : If $(Y,*,1)$ is a BCL^+ algebra, then $Y = \{1\}$.

4. Main Results in the Case of Funnels

Let $A, B \subseteq Y$ be any two subsets of BCL^+ algebras $(Y;*,1)$. We can write

$$A * B = \{x * y | x \in A, y \in B\}.$$

Definition 9 : Let H be a nonempty subset of a BCL^+ algebra $(Y;*,1)$and satisfies $Y * H = H$. Then H is a funnel of BCL^+ algebras. (Of course, the Y and$\{1\}$ itself are funnels of BCL^+ algebras.)
Funnels that are BCL^+ algebras are given in the next basic example.
Example 10 : Let $Y = \{1, a, b, c\}$ and the operation $*$ on Y can be represented by

Table 1 : BCL^+ operation

$*$	1	a	b	c
1	1	1	1	1
a	a	1	1	c
b	b	1	1	c
c	c	1	c	1

By Definition 1 (BCL^+ 3).

(i) The left side of the equation is

$$((a * b) * c) * ((a * c) * b) = (1 * c) * (c * b) = 1 * c = 1.$$

(ii) The right side of the equation is

$$(c * b) * a = c * a = 1.$$

Then $(Y; *, 1)$ is a BCL^+ algebra, and so $H = \{1, c\}$ is a subset of Y, it is also a funnel.

Theorem 11 : If $(Y; *, 1)$ is a BCL^+ algebra, then Y is a funnel.

Proof : The proof is simple. Let $1 = H$. By Theorem 8. Since $1 = Y * Y = H$, by Definition 9, we have $Y = H$. □

Theorem 12 : Let H be a funnel of BCL^+ algebras. Then there exists $1 \subseteq H$.

Proof : Let H be a funnel of BCL^+ algebras. Then $0 \neq H = Y * H$. By Definition 9, if $a \subseteq H$, then

$$1 = a * a \subseteq Y * H = H$$

and so $1 \subseteq H$. □

Theorem 13 : Let H be a funnel of BCL^+ algebras $(Y; *, 1)$. Then H is a subalgebra of BCL^+ algebras $(Y; *, 1)$.

Proof : Let $a, b \in H$. Then by Definition 9, we have

$$a * b \in H * H \subseteq Y * H = H \subseteq Y,$$

therefore, this proves the funnel H is a subalgebra of BCL^+ algebras.

Remark 14 : As we shall see, the converse of Theorem 13 is not true. For example, $H = \{1, a\}$ is a subset of Y, but it is not a funnel of BCL^+ algebras $(Y; *, 1)$ in Example 10.

Next, we discuss the unions and intersections of funnels in BCL^+ algebras.

Theorem 15 : Let A and B be two funnels of a BCL^+ algebra $(Y; *, 1)$. Then $A \cap B$ and $A \cup B$ are funnels of BCL^+ algebras $(Y; *, 1)$.

Proof : For $\Psi \in Y$, we see that if $\Psi = A \cap B$, then by Definition 9, we have

$$\Psi \subseteq Y * \Psi = Y * (A \cap B) \subseteq (Y * A) \cap (Y * B) = A \cap B = \Psi$$

and so $Y * \Psi = \Psi$. Thus $A \cap B$ is a funnel of BCL^+ algebras $(Y; *, 1)$.

Obviously, we also use the same method for $A \cup B$. □

Remark 16 : Let M be funnel of BCL^+ algebras $(Y; *, 1)$ and suppose N is a subalgebra of BCL^+ algebras $(Y; *, 1)$. Then $M \cup N$ not necessarily the funnel of BCL^+ algebras, but for $M \cap N$, we prove the following theorem.

Theorem 17 : Let M be funnel of BCL^+ algebras $(Y;*,1)$, and suppose N is a subalgebra of BCL^+ algebras $(Y;*,1)$. Then $M \cap N$ is a funnel of N.

Proof : Write

$$N*(M\cap N) \subseteq (N*M)\cap(N*N) \subseteq (Y*M)\cap(N*N).$$

By Theorem 15, let $N = Y$, then

$$(Y*M)\cap(N*N) = M\cap N \subseteq N*(M\cap N),$$

and so $N*(M\cap N) = M\cap N$. □

The following corollary is crucial for what want to do.

Corollary 18 : Let M be funnel of BCL^+ algebras $(Y;*,1)$ and assume that M is a subalgebra of BCL^+ algebras $(Y;*,1)$. If $E \subseteq M$, then E is a funnel of Y iff E is a funnel of M.

Corollary 19 : Let M be funnel of BCL^+ algebras $(Y;*,1)$ and suppose N is a subalgebra of BCL^+ algebras $(Y;*,1)$. If $M\cap N = \phi$, then there exists a deductive system D and $M \subseteq D$ such that $D \cap N = \phi$.

Theorem 20 : Let $\Im = (Y;*,1)$ and $\Re = (X;*,1)$ be two BCL^+ algebras, define $\theta : \Im \to \Re$ is a morphism. Let θ be onto and let M be funnel of $\Im$. Then $\theta(M)$ is a funnel of $\Re$.

Proof : We may assume that θ be onto and let M be funnel of $\Im$. By Corollary 18, we have

$$\theta(M) \subseteq X*\theta(M).$$

Now let $a \in \theta(M)$ and $x \in X$ for some $b \in M$ and $y \in Y$, then $a = \theta(b)$ and $x = \theta(y)$, we see that

$$x*a = \theta(y)*\theta(b) = \theta(y*b) \in \theta(Y*M) = \theta(M),$$

and so $X*\theta(M) \subseteq \theta(M)$, we have $\theta(M)$ is a funnel of $\Re$. □

Our next goal is somewhat more ambitions.

Theorem 21 : Let H be a filtration of BCL^+ algebras $(Y;*,1)$ and let H be a deductive system. Then H is a funnel of Y.

Proof : We may assume that H be a deductive system of BCL^+ algebras $(Y;*,1)$.

By Definition 3 we have $1 \in H$ and H be a nonempty subset of a BCL^+ algebra $(Y;*,1)$. Suppose that $a \in H$ and $b \in Y$, hence $a \leq b*a$, then $b*a \in H$ and so $Y*H \in H$. We thus have

$$H = \{1\} * H \subseteq Y * H$$

and therefore $Y*H = H$. We deduce H be a deductive system and H is a funnel. Therefore H is a filtration by Theorem 7. □

Theorem 22 : Let H be a deductive system of BCL^+ algebras $(Y;*,1)$. Then for every nonempty subset K of Y and $K*H$ is a funnel of BCL^+ algebras $(Y;*,1)$.

Proof : By Theorem 21 and by Lemma 5 (Y1), we have

$$Y*(K*H) = K*(Y*H) = K*H,$$

and $K*H$ is a funnel of Y. □

Theorem 23 : Let M be a subalgebra of BCL^+ algebras $(Y;*,1)$. Then M is a funnel iff $a \in M$ and $1*a \in M$.

Proof : Let M be a subalgebra of BCL^+ algebras $(Y;*,1)$ and let M is a funnel of BCL^+ algebras $(Y;*,1)$, then $a \in M$ and $1 \in M$ and so $1*a \in M$.

Conversely, let $a, b \in M$, by Lemma 5 (Y2), we have

$$(a*b)*a = (a*a)*b = 1*b \in M.$$

Since $a \in M$ and M be a funnel, thus $a*b \in M$. By Theorem 12 and Theorem 13, we have

$$a*b \in M \subseteq Y*M,$$

it is easy to see $Y*M = M$, and so M be a subalgebra of BCL^+ algebras $(Y;*,1)$, it is also a funnel. □

5. Conclusion

Relation between filtrations and deductive systems is very interesting in BCL^+ algebras. In this paper, we introduce the concepts of funnels for BCL^+ algebras in Definition 9. We show some useful properties of these funnels that give various methods how to get funnels from them. Although the funnel is a basic concept in BCL^+ algebras, the idea in this paper, but can be applied to other algebraic logic, like BL-algebras in [9], Heyting algebras in [10], Hilbert algebras in [11], etc.

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