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ON ARITHMETIC GRAPHS

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Abstract

Let G be a graph with q edges and let k and d be positive integers. A labeling f of G is said to be (k, d)-arithmetic if the vertex labels are distinct non-negative integers and the edge labels induced by $f(x) + f(y)$ for each xy are $k, k+d, k+2d, \cdots, k+(q-1)d$. A graph is called arithmetic if it is (k, d)-arithmetic for some k and d. In this paper we prove that H_n-graph, generalized H_n graph, $H_n \odot S_m, P_m(P_n)$ are arithmetic graphs.

1. Introduction

For all terminology and notations in Graph Theory we follow Harary [2]. Unless mentioned or otherwise a graph in this paper shall mean a simple finite graph without isolated vertices.

Let G be (p, q) graph. Let $V(G), E(G)$ denote respectively the vertex set and edge set of G. Consider an injective function $g : V(G) \to X$, where $X = \{0, 1, 2, \cdots, q\}$ if G is a tree and $X = \{0, 1, 2, \cdots, q-1\}$ otherwise. Define the function $g^* : E(G) \to N$, the set of all natural numbers such that $g^*(uv) = g(u) + g(v)$ for all edge (u, v). If $g^*(E(G))$ is

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a sequence of distinct consecutive integers, say $\{k, k+1, \cdots, k+q-1\}$ for some k, then the function g is said to be sequential labeling and the graph which admits such a labeling is called sequential graph [5].

In 1989 Acharya and Hegde [1] introduced a new version of sequential graph known as arithmetic graph and is defined as follows: Let G be a graph with q edges and let k and d be positive integers. A labeling f of G is said to be (k, d)-arithmetic if the vertex labels are distinct nonnegative integers and the edge labels induced by $f(x)+f(y)$ for each xy are $k, k+d, k+2d, \cdots, k+(q-1)d$. A graph is called arithmetic if it is (k, d)-arithmetic for some k and d.

Definition 1.2 : Let H_n-graph of a path P_n is the graph obtained from two copies of P_n with vertices $v_1, v_2, \cdots, v_n$ and $u_1, u_2, \cdots, u_n$ by joining the vertices $v_{\frac{n+1}{2}}$ and $u_{\frac{n+1}{2}}$ by means of an edge if n is odd and the vertices $v_{\frac{n}{2}+1}$ and $u_{\frac{n}{2}}$ if n is even.

Theorem 1.3 : H_n-graph admits a $(2t+1, 2)$, $t \geq 0$ arithmetic labeling.

Proof : H_n-graph is obtained from two paths $u_1u_2u_3\cdots u_n$ and $v_1v_2v_3\cdots v_n$ of equal length by joining an edge $u_{\frac{n+1}{2}}v_{\frac{n+1}{2}}$ when n is odd or $u_{\frac{n}{2}1}v_{\frac{n}{2}}$ when n is even.

Note that H_n has $2n$ vertices and $2n-1$ edges.

Let t be an integer such that $t \geq 0$.

Define a function $f : V(H_n) \to N$ (set of non-negative integers) as follows

$$f(u_i) = i + t - 1, \quad 1 \leq i \leq n$$

$$f(v_i) = n + i + t - 1, \quad 1 \leq i \leq n.$$

Then the edges get labels

$$\begin{aligned}
f\left(u_{\frac{n+1}{2}}v_{\frac{n+1}{2}}\right) &= 2t + 2n - 1 \quad \text{when } n \text{ is odd.} \\
f\left(u_{\frac{n}{2}+1}v_{\frac{n}{2}}\right) &= 2t + 2n - 1 \quad \text{when } n \text{ is even.} \\
f(u_iu_{i+1}) &= 2(t+i) - 1,‘ \quad 1 \leq i \leq n-1 \\
f(v_iv_{i+1}) &= 2n + 2(t+i) - 1, \quad 1 \leq i \leq n-1.
\end{aligned}$$

Thus the value of the edge $u_{\frac{n+1}{2}}v_{\frac{n+1}{2}}$ (when n is odd) or $u_{\frac{n}{2}+1}v_{\frac{n}{2}}$ (when n is even) is $2t+2n-1$.

The set of labels of the edges $u_iu_{i+1}, 1 \leq i \leq n-1 = \{2t+1, 2t+3, 2t+5, \cdots, 2t+2n-3\}$.

The set of labels of the edges $v_iv_{i+1}, 1 \leq i \leq n-1 = \{2t+2n+1, 2t+2n+3, 2t+2n+5, \cdots, 2t+4n-3\}$.

Also edge values are distinct and the values of the edge set of H_n are

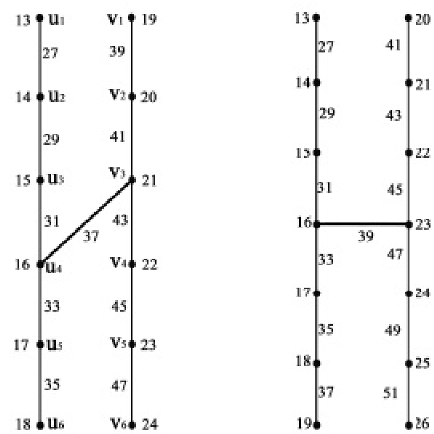
$$\{2t+1, 2t+3, \cdots, , 2t+2n-1, 2t+2n+1, 2t+2n+3, \cdots, 2t+4n-3\}. \quad (1)$$

Take $k = 2t+1, d = 2$. Now

$k+(q-1)d = 2t+1+(2n-1-1)2 = 2t+1+(2n-2)2 = 2t+1+4n-4 = 2t+4n-3.$

Therefore (1) is of the form $\{k, k+d, k+2d, ., k+(q-1)d\}$ where $k = 2t+1$ and $d = 2$. Thus f is an arithmetic labeling. Hence H_n is a $(2t+1, 2)$-arithmetic graph.

Illustration 1.4 : (27, 2) Where $t = 13$. Arithmetic labeling of H_6 and H_7 are shown below.



Construction 1.5 : Let P_n be a path on n vertices. Take k copies of P_n. Let the vertices of k copies of P_n be $v_{i,j}, 1 \leq i \leq k, 1 \leq j \leq n$. Join the vertices $v_{i,\frac{n}{2}+1}$ and $v_{i+1,\frac{n}{2}}, 1 \leq i \leq k-1$ if n is even or $v_{i,\frac{n+1}{2}}$ and $v_{i+1,\frac{n+1}{2}}, 1 \leq i \leq k-1$ if n is odd. The resultant graph said to be generalized H_n graph and is denoted by kH_n.

Theorem 1.6 : The graph kH_n is a $(2t+1, 2), t \geq 0$ arithmetic graph.

Proof : Let the vertices of k copies of P_n be $v_{i,j}, 1 \leq i \leq k$ and $1 \leq j \leq n$. The edge set of kH_n is $E(kH_n) = \{v_{i,j}v_{i,j+1}/1 \leq i \leq k$ and $1 \leq j \leq n-1\} \cup \{v_{i,\frac{n}{2}+1}v_{i+1,\frac{n}{2}}, (1 \leq i \leq k-1$ if n is even) or $v_{i,\frac{n+1}{2}}v_{i+1,\frac{n+1}{2}}(1 \leq i \leq k-1$ if n is odd)$\}$.

Note that kH_n has kn vertices and $kn-1$ edges.

Define a function $f : V(kH_n) \to N$ as follows.

$$f(v_{i,j}) = (i-1)n + j + t - 1 \quad t \geq 0, \quad 1 \leq i \leq k, \quad 1 \leq j \leq n.$$

The vertex labels are distinct. Then the edges get labels

$$\begin{aligned} f\left(v_{i,\frac{n}{2}+1}v_{i+1,\frac{n}{2}}\right) &= 2t+2in-1 \quad 1\le i\le k-1 \text{ if } n \text{ is even} \\ f\left(v_{i,\frac{n+1}{2}}v_{i+1,\frac{n+1}{2}}\right) &= 2t+2in-1 \quad 1\le i\le k-1 \text{ if } n \text{ is odd} \\ f(v_{i,j}v_{i,j+1}) &= 2n(i-1)+2j+2t-1 \quad 1\le i\le k \text{ and } 1\le j\le n-1. \end{aligned}$$

Thus the labels of the edges $v_{i,\frac{n}{2}+1}v_{i+1,\frac{n}{2}}$ $(1\le i\le k-1$ if n is even) or $v_{i,\frac{n+1}{2}}v_{i+1,\frac{n+1}{2}}$ $(1\le i\le k-1$ if n is odd) os

$$2t+2n-1, 2t+4n-1, 2t+6n-1, \cdots, 2t+2(k-1)n-1.$$

The values of edges in the k copies of P_n are given below.

$$\begin{aligned} &2t+1, 2t+3, 2t+5, \cdots, 2t+2n-3 \\ &2t+2n+1, 2t+2n+3, 2t+2n+5, \cdots, 2t+4n-3. \\ &2t+4n+1, 2t+4n+3, 2t+4n+5, \cdots, 2t+6n-3 \\ &\vdots \\ &2t+2(k-2)n+1, 2t+2(k-2)n+3, 2t+2(k-2)n+5, \cdots, 2t+2(k-1)n-3 \\ &2t+2(k-1)n+1, 2t+2(k-1)n+3, 2t+2(k-1)n+5, \cdots, 2t+2kn-3. \end{aligned}$$

Hence the values of the edge set of kH_n is $2t+1, 2t+3, 2t+5, 2t+7, \cdots, 2t+2n-1, 2t+2n+1, \cdots, 2t+4n+1, \cdots, 2t+2kn-3$. (1)

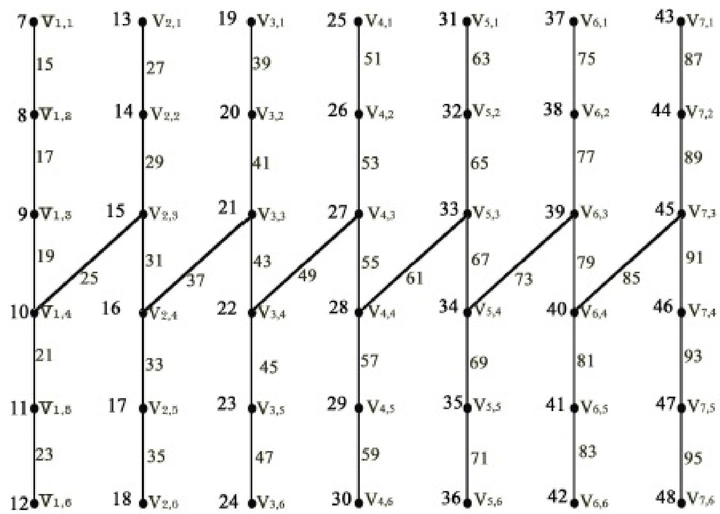
Take $k = 2t+1$ and $d = 2$.

Now $k+(q-1)d = 2t+1+(kn-1-1)2 = 2t+1+2kn-4 = 2t+2kn-3$.

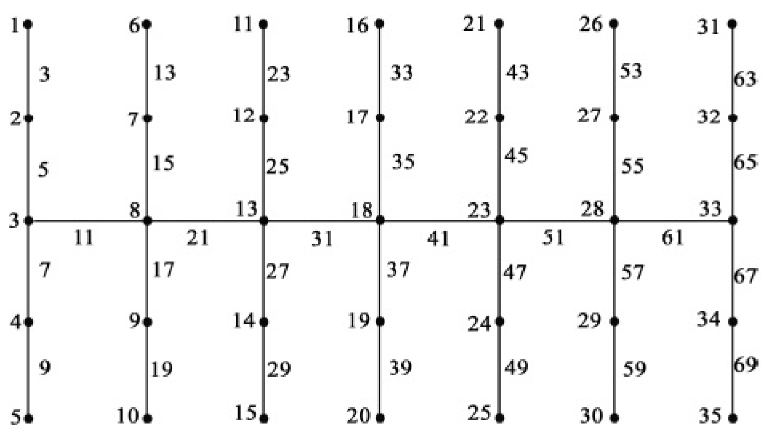
Therefore (1) is of the form $k, k+d, k+2d, \cdots, k+(q-1)d$ where $k = 2t+1$ and $d = 2$.

Thus f is an arithmetic labeling. Hence kH_n is a $(2t+1, 2)$-arithmetic graph.

Illustration 1.7 : (15,2) Arithmetic labeling of $7H_6$ is given below.



(3,2) Arithmetic labeling of $7H_5$ is given below.



Definition 1.8 : The graph $H_n \odot S_m$ is obtained from H_n by identifying the centre vertex of the star S_m at each vertex of H_n.

Theorem 1.9 : The graph $H_n \odot S_m$ is an arithmetic graph.

Proof : Let $G = H_n \odot S_m$.

The vertex set of G is

$$V(G) = \{u_i/1 \leq i \leq n, v_i/1 \leq i \leq n\} \cup \{u_{ij}, v_{ij}/1 \leq i \leq n, 1 \leq j \leq m\}.$$

The edge set of G is $V(G) = \{u_i u_{i+1}, v_i v_{i+1}/1 \leq i \leq n-1\} \cup \{u_{\frac{n}{2}+1} v_{\frac{n}{2}}$ (if n is even) or $u_{\frac{n+1}{2}} v_{\frac{n+1}{2}}$ (if n is odd)$\}$ $\cup\{u_i u_{ij}, v_i v_{ij}/1 \leq i \leq n, 1 \leq j \leq m\}$.
Note that G has $2n(m+1)$ vertices and $2n(m+1)-1$ edges.
Let t be an integer such that $t \geq 0$.
Define $f : V(G) \to N$ as follows

$$\begin{aligned}
f(u_i) &= i+t-1, && 1 \leq i \leq n \\
f(v_i) &= n+i+t-1, && 1 \leq i \leq n \\
f(u_{11}) &= (2m+3)n+t+4 && \\
f(u_{i1}) &= (2m+3)n+(i-1)+t+4, && 2 \leq i \leq n \\
f(u_{ij}) &= f(u_{i1})+(j-1)2n+t-1, && 1 \leq i \leq n, 2 \leq j \leq m \\
f(v_{11}) &= 4n+t+2 && \\
f(v_{i1}) &= f(v_{i-1,1})+t-4, && 2 \leq i \leq n \\
f(v_{ij}) &= f(v_{i1})+(j-1)2n+t-1, && 1 \leq i \leq n, 2 \leq j \leq m.
\end{aligned}$$

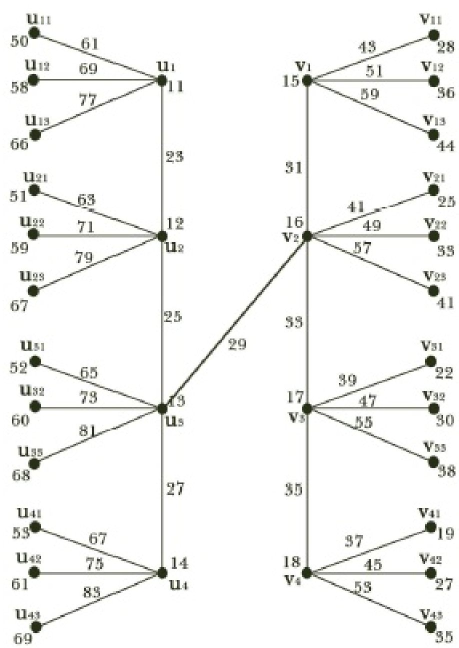
The vertex labels are distinct. Then the edges get labels

$$\begin{aligned}
f(u_i u_{i+1}) &= 2i+2t-1, 1 \leq i \leq n-1 \\
f(v_i, v_{i+1}) &= 2n+2(i+t)-1, \quad 1 \leq i \leq n-1 \\
f(v_1 v_{11}) &= 5n+2t+2 \\
f(v_i v_{i1}) &= 2t+n+i+f(v_{i-1,1})-3, \quad 2 \leq i \leq n. \\
f(v_i v_{ij}) &= 2t+n(2j-1)+f(v_{i1})-2, \ 1 \leq i \leq n, \quad 2 \leq j \leq m. \\
f(u_1 u_{11}) &= 2t+(2m+3)n+4 \\
f(u_i u_{i1}) &= 2t+2(i-1)+(2m+3)n+4, \quad 2 \leq i \leq n. \\
f(u_i u_{ij}) &= 2t+i+2n(j-1)+f(u_{i1})-2, \quad 1 \leq i \leq n, 2 \leq j \leq m.
\end{aligned}$$

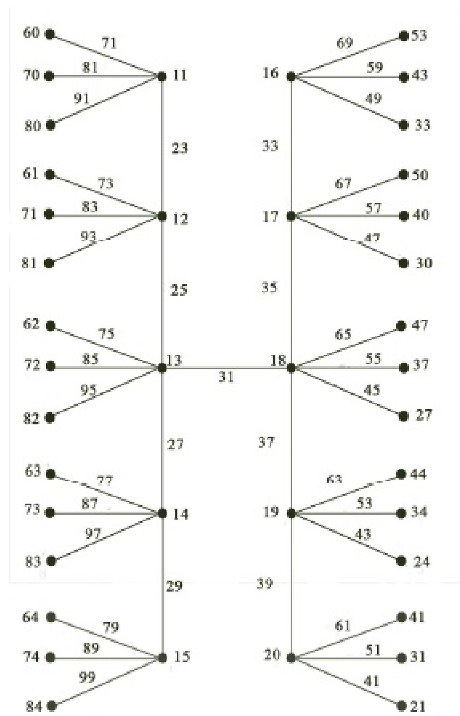
Then the values of the edges $u_i u_{i+1}, 1 \leq i \leq n-1$ are $2t+1, 2t+3, \cdots, 2t+2n-5, 2t+2n-3$.
The values of the edges $v_i v_{i+1}, 1 \leq i \leq n-1$ are $2t+2n+1, 2t+2n+3, 2t+2n+5, \cdots, 2t+4n-5, 2t+4n-3$.

The value of the edge $u_{\frac{n}{2}+1}v_{\frac{n}{2}}$ (if n is even) or $u_{\frac{n+1}{2}}v_{\frac{n+1}{2}}$ (if n is odd) is $2t+2n-1$.
The values of the edges $v_iv_{ij}, 1 \le i \le n, 1 \le j \le m$ are $2t+4n-1, 2t+4n+1, 2t+4n+3, 2t+4n+5, \cdots, 2t+(2m+3)n+2$.
The values of the edges $u_iu_{ij}, 1 \le i \le n, 1 \le j \le m$ are $2t+(2m+3)n+4, 2t+(2m+3)n+6, 2t+(2m+3)n+8, \cdots, 2t+4nm+4n-3$.
Therefore the set of values of the edge set of G is $\{2t+1, 2t+3, 2t+5, 2t+7, \cdots, 2t+2n-1, 2t+2n+1, 2t+2n+3, \cdots, 2t+4n+1, \cdots, 2t+(2m+3)n+4, \cdots, 2t+4nm+4n-3\}$ (1)
Take $k = 2t+1,\ d = 2$.
Now $k+(q-1)d = 2t+1+[2n(m+1)-2]2 = 2t+1+4nm+4n-4 = 2t+4nm+4n-3$.
Hence (1) is of the form $\{k, k+d, k+2d, \cdots, k+(q-1)d\}$ where $k = 2t+1,\ d = 2$.
Thus f is an arithmetic labeling. Therefore $H_n \odot S_m$ is a $(2t+1, 2), t \ge 0$ arithmetic graph.

Illustration 1.10 : (23,1) Arithmetic labeling of $H_4 \odot S_3$.



(23,1) **Arithmetic labeling of** $H_5 \odot S_3$



Construction 1.11 : Let P_m and P_n be two paths. Let $u_i, 1 \leq i \leq m$ be the vertices of P_m. Take m copies of P_n. Let the vertices of m copies of P_n be v_{ij}, $1 \leq i \leq m, 1 \leq j \leq n$. Adjoining the end vertex v_{i1} of P_n to each w_i for $i = 1, 2, \cdots, m$. The resultant graph is denoted by $P_m(P_n)$. Number of vertices of $P_m(P_n)$ is mn and the number of edges is $mn - 1$.

Theorem 1.12 : The graph $P_m(P_n)$ is a $(2t + 1, 2), t \geq 0$ arithmetic graph.

Proof : Let $u_1, u_2, \cdots, u_m$ be the vertices of P_m and vertices of m copies of P_n are $v_{ij}, 1 \leq i \leq m, 1 \leq j \leq n$, where v_{i1}, $(1 \leq i \leq m)$ are identified with the corresponding u_i $(1 \leq i \leq m)$.

Note that $P_m(P_n)$ has mn vertices and $mn - 1$ edges.

Let t be an integer such that $t \geq 0$.

Define $f : V(P_m(P_n)) \to N$ as follows.

$$f(v_{ij}) = (i-1)n + j + t - 1, \quad 1 \leq i \leq m \text{ and even } i, 1 \leq j \leq n$$

$$f(v_{ij}) = ni - j + t, \quad 1 \leq i \leq m \text{ and odd } i, 1 \leq j \leq n.$$

The vertex labels are distinct. Then the edges get labels

$$\begin{aligned} f(v_{i1}v_{i+11}) &= 2t+2ni-1, && 1\le i\le m-1.\\ f(v_{ij}v_{ij+1}) &= 2t+2ni-2j-1, && 1\le i\le m \text{ and odd } i, 1\le j\le n-1.\\ f(v_{ij}v_{ij+1}) &= 2t+2n(i-1)+2j-1, && 1\le i\le m \text{ and even } i, 1\le j\le n-1.\end{aligned}$$

Thus the edge labels of the path P_m are $2t+2n-1, 2t+2(2n)-1, \cdots, 2t+(m-1)2n-1$. Labels of the edges in the m copies of P_n are given below

$$\begin{aligned} &2t+1, 2t+3, 2t+5, \cdots, 2t+2n-3.\\ &2t+2n+1, 2t+2n+3, 2t+2n+5, \cdots, 2t+2(2n)-3.\\ &2t+2(2n)+1, 2t+2(2n)+3, \cdots, 2t+3(2n)-3.\\ &\vdots\\ &2t+(m-2)2n+1, 2t+(m-2)2n+3, \cdots, 2t+(m-1)2n-3\\ &2t+(m-1)2n+1, 2t+(m-1)2n+3, \cdots, 2t+m2n-3\end{aligned}$$

Hence the values in the edge set of

$$P_m(P_n) = \{2t+1, 2t+3, 2t+5, \cdots, 2t+2n-3, 2t+2n-1, \cdots, 2t+2(2n)-3, \cdots, 2t+2mn-3\} \tag{1}$$

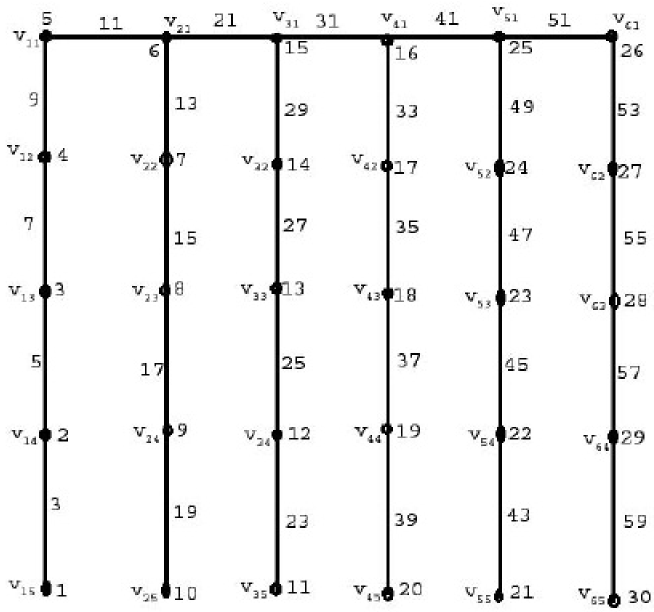
Here $k = 2t+1$ and $d = 2$.

Now $k+(q-1)d = 2t+1+(mn-1-1)2 = 2t+1+2mn-4 = 2t+2mn-3$.

Hence (1) is of the form $\{k, k+d, \cdots, k+(q-1)d\}$ where $k = 2t+1$ and $d = 2$.

Therefore f is an arithmetic labeling. Hence $P_m(P_n)$ is a $(2t+1, 2)$ $t \ge 0$ arithmetic graph.

Illustration 1.13 : (3,2)-Arithmetic labeling of $P_6(P_5)$.



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