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FINITE ELEMENT METHOD TO DETERMINE STRESS INTENSITY FACTOR USING J INTEGRAL APPROACH

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Abstract

Finite element methods (FEM) are widely employed in Damage Tolerance (DT) assessment of structural components. It has become very important concept in predicting strength and life of cracked structures. For assessing the strength of flawed structures, the evaluation of the stress intensity factor is essential. This paper describes the finite element procedure to determine the stress intensity factor using J integral approach. A comparative study is made on the stress intensity factor of cracked configurations utilizing a plane82 element of ANSYS. The computed values of strain energy release rate (G_I), J-integral (J_I) and stress intensity factor (K_I) for tensile cracked configurations are found to be in good agreement with those of handbook solutions. A plane 82 element is proved to be well suited to model curved boundaries. Finite element formulation and fracture analysis procedures are explained briefly.

1. Introduction

Fracture mechanics has developed into a useful discipline for predicting strength and

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life of cracked structures. The important aspect of linear elastic fracture mechanics is the “Damage Tolerance”' (DT) assessment. It is a procedure that defines whether a crack can be sustained safely during the service life of the structure. DT assessment is therefore required as a basis for any fracture control plan, generating the required information about the allowable size of the crack that would withstand maximum service loads without failure. In such DT assessment, the overall idea is to determine the stress intensity factor (K_1) for various cracked configurations using Finite Element Methods and compares such FEM results with the analytical solutions. Finite element methods are widely used to study the behavior of the cracks present in the structure and provide better solution to the complex situations wherever experiments become impossible/costlier.

The finite element method has become one of the most popular and general numerical methods of structural analysis. The method has the capability to deal with complex loading conditions, material behavior and practical geometries. The three main aspects of the FEM are (a) modeling the crack region, (b) calculating J integral and (c) calculating of stress intensity factor using J integral [1]. Stress intensity factors may correspond to three basic modes of fracture and it measures the degree of resistance to fracture, J integral, a path-independent line integral that measures the strength of the singular stresses and strains near a crack tip and the energy release rate represents the amount of work associated with a crack opening.

This paper addresses the FEM procedure to model the stress intensity factor for three cracked configurations viz., single edge crack, double edge crack and centre crack specimens and the results obtained from FEM are compared with the results of analytical methods.

2. Element Formulation and Fracture Analysis

Plane 82 is a higher order of the 2-D four node element The plane 82 element is well suited to model curved boundaries. The 8-node element is defined by eight nodes having two degrees of freedom at each node, translation in the modal x and y direction. The elements may be used as a plane element or as an axisymmetric element [1, 2]. In fracture mechanics, the quadratic elements Plane 82 are common in use. After meshing the crack tip, elements surrounding the crack tip are shown in the Figure- 1.0.

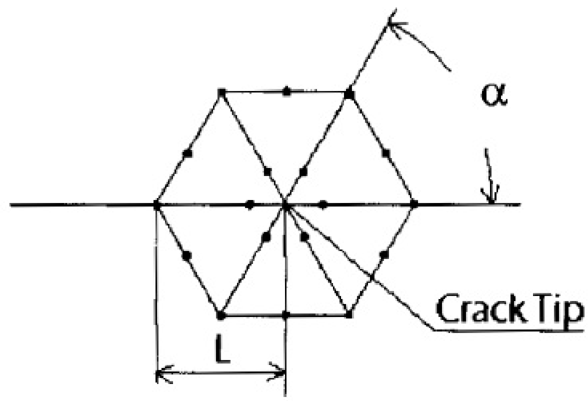


Figure 1.0. Elements surrounding the crack tip

The finite element formulation for plan 82 element is described below. The elements displacement function can be written in the form

$$\left\{\begin{array}{c} u \\ \\ v \end{array}\right\} = \left(\begin{array}{cccccccccc} N_1 & 0 & N_2 & 0 & N_3 & 0 & \cdots & N_8 & 0 \\ \\ 0 & N_1 & 0 & N_2 & & \cdots & 0 & N_8 & \end{array}\right) \left\{\begin{array}{c} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \\ u_n \\ v_n \end{array}\right\} \tag{1.1}$$

i.e. $\left\{\begin{array}{c} u \\ v \end{array}\right\} = [N]\{\delta\}_e$ Where $[N]$ is the element shape function matrix and $\{\delta\}_e$ is the element nodal displacement vector. The shape functions $N_1, N_2, \cdots$ and N_8 in element natural coordinates (ξ, η) are given by

$$\begin{aligned} N_1 &= -\tfrac{(1-\xi)(1-\eta)(1+\xi+\eta)}{4} \\ N_2 &= -\tfrac{(1-\xi^2)(1-\eta)(1-\xi+\eta)}{4} \\ N_3 &= -\tfrac{(1+\xi)(1+\eta)(1-\xi-\eta)}{4} \\ N_4 &= -\tfrac{(1-\xi)(1+\eta)(1+\xi-\eta)}{4} \\ N_5 &= \tfrac{(1-\xi^2)(1-\eta)}{2} \end{aligned}$$

$$\begin{aligned} N_6 &= \tfrac{(1+\xi)(1-\eta^2)}{2} \\ N_7 &= \tfrac{(1-\xi^2)(1+\eta)}{2} \\ N_8 &= \tfrac{(1-\xi)(1-\eta^2)}{2} \end{aligned} \tag{1.2}$$

The shape functions N_1 to N_8 are quadratic polynomials. The element displacements given by equation (1.1) have a quadratic variation and involve only the nodal displacements as unknown. The same set of shape functions are used to describe the element geometry as

$$\begin{aligned} x &= \sum_{i=1}^{8} N_i x_i \\ y &= \sum_{i=1}^{8} N_i y_i. \end{aligned} \tag{1.3}$$

The strains $\{\epsilon\}$ and stress $\{\sigma\}$ within an element are obtained as

$$\begin{aligned} \{\epsilon\} &= \left\{ \begin{array}{c} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{array} \right\} = \begin{pmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{pmatrix} \left\{ \begin{array}{c} u \\ v \end{array} \right\} \\ &= [L][N]\{\delta\}_e \\ &= [B]\{\delta\}_e \end{aligned} \tag{1.4}$$

where

$$[B] = \begin{pmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \cdots & \frac{\partial N_8}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & \cdots & 0 & \frac{\partial N_8}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \cdots & \frac{\partial N_8}{\partial y} & \frac{\partial N_8}{\partial x} \end{pmatrix}. \tag{1.5}$$

Stress in the element is written as

$$\begin{aligned} \{\sigma\} &= [D][B]\{\delta\}_e \\ \{\sigma\}^T &= \{\sigma_x, \sigma_y, \tau_{xy}\}^T \end{aligned} \tag{1.6}$$

The stress-strain matrix $[D]$ is given by

$$[D] = \frac{E}{(1+\gamma)(1-\gamma)} \begin{pmatrix} 1-\gamma & \gamma & 0 \\ \gamma & 1-\gamma & 0 \\ 0 & 0 & 0.5-\gamma \end{pmatrix}. \tag{1.7}$$

Now the potential energy π is obtained from

$$\pi = \frac{1}{2}\int \{\delta\}^T [B]^T [D][B]\{\delta\} dv - \{\delta\}^T \{fe\} \tag{1.8}$$

where $\{fe\} = \{F\}$ is the externally applied elemental force vector.
According to the theorem of minimum potential energy, variation in π is zero.

$$\therefore \quad \frac{1}{2}\int \Delta\{\delta\}^T [B]^T [D][B]\{\delta\} + \frac{1}{2}\int \{\delta\}^T [B]^T [D][B]\Delta\delta - \{\Delta\delta\}^T \{F\} = 0 \tag{1.9}$$

$$\begin{aligned} &= \textstyle\int \Delta\{\delta\}^T [B][D][B]\{\delta\} dv - \Delta\{\delta\}^T \{F\} = 0 \\ &= \Delta\{\delta\}[\textstyle\int_v [B]^T [D][B] dv\{\delta\} - \{F\}] = 0 \end{aligned} \tag{1.10}$$

$$\text{i.e.} \quad \int_v [B]^T [D][B] dv\{\delta\} = \{F\}$$

$$\therefore [K]\{\delta\} = \{F\}$$

where

$$\int_v [B]^T [D][B] dv = [K]. \tag{1.11}$$

The matrix $[K]$ is obtained through a suitable numerical integration scheme, usually the Gauss quadrature procedure. Relation (1.11) represents a set of simultaneous equations involving the nodal displacement for the entire domain. These equations are infact, the overall equilibrium equations. The overall stiffness matrix is a singular matrix. It is made non-singular by incorporating the specified fixed boundary conditions. The solution of equations (1.11) constitutes a major part of the required computer time.

3. Evaluation of Stress Intensity Factor

Finite element method has been extensively used for the determination of stress intensity factor (K) [3, 4]. There are numerous special elements which can be discretised to simulate the singular stress field around a crack-tip [5]. The quarterpoint singularity elements [5], which meet all the convergence requirements, have the advantage that they can be most easily included in a computer program of 8-noded quadrilaterals [5]. Rice J.R. and Neale, B. K. [7, 6] has derived the J-integral expression in a systematic procedure.

The J-integral is found to be equal to the strain energy release rate G, in the linear elastic range and there exists a relation between K and G. the stress intensity factor K can be obtained utilizing the J- integral value [7].

$$J = \int_r \left[U dy - t \frac{\partial u}{\partial x} ds \right] \tag{1.12}$$

where U is strain energy, t is outward traction vector along the normal n_j to the contour Γ and $\frac{\partial u}{\partial x}$ is the strain vector. The J integral coordinate system is shown in Figure-2.

Theoretically J-integral is path-independent. But the finite element solution is approximate and the integral becomes path-dependent. It is recommended that the maximum value of J should be used to evaluate K, like the method based on the strain energy release rate, the J-integral method has an accuracy which is not very dependent on mesh refinement and it permits elimination for the extrapolation step usually associated with the displacement and stress methods.

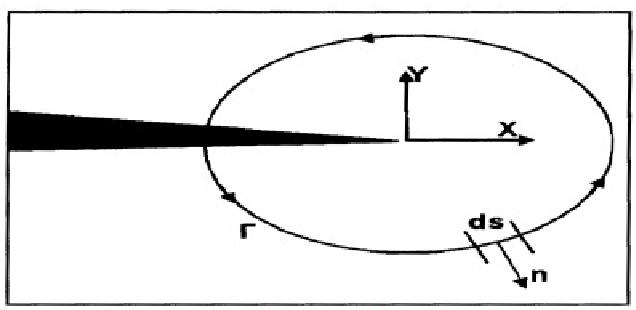


Figure -2: Typical line integral coordinate system

The total J integral value is the summations of the contributions of all elements forming the integral path and the stress intensity factors due to first, second and third modes.

4. Case Studies

Using ANSYS FEM software, finite element analysis was carried out using plane-82 element, for the following three cases to compute stress intensity factor K_I Strain energy release rate G_I and J-integral J_I.

Case I : Tensile specimen with a single edge crack (Figure 3.0)

Case-II : Tensile specimen with a double edge crack (Figure 4.0)

Case-III : Tensile specimen with a centre crack (Figure 5.0)

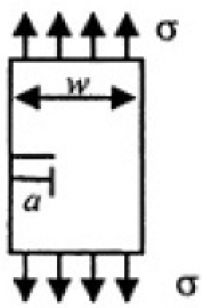


Figure 3.0

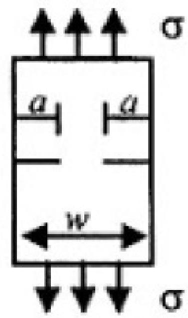


Figure 4.0

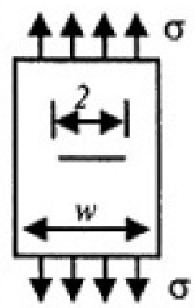
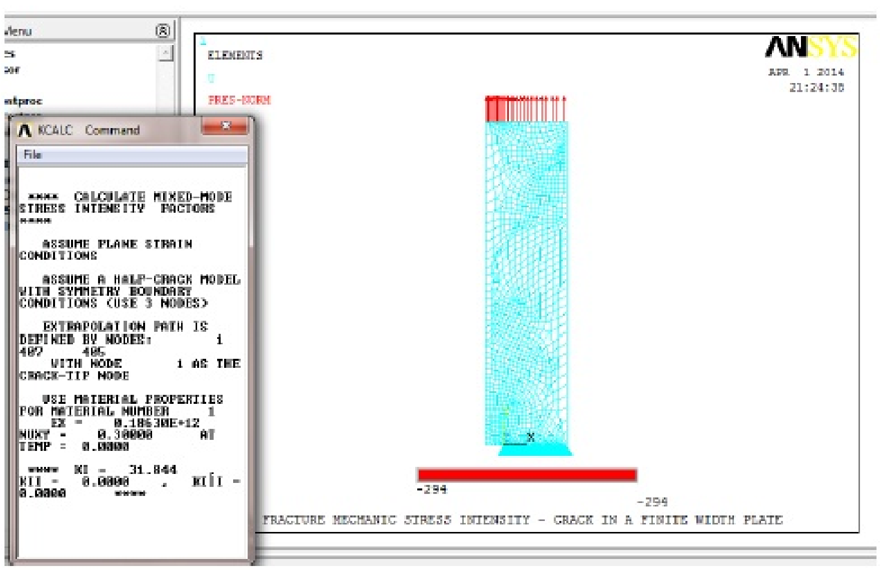


Figure 5.0

Case - 1 : Single edge crack tensile specimens have the dimensions :
Length $(L) = 0.08$m; width $(W) = 0.01$m; thickness $(t) - 0.01$m and crack length $(a) = 0.002$m. The number of elements considered for FEM analysis is 1316 elements. Finite element analysis has been carried out and the stress intensity factor and the Strain energy release rate have been obtained. The FEM mesh generated for single edge crack is shown below.

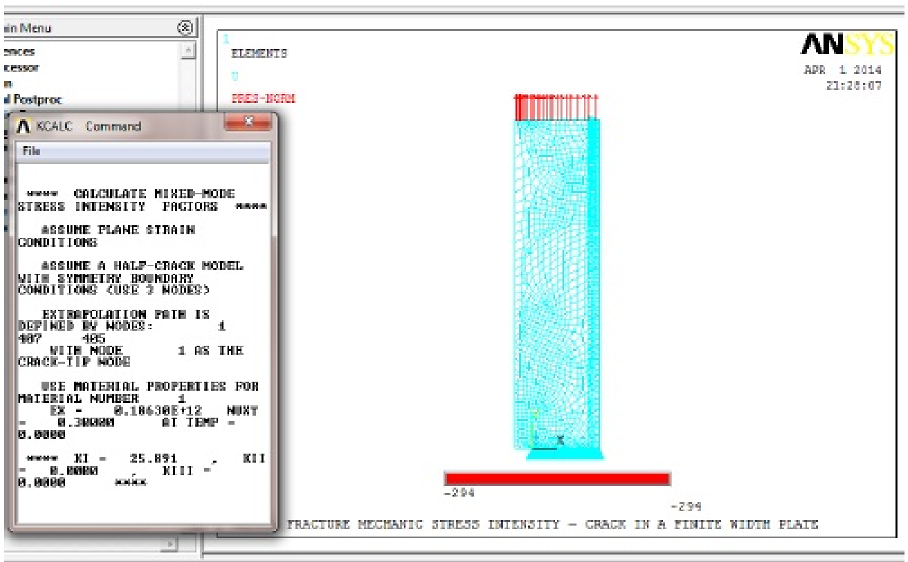
Single edge crack specimen :



Case - 2 : Double edge crack specimen.

Double edge crack tensile specimen has the dimensions: Length (L) = 0.08m; width (W) = 0.01m; thickness (t) = 0.01m and crack length (a) = 0.002m. The number of elements considered for FEM analysis is 1316 elements. The FEM mesh generated for double edge crack specimen is shown below.

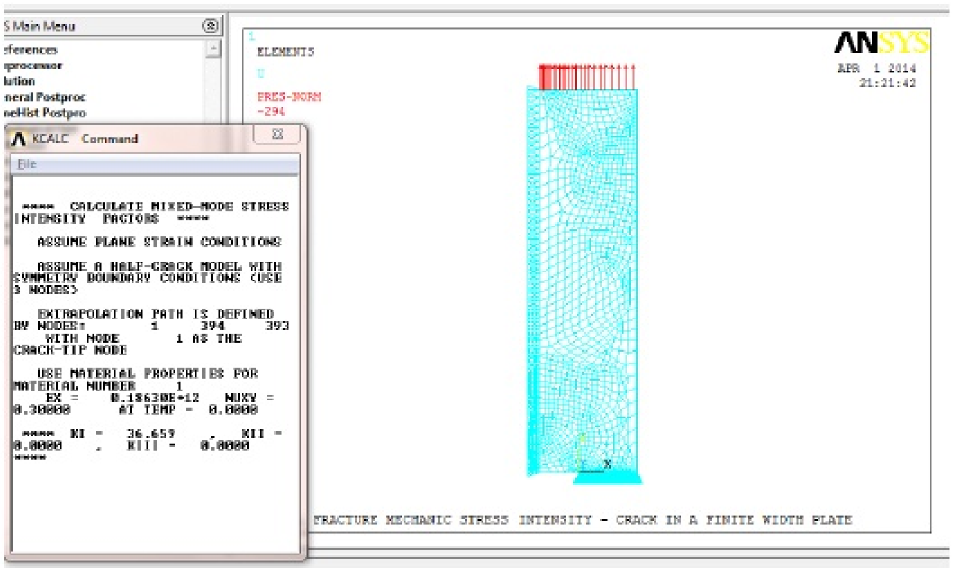
Double edge crack.



Case - 3 : Centre crack specimen

Centre crack tensile specimen has the dimensions : Length (L) = 0.08m; width (W) = 0.01m; thickness (t) – 0.01m and crack length (a) = 0.004m. The number of elements considered for FEM analysis is 1260 elements. Finite element analysis has been carried out to determine the stress intensity factor and the Strain energy release rate. The FEM mesh generated for centre crack specimen is shown below.

M250 grade maraging steel material used for rocket motor case fabrication has been considered in the present study. The mechanical property of the material is given in Table-1.0. Analysis was carried out to evaluate stress intensity factor K_I, strain energy release rate and J-integral For the above three cracked configurations for an applied stress value of 294 MPa, Stress intensity factor, strain energy release rate and J-integral for these cases were also obtained using the following analytical expressions.



Single edge cracked specimen (Figure 3.0)

$$K_1 = \sigma\sqrt{\partial a}\,Y\frac{a}{W} \tag{2.0}$$

$$Y(\xi) = 1,12 - 0231\xi + 10.55\xi^2 - 21.71\xi^2 + 30.39\xi^4, \xi \leq 0.6.$$

Double edge cracked specimen (Figure 4.0)

$$K_1 = \sigma\sqrt{\pi a}\,Y\frac{2a}{W} \tag{3.0}$$

$$Y(\xi) = 1.12 + 0.203\xi - 1.197\xi^2 + 1.930\xi^2.$$

Centre cracked specimen (Figure 5.0)

$$K_1 = \sigma\sqrt{\pi a}\,Y\frac{a}{W} \tag{4.0}$$

$$\gamma(\xi) = \sqrt{\sec(\pi\xi)}. \tag{4.0}$$

Table 1.0 : Mechanical properties of M250 maraging steel.

Material	**Young's Modulus** $E(GPa)$	**Poisson's ratio** v	**Rigidity Modulus** $\mu = \frac{E}{2(1+v)}(GPa)$
M250 Steel	186.3	0.3	71.7

Table -2.0 presents a good comparison of $(K_1), (G_1)$ and $J_1)$, obtained from finite element method and analytical methods for the three cases of cracked specimens.
From the Table-2, it is known that the (G_1) and (J_1), values are material dependent, whereas (K_I) values almost material independent and it is also seen that the J-integral is equal to the strain energy release Rate (G) in the linear elastic range.

Table 2 : Comparisons of stress intensity facto, Strain energy release rate and J-integral obtained from FEM and analytical methods.

Material	Finite element solution			**Analytical solution**
	$K_1 Mpa\sqrt{m}$	$G_1 MPa\ m$	$J_1 MPam$	$K_1 MPa\sqrt{m}$
single edge crack	31.844	0.0054	0.0049	31.944 (equ.2)
Double edge crack	25.891	0.00359	0.003584	26.400 (equ.3)
Centre crack	32.960	0.00721	0.00719	32.96 (equ.4)

5. Results and Discussion

The stress intensity factor evaluation is essential for assessing the strength of flawed structures. A comparative study is made on the stress intensity factor of cracked configurations utilizing a plane 82 element of ANSYS. The computed values of strain energy release rate (G_I), J-integral (J_I) and stress intensity factor (K_I) for tensile cracked configurations are found to be in good agreement with those of handbook solutions. This confirms the validation of finite element modelling of cracked configurations. For any complex cracked configuration and complex loading conditions, it is possible to obtain finite element solution utilizing the commercial software package ANSYS. To assess the fracture strength of cracked bodies, the evaluated stress intensity factor should be compared with the fracture toughness or the critical stress intensity factor of the material.

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