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A FIXED POINT APPROACH FOR SOLVING A SYSTEM OF EXTENDED GENERAL VARIATIONAL INEQUALITY PROBLEMS

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Abstract

In this paper, we consider a system of extended general variational inequality problems (in short, SEGVIP) in real Hilbert spaces. Using fixed point theorem and projection operator technique, it is observed that the SEGVIP is equivalent to the system of projection equations. This alternative equivalence formulation is used to prove the existence of a unique solution of SGEVIP. The approach used in this paper may be treated as an extension and unification of approaches for studying existence results for various important classes of system of variational inequality problems given by many authors, see for example [1, 3, 5-9, 11].

1. Introduction

Variational inequality theory introduced by Stampacchia [10] and Fichera [2], has become a rich source of inspiration and motivation for the study of a large number of

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problems arising in mechanics, elasticity and optimization etc., see [1-12]. In the last four decades, considerable interest has been shown in developing numerical methods which provide an efficient and implementable algorithm for solving variational inequalities and its generalization. This theory provides a simple, natural and unified framework for a general treatment of unrelated problems, which have motivated a large number of mathematicians to generalize and extend the variational inequalities and related optimization problems in several directions using novel techniques, see for example [1, 3, 5-9, 11].

By using the projection technique, Noor [5,6], Noor *et al.* [8] and Verma [11] studied the existence of solutions for some classes of extended general variational inequalities in Hilbert and Banach spaces. Very recently, by using the projection technique, Cho *et al.* [1], Huang *et al.* [3], Noor *et al.* [7] and Saleh *et al.* [9] studied the existence of solutions for some classes of system of general extended variational inequality problems in Hilbert and Banach spaces.

Inspired by recent research going on in this area, in this paper, we consider a system of extended general variational inequality problems (SEGVIP) in real Hilbert spaces. Using fixed point theorm and projection operator technique, it is observed that the SEGVIP is equivalent to the system of projection equations. This alternative equivalence formulation is used to prove the existence of a unique solution of SGEVIP.

2. Preliminaries

From now onwards, unless or otherwise stated, let $I = \{1, 2\}$ be an index set and for each $i \in I$, let H_i be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle_i$ and $\|.\|_i$, respectively. Let $N_i : H_1 \times H_2 \to h_i\ g_i, H_i : H_i \to h_i$ be nonlinear mappings. We consider the following system of extended general variational inequality problems (in short, SEGVIP):

Find $(x, y) \in H_1 \times H_2 : h_1(x) \in K_1$, $h_2(y) \in K_2$ such that

$$\langle N_1(x, y) + h_1(x) - g_1(x),\ g_1(v_1) - h_1(x)\rangle_1 \geq 0, \quad \forall v_1 \in H_1 : g_1(v_1) \in K_1, \tag{2.1}$$

$$\langle N_2(x, y) + h_2(y) - g_2(y),\ g_2(v_2) - h_2(y)\rangle_2 \geq 0, \quad \forall v_2 \in H_2 : g_2(v_2) \in K_2. \tag{2.2}$$

Similar type problems have been studied by many authors, see for example [5-9].

Now, we give the following known concepts and results which are needed in the sequel:

Lemma 2.1 [2,8-10] : Let K be a closed and convex set in H. Then for a given $z \in H$, $u \in K$ satisfies the inequality

$$\langle u - z, v - u \rangle \geq 0, \quad \forall v \in K,$$

if and only if $u = P_K(z)$, where P_K is the projection of H onto the closed convex set K in H.

It is well known that the projection operator P_K is nonexpansive i.e.,

$$\|P_K(x) - P_K(y)\| \leq \|x - y\|, \ \ \forall x, y \in H.$$

Definition 2.1 [7-9] : A mapping $g : H \to H$ is said to be:

(a) σ-strongly monotone if there exists a constant $\sigma > 0$ such that

$$\langle g(x) - g(y), x - y \rangle \geq \sigma \|x - y\|^2;$$

(b) δ-Lipschitz continuous if there exists a constant $\delta > 0$ such that

$$\|g(x) - g(y)\| \leq \delta \|x - y\|.$$

Definition 2.2 [6-9] : A mapping $N_1 : H_1 \times H_2 \to H_1$ is said to be:

(a) α_1-strongly monotone in the first argument if there exists a constant $\alpha_1 > 0$ such that

$$\langle N_1(x_1, y) - N_1(x_2, y), x_1 - x_2 \rangle_1 \geq \ \alpha_1 \|x_1 - x_2\|_1{}^2, \ \ \forall x_1, x_2 \in H_1, y \in H_2;$$

(b) (β_1, γ_1)-mixed Lipschitz continuous if there exist constants $\beta_1, \gamma_1 > 0$ such that

$$\|N_1(x_1, y_1) - N_1(x_2, y_2)\|_1 \ \leq \beta_1 \|x_1 - x_2\|_1 + \gamma_1 \|y_1 - y_2\|_2, \ \forall x_1, x_2 \in H_1, y_1, y_2 \in H_2.$$

Lemma 2.2 [9,11] : Let H be a Hilbert space. Then for any $x, y \in H$, we have

$$\|x + y\|^2 \leq \|x\|^2 + \langle y, x + y \rangle.$$

3. Fixed Point and Fixed Point Theorem

In this section, we present fixed point and contraction mapping which are needed in the sequel.

Definition 3.1 [9,12] : Let (X, d) be a metric space and let $T : X \to X$ be a mapping. A point $x \in X$ is said to be fixed point of T if $Tx = x$.

Definition 3.2 [9,12] : A mapping $T : X \to X$ is said to be contraction if

$$d(T(x), T(y)) \ \leq \ \alpha \ d(x, y), \quad \forall x, y \in X,$$

for some α, $0 \leq \alpha < 1$. If $\alpha = 1$, then the mapping T is non-expansive.

Now, we state a fixed point theorem, which is a natural generalization of Banach contraction theorem, see [4, 8, 9, 12].

Theorem 3.1 (Fixed Point Theorem) : Let X be a Banach space. If T is contraction mapping on X into itself. Then T has a unique fixed point.

4. Main Result

First we establish an equivalence between the SEGVIP (2.1)-(2.2) and the system of projection equations and then using this equivalence to prove the existence of a unique solution of SEGVIP (2.1)-(2.2).

Lemma 4.1 : For any given $(x, y) \in H_1 \times H_2$, (x, y) is a solution of SEGVIP (2.1)-(2.2) if and only if (x, y) satisfies the system of projection equations

$$h_1(x) = P_{K_1}[g_1(x) - \rho_1 N_1(x, y)], \tag{4.1}$$

$$h_2(y) = P_{K_2}[g_2(y) - \rho_2 N_2(x, y)], \tag{4.2}$$

where $\rho_1, \rho_2 > 0$ are constants.

Theorem 4.1 : For each $i = 1, 2$, let $N_i : H_1 \times H_2 \to H_i$, $g_i, H_i : H_i \to H_i$ be nonlinear mappings. Let the mapping N_1 is α_1-strongly monotone in the first argument and (β_1, γ_1)-mixed Lipschitz continuous and N_2 is α_2-strongly monotone in the second argument and (β_2, γ_2)-mixed Lipschitz continuous. Let the mappings h_i is μ_i-strongly monotone and η_i-Lipschitz continuous and g_i is σ_i-strongly monotone and δ_i-Lipschitz continuous. Suppose that $\rho_1, \rho_2 > 0$ satisfy the following condition:

$$U_1 + V_1 + W_1 + \rho_2\beta_2 \leq \ 1 \ \ ; \ \ U_2 + V_2 + W_2 + \rho_1\gamma_1 \leq \ 1, \tag{4.3}$$

where $U_1 := \sqrt{1-2\sigma_1+\delta_1^2}$; $V_1 := \sqrt{1-2\mu_1+\eta_1^2}$; $W_1 := \sqrt{1-2\alpha_1\rho_1+\beta_1^2\rho_1^2}$;

$$U_2 := \sqrt{1-2\sigma_2+\delta_2^2}\ ;\ V_2 := \sqrt{1-2\mu_2+\eta_2^2}\ ;\ W_2 := \sqrt{1-2\alpha_2\rho_2+\gamma_2^2\rho_2^2}\ .$$

Then SEGVIP (2.1)-(2.2) has a unique solution.

Proof : For given $\rho_i > 0$ $(i = 1, 2)$ and for all $(x, y) \in H_1 \times H_2$, define the mappings $R : H_1 \times H_2 \to H_1$ and $S : H_1 \times H_2 \to H_2$ by

$$R(x,y) \ = \ x - h_1(x) + P_{K_1}[g_1(x) - \rho_1 N_1(x,y)], \tag{4.4}$$

$$S(x,y) \ = \ y - h_2(y) + P_{K_2}[g_2(y) - \rho_2 N_2(x,y)]. \tag{4.5}$$

For given $(i = 1, 2)$ and for all $(x_i, y_i) \in H_1 \times H_2$, it follows from (4.4) that

$$\begin{aligned}
\|R(x_1,y_1)-R(x_2,y_2)\|_1 \leq\ & \|x_1-x_2-(h_1(x_1)-h_1(x_2))\|_1 \\
& +\|P_{K_1}(g_1(x_1)-\rho_1N_1(x_1,y_1)) - P_{K_2}(g_1(x_2)-\rho_1N_1(x_2,y_2))\|_1 \\
\leq\ & \|x_1-x_2-(h_1(x_1)-h_2(x_2))\|_1 + \|x_1-x_2-(g_1(x_1)-g_2(x_2))\|_1 \\
& +\|x_1-x_2-\rho_1(N_1(x_1,y_1)-N_1(x_2,y_1))\|_1 \\
& +\rho_1\|N_1(x_2,y_1)-N_1(x_2,y_2))\|_1.
\end{aligned} \tag{4.6}$$

Since N_1 is α_1-strongly monotone in the first argument and (β_1, γ_1)-mixed Lipschitz continuous, it follows that

$$\begin{aligned}
\|x_1-x_2-\rho_1(N_1(x_1,y_1)-N_1(x_2,y_1))\|_1^2 \leq\ & \|x_1-x_2\|_1^2 - 2\rho_1\langle N_1(x_1,y_1)-N_1(x_2,y_1), x_1-x_2\rangle_1 \\
& +\rho_1^2\|N_1(x_1,y_1)-N_1(x_2,y_1)\|^2 \\
\leq\ & (1-2\rho_1\alpha_1+\rho_1^2\beta_1^2)\|x_1-x_2\|_1^2.
\end{aligned} \tag{4.7}$$

Similarly, we estimate:

$$\|x_1-x_2-(g_1(x_1)-g_1(x_2))\|_1^2 \ \leq (1-2\sigma_1+\delta_1^2)\|x_1-x_2\|_1^2, \tag{4.8}$$

$$\|x_1-x_2-(h_1(x_1)-h_1(x_2))\|_1^2 \ \leq (1-2\mu_1+\eta_1^2)\|x_1-x_2\|_1^2, \tag{4.9}$$

where g_1 is σ_1-strongly monotone and δ_1-Lipschitz continuous and h_1 is μ_1-strongly monotone and η_1-Lipschitz continuous.

From (4.4)-(4.9), we have

$$\|R(x_1,y_1)-R(x_2,y_2)\|_1 \le \left(\sqrt{1-2\sigma_1+\delta_1^2}+\sqrt{1-2\mu_1+\eta_1^2}\right.$$
$$\left.+\sqrt{1-2\alpha_1\rho_1+\beta_1^2\rho_1^2}\right)\|x_1-x_2\|_1+\rho_1\gamma_1\|y_1-y_2\|_2. \tag{4.10}$$

Also, it follows from (4.5) that

$$\|S(x_1,y_1)-S(x_2,y_2)\|_2 \le \|y_1-y_2-(h_2(y_1)-h_2(y_2))\|_2+\|y_1-y_2-(g_2(y_1)-g_2(y_2))\|_2+$$
$$+\|y_1-y_2-\rho_2(N_2(x_1,y_1)-N_2(x_1,y_2))\|_2$$
$$+\rho_2\|(N_2(x_1,y_2)-N_2(x_2,y_2))\|_2. \tag{4.11}$$

Since N_2 is α_2-strongly monotone in the second argument and (β_2,γ_2)-mixed Lipschitz continuous, it follows that

$$\|y_1-y_2-\rho_2(N_2(x_1,y_1)-N_2(x_1,y_2))\|_2 \le (1-2\alpha_2\rho_2+\gamma_2^2\rho_2^2)\ \|y_1-y_2\|_2^2. \tag{4.12}$$

Similarly, we estimate:

$$\|y_1-y_2-(g_2(y_1)-g_2(y_2))\|_2 \le (1-2\sigma_2+\delta_2^2)\ \|y_1-y_2\|_2^2, \tag{4.13}$$

$$\|y_1-y_2-(h_2(y_1)-h_2(y_2))\|_2 \le (1-2\mu_2+\eta_2^2)\ \|y_1-y_2\|_2^2, \tag{4.14}$$

where g_2 is σ_2-strongly monotone and δ_2-Lipschitz continuous and h_2 is μ_2-strongly monotone and η_2-Lipschitz continuous.
From (4.11)-(4.14), we have

$$\|S(x_1,y_1)-S(x_2,y_2)\|_2 \le \left(\sqrt{1-2\sigma_2+\delta_2^2}+\sqrt{1-2\mu_2+\eta_2^2}\right.$$
$$\left.+\sqrt{1-2\alpha_2\rho_2+\gamma_2^2\rho_2^2}\right)\|y_1-y_2\|_2+\rho_2\beta_2\|x_1-x_2\|_1. \tag{4.15}$$

Also from (4.10) and (4.15), we have

$$\|R(x_1,y_1)-R(x_2,y_2)\|_1+\|S(x_1,y_1)-S(x_2,y_2)\|_2$$
$$\le (U_1+V_1+W_1+\rho_2\beta_2)\|x_1-x_2\|_1+(U_2+V_2+W_2+\rho_1\gamma_1)\|y_1-y_2\|_2$$
$$\le k_1\|x_1-x_2\|_1+k_2\|y_1-y_2\|_2$$

$$\leq k(\|x_1 - x_2\|_1 + \|y_1 - y_2\|_2), \tag{4.16}$$

where $k := \max\{k_1, k_2\}$; $k_1 := U_1 + V_1 + W_1 + \rho_2\beta_2$; $k_2 := U_2 + V_2 + W_2 + \rho_1\gamma_1$;

$U_1 := \sqrt{1 - 2\sigma_1 + \delta_1^2}$; $V_1 := \sqrt{1 - 2\mu_1 + \eta_1^2}$; $W_1 := \sqrt{1 - 2\alpha_1\rho_1 + \beta_1^2\rho_1^2}$;

$U_2 := \sqrt{1 - 2\sigma_2 + \delta_2^2}$; $V_2 := \sqrt{1 - 2\mu_2 + \eta_2^2}$; $W_2 := \sqrt{1 - 2\alpha_2\rho_2 + \gamma_2^2\rho_2^2}$.

Now, define the norm $\|\cdot\|_*$ on $H_1 \times H_2$ by

$$\|(x, y)\|_* = \|x\|_1 + \|y\|_2, \ \forall (x, y) \in H_1 \times H_2. \tag{4.17}$$

It is easy to observe that $(H_1 \times H_2, \|\cdot\|_*)$ is a Banach space. Define a mapping

$Q(x, y) : H_1 \times H_2 \to H_1 \times H_2$ by

$$Q(x, y) = (R(x, y), S(x, y)), \quad \forall (x, y) \in H_1 \times H_2. \tag{4.18}$$

Since $k = \max\{k_1, k_2\} < 1$ by (4.3). Hence, it follows from (4.16)-(4.18) that

$$\|Q(x_1, y_1) - Q(x_2, y_2)\|_* \leq k\|(x_1, y_1) - (x_2, y_2)\|_*. \tag{4.19}$$

This proves that the mapping Q is a contraction mapping. Hence, by Banach contraction principle, there exists a unique $(x, y) \in H_1 \times H_2$ such that $Q(x, y) = (x, y)$, which implies that

$$h_1(x) = P_{K_1}[g_1(x) - \rho_1 N_1(x, y)],$$

$$h_2(y) = P_{K_2}[g_2(y) - \rho_2 N_2(x, y)].$$

It follows from Lemma 4.1 that (x, y) is the unique solution of SEGVIP (2.1)-(2.2). This completes the proof.

Remark 4.1 : For $i = 1, 2$, it is clear that $\sigma_i \leq \delta_i$, $\mu_i \leq \eta_i$ and $\rho_1, \rho_2 > 0$. Further, $\theta < 1$ and condition (4.3) of Theorem 4.1 holds for some suitable set values of constants, for example,

- $\alpha_1 = .3,\ \beta_1 = .4,\ \gamma_1 = .1,\ \sigma_1 = .1,\ \delta_1 = .2,\ \mu_1 = .1,\ \eta_1 = .2,\ \rho_1 = .2.$
- $\alpha_2 = .2,\ \beta_2 = .3,\ \gamma_2 = .2,\ \sigma_2 = .2,\ \delta_2 = .3,\ \mu_2 = .2,\ \eta_2 = .3,\ \rho_2 = .1.$

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