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ENFORCEMENT OF ESSENTIAL BOUNDARY CONDITIONS IN THE MESHLESS LOCAL PETROV-GALERKIN METHOD FOR ELECTROMAGNETIC COMPUTATIONS

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Abstract

In meshless methods the enforcement of essential boundary conditions is an important step for solving the partial differential equations successfully due to the loss of Kronecker property in meshless shape functions. The purpose of this paper is to study the imposition of essential boundary conditions by introducing several techniques to the meshless local Petrov-Galerkin (MPLG) method, which is a kind of truly meshless method. Detail comparison investigation is given for an overview about these techniques. All the methods are applied for the electromagnetic field computation and numerical results verify the effectiveness of the proposed methods.

1. Introduction

Meshless methods have been developed as a powerful alternative to the well established mesh-based methods for a wide range of engineering applications, such as mechanics

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problems [1], electromagnetic field computations [2]. Meshless methods only use nodal points instead of element meshes for the approximation of unknown quantities. Meshless methods are also characterized by their wide adaptability and the low cost of preparing input and output data for numerical analysis. Among meshless methods, the meshless local Petrov-Galerkin (MLPG) method developed by Atluri and Zhu [3,4] is a truly meshless method, which requires neither nodal connectivity nor background cells, for either the interpolation or the integration purposes. The most significant difference between the MLPG and other meshless methods is that the local weak forms are generated on a set of overlapping local subdomains with simple geometrical shapes, instead of the global weak form. Based on the local weak forms, the MLPG method avoids background integral cells besides employing meshless interpolation. When a local weak form is used for a field node, the numerical integrations are performed over a local quadrature domain defined for the node, which can also be the local domain where the test function is defined. The local domain usually is regular and simple shape for an internal node such as a circular and square, and the integration is performed numerically within the local domain. Hence the domain and boundary integrals in the weak form can be easily evaluated over regularly shaped sub-domains and their boundaries. However, because the MLPG method has the drawback that the meshless shape functions are not interpolation functions in general, it is difficult to impose essential boundary conditions for solving partial differential equations by using the MLPG method.

In this paper, we investigate some useful techniques for the imposition of essential boundary conditions in detail, which can be easily extend to other meshless methods. The penalty method [3] is applicable to a number of problems conveniently. But the essential boundary condition is weakly imposed since the penalty parameter controls how well the essential boundary conditions are set. In addition, the coefficient matrix is often poorly conditioned because the condition number increases with the penalty parameter. Nevertheless, the symmetric weak formulation can be obtained by using the penalty approach to enforce boundary conditions. The coupling with finite elements [4] employs a string of elements along essential boundaries and combines the finite element shape function defined on this string with the meshless one by a ramp function. Usually the approximation itself is continuous, but its derivative undergoes a jump across the interface. In this work, a transformation method (TM) and a boundary singular weight function method (BSW) are introduced for the MLPG method to solve the PDEs. Both the techniques can be directly to enforce the essential boundary conditions, as well as be further extended to other meshless methods such as the element free Galerkin method, the partition of unity method and diffuse element method. Finally, all the methods are used to solve the electromagnetic filed computations. From the detailed comparison investigation, it can be demonstrated that the transformation method is accurate and robust, and the singular weight function method can save computational time to a grate extent. Numerical examples for some electromagnetic field models are presented to validate the efficiency of the proposed approaches.

2. The MLPG Method

2.1 Moving Least Square (MLS) Approximation

In general, a meshless method represents the trial function with the values of the unknown variable at some randomly located nodes by using the local approximation. The MLS approximation is one of the most popular method. Consider a function $u(x)$ in Ω. The MLS approximation $u^h(x)$ is defined by

$$u^h(x) = \sum_{j=1}^{m} p_j(x)a_j(x) = p^T(x)a(x). \tag{1}$$

where $p^T(x) = (p_1(x)p_2(x)\cdots p_m(x))$ is a complete monomial basis of order m, and $a(x)$ is a vector with the components $a_j(x), j = 1, 2, \cdots, m$. Here m is the number of basis functions. The coefficient vector $a(x)$ can be determined by minimizing the weighted discrete L_2 norm with the least square theorem, which is defined as follows,

$$J(x) = \sum_{i=1}^{n} w(x - x_i)(p^T(x_i)a(x) - \hat{u}_i)^2 = (Pa(x) - \hat{u})^T W(Pa(x) - \hat{u}). \tag{2}$$

where $w(x - x_i)$ is the weight function associated with the node i. n is the number of nodes in the sub-domain where the weight function $w(x-x_i) > 0$, $P = (p(x_1), p(x_2), \cdots, p(x_n))^T$, $W = diag(w(x - x_1), w(x - x_2), \cdots, w(x - x_n))$ and $\hat{\mathbf{u}} = (\hat{u}_1, \hat{u}_2, \cdots, \hat{u}_n)^T$. $\hat{u}_i$ is the fictitious nodal value associated with x_i. The stationary of J in (2) with respect to $a(x)$ leads to the following linear relation between $a(x)$ and $\hat{\mathbf{u}}$.

$$A(x)a(x) = B(x), \tag{3}$$

where the matrices $A(x)$and $B(x)$ are defined by $A(x) = P^T W P = B(x)P = \sum_{i=1}^{n} w_i(x)p(x_i)p^T(x_i)$, $B(x) = P^T W = (w_1(x)p(x_1), w_2(x)p(x_2), \cdots, w_n(x)p(x_n))$. The MLS approximation is well defined once the matrix $A(x)$ is not singular. Obviously, this is the case if and only if the rank of P equals m. Hence it is necessary that at least m weight functions are non-zero (i.e. $n > m$) for each sample node $x \in \Omega$. The Gaussian weight function is used in this work,

$$w_i(x) = \begin{cases} \frac{\exp[-(d_i/c_i)^2]-\exp[-(r_i/c_i)^2]}{1-\exp[-(r_i/c_i)^2]} & 0 \le d_i \le r_i, \\ \\ 0 & d_i > r_i, \end{cases} \tag{4}$$

where $d_i = \|x - x_i\|$. c_i is a constant controlling the shape of the weight function w_i and r_i is the size of the support domain. The size of support domain should be large enough to have sufficient number of nodes to ensure the regularity of the matrix A. $a(x)$ can be given from (3), and the following relation can be written by substituting $a(x)$ into (1). Then we have

$$u^h(x) = \Phi^T(x)\hat{u} = \sum_{I=1}^{N} \Phi_I(x)u^*(x_I), \tag{5}$$

where

$$\Phi_I(x) = \sum_{j=1}^{m} p_j(x)(A^{-1}(x)B(x))_{jI}. \tag{6}$$

Here $\Phi_I(x)$ is usually called the shape function of the MLS approximation corresponding to the nodal point x_I. It can be seen that the MLS shape functions do not possess Kronecker delta property, i.e. $\Phi_I(x_J) \neq \delta_{IJ}$, which leads to the difficulty of the imposition of essential boundary conditions. In this work we focus on the enforcement of essential boundary conditions.

2.2 Local Petrov-Galerkin Integral Equation

For a two dimensional electrostatic problem on the domain Ω, which is bounded by the boundary Γ. The governing equation is given by

$$\begin{cases} -\Delta u = f & \text{in} \quad \Omega \\ \\ u = \overline{u} & \text{on} \quad \Gamma_u \\ \\ \frac{\partial u}{\partial n} = q = \overline{q} & \text{on} \quad \Gamma_q, \end{cases} \tag{7}$$

where $\overline{u}$ and $\overline{q}$ are the prescribed potential and normal flux, respectively,and n is the outward normal direction to the boundary Γ. A generalized local weak formulation over a local subdomain Ω_s can be written as

$$\int_{\Omega_s} (\Delta u + f) v d\Omega = 0. \tag{8}$$

We denote $\partial\Omega_s$ as the boundary of the local sub-domain Ω_s. $\partial\Omega_s = \Gamma_s \cup L_s$. Γ_{su} and Γ_{sq} are the parts of $\partial\Omega_s$, over which the essential and natural boundary condition are specified separately. We choose $v(x, x_I)$ as the test function in every sub-domain, and $v(x, x_I)$ can be the weight function, Dirac's Delta function, Heaviside step function and so on [4]. Using the divergence theorem, we can obtain the linear system

$$\mathbf{K} \bullet \mathbf{u}^* = f \tag{9}$$

and

$$K_{IJ} = \int_{\Omega_s} (\Phi_{J,x}(x) v_{,x}(x, x_I) + \Phi_{J,y}(x) v_{,y}(x, x_I)) d\Omega - \int_{\Gamma_{su}} \frac{\partial \Phi_J(x)}{\partial n} v(x, x_I) ds, \tag{10}$$

$$f_I = \int_{\Gamma_{sq}} \overline{q} v(x, x_I) ds + \int_{\Omega_s} f v(x, x_I) d\Omega, \tag{11}$$

where $I = 1, 2, \cdots, N, J = 1, 2, \cdots, M$. N denotes the total number of nodes of Ω and M is the number of test function centered at x_I, which do not vanish at x_J.

3. The Enforcement of Essential Boundary Conditions

Like other meshless methods, the shape function in the MLPG method does not satisfy the Kronecker-δ property, which causes that the essential boundary conditions need to be treated with additional efforts. Many methods have been proposed to deal with the difficulty. The penalty method is widely used for imposition of essential boundary conditions in the MLPG method with the penalty term of $\alpha \int_{\Gamma_u} (u - \overline{u}) v ds$. However, if the penalty parameter α is too small, it cannot effectively impose essential boundary conditions. On the other hand, if α is too large, it will lead to the ill conditioning of the coefficient matrix. A suitable value for the penalty parameter is not easy to choose in advance. Another technique is motivated by the coupled element free Galerkin and finite element method presented by Belystchko [5]. In this approach, the computational domain is divided into two regions where the finite element method and the meshless method are used separately. A transition domain between the two regions is defined, where a ramp function is chosen to combine the shape functions of the two methods. In this approach, the continuity condition of potential variables is ensured. But their derivatives may undergo a jump across the interface. Although the coupled MLPG and finite element method presented in [6] has avoided the discontinuity, the complexity of the modified shape function in the transition domain still needs to be improved.We introduce the following techniques to impose the essential boundary condition in the MLPG method.

3.1 The Transformation Method

The potential viable for static fields is written as following

$$u^h(x_J) = \sum_{I=1}^{N} \Phi_I(x_J) u_I = \mathbf{A}_J^T \mathbf{u}, \tag{12}$$

where $\mathbf{A}_J = [\Phi_1(x_J)\ \Phi_2(x_J) \cdots \Phi_N(x_J)]^T$ and $\mathbf{u} = [u_1 \quad u_2 \cdots u_N]^T$. Here assume

$$\hat{\mathbf{u}} = \Lambda^T \mathbf{u}, \tag{13}$$

where we call $\mathbf{u}$ the general potential vector. $\hat{\mathbf{u}} = [u^h(x_1) \quad u^h(x_2) \cdots u^h(x_N)]^T$ is the nodal potential vector, and $\Lambda = [\mathbf{A}_1 \quad \mathbf{A}_2 \cdots \mathbf{A}_N]$ is a transformation matrix,

$$\Lambda = \begin{bmatrix} \Phi_1(x_1) & \Phi_1(x_2) & \cdots & \Phi_1(x_N) \\ \phi_2(x_1) & \Phi_2(x_2) & \cdots & \Phi_2(x_N) \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_N(x_1) & \Phi_N(x_2) & \cdots & \Phi_N(x_N) \end{bmatrix}. \tag{14}$$

Further we can obtain $\mathbf{u} = \Lambda^{-T}\hat{\mathbf{u}}$, i.e. $u_I = \sum\limits_{K=1}^{N} \Lambda_{IK}^{-T} \hat{u}_K$. Λ^{-T} is the transpose of inverse of the matrix Λ. Then we have

$$u^h(x) = \sum_{I=1}^{N} \sum_{K=1}^{N} \Phi_I(x) \Lambda_{IK}^{-T} \hat{u}_K = \sum_{K=1}^{N} \Psi_k(x) \hat{u}_K, \tag{15}$$

where $\Psi_k(x) = \sum\limits_{I=1}^{N} \Phi_I(x) \Lambda_{IK}^{-T}$ is the modified shape function, and it satisfies Kronecker-δ condition, i.e.

$$\Psi_I(x_J) = \sum_{K=1}^{N} \Phi_K(x_J) \Lambda_{KI}^{-T} = \sum_{K=1}^{N} \Lambda_{IK}^{-1} \Phi_K(x_J) = \delta_{IJ}. \tag{16}$$

Thus $\Phi_I(x)$ is replaced by $\Psi_I(x)$ in (16), and the essential boundary condition can be imposed exactly. During the numerical implementation, the construction of resulting stiffness matrix will involve the inversion of the matrix Λ and it will not increase too much computational cost.

3.2 The Boundary Singular Weight Method

Lancaster and Salkauskas [7] first suggested that the MLS approximation would lead to the interpolation by introducing a singularity into the weight function. Then this idea was introduced by Kaljevic and Saigal [8] to the element free Galerkin method. In their approach, singular weight functions are employed all discrete nodes, and Kronecker-δ properties are recovered in the MLS shape functions. In this paper, the MLPG shape functions constructed with singularities introduced only to the constrained boundary nodes. As such, this method does not generate interpolation functions at the interior nodes, but it is sufficient to obtain nodal values at the restrained boundary nodes for direct imposition of essential boundary conditions.

A singularity is introduced to the weight functions with a designated node I located at $(\tilde{x}_I, \tilde{y}_I)$ on the essential boundary,

$$\tilde{w}(x - \tilde{x}_I, y - \tilde{y}_I) = \frac{w(x - \tilde{x}_I, y - \tilde{y}_i)}{f(x - \tilde{x}_I, y - \tilde{y}_I)}, \tag{17}$$

where $f(0,0) = 0$ and the superposed $\sim$ on the nodal coordinate denotes a node with singularity imposed in the associated shape function. The function f is chosen to have the following form [9]

$$f(x - \tilde{x}_I, y - \tilde{y}_I) = \left[\left(\frac{x - \tilde{x}_I}{a_x}\right)^2 + \left(\frac{y - \tilde{y}_I}{a_y}\right)^2\right]^p, \quad p > 0 \tag{18}$$

where $2p$ reflects the order of singularity, and a_x, a_y are the parameters to adjusting the size of influence domain. Using (17), we get the shape function associated with the weight $\tilde{w}(x - \tilde{x}_I, y - \tilde{y}_I)$ as following

$$\tilde{\Phi}_I(x) = p^T(x)\tilde{\mathbf{A}}^{-1}(x)p(\tilde{x}_I)\tilde{w}(x - \tilde{x}_I), \tag{19}$$

where

$$\begin{aligned}\tilde{\mathbf{A}}(x) \quad &= \quad \sum_{J \notin \Gamma_u} p(x_J)p^T(x_J)w(x - x_J) \\ &\quad + \sum_{K \in \Gamma_u, K \notin I} p(\tilde{x}_K)p^T(\tilde{x}_K)w(x - \tilde{x}_K) + p(\tilde{x}_I)p^T(\tilde{x}_I)\tilde{w}(x - \tilde{x}_I).\end{aligned}$$

Other shape functions are

$$\Phi_J(x) = p^T(x)\tilde{\mathbf{A}}^{-1}(x)p(x_J)w(x - x_J), \quad J \notin \Gamma_u, \tag{20}$$

$$\Phi_K(x) = p^T(x)\tilde{\mathbf{A}}^{-1}(x)p(\tilde{x}_K)\tilde{w}(x - \tilde{x}_K), \quad K \in \Gamma_u, \quad K \neq I. \tag{21}$$

The singular weight shape functions $\tilde{\Phi}_I(x)$ have the following property $\tilde{\Phi}_I(x \to \tilde{x}_I) = 1$. Other shape functions have the following property $\Phi_J(x \to \tilde{x}_I) = 0, J \notin \Gamma_u$, $\tilde{\Phi}_K(x \to \tilde{x}_I) = 0, K \in \Gamma_u, K \neq I$. Recall the approximation of the potential variable $u^h(\tilde{x}_I) = \sum\limits_{J=1,J\neq I}^{N} \Phi_J(\tilde{x}_I)u_J + \tilde{\phi}_I(\tilde{x}_I)u_I = u_I$. Comparing the approach by Kaljevic [8], the proposed boundary singular weight method provides exact nodal values at the constraint boundary nodes.

3. Numerical Example

3.1 Comparison Analysis

A simple electrostatic model shown in Fig.2 is used for the comparison analysis of the accuracy and CPU time of the proposed and existing methods. 441 nodes are used to discrete the whole domain. The following relative error is defined as:

$$Re = \sqrt{\frac{\sum\limits_{i=1}^{N}(\overline{\phi}_i - \phi_i)^2}{\sum\limits_{i=1}^{N}\phi_i^2}}, \tag{22}$$

where $\overline{\phi}$ and ϕ are the numerical solution by the numerical method and the analytic method separately, and N is the number of nodes in the computational domain.

The comparison of the normalized error and the CPU time using these techniques separately are listed in Table 1. In the table, the computational error and CPU time are normalized by those in the case of the penalty method. The results indicate that the boundary singular weight (BSW) method has saved computational effort in a degree, and the transformation method (TM) has given good solution although it needs the cost of coordinate transformation.

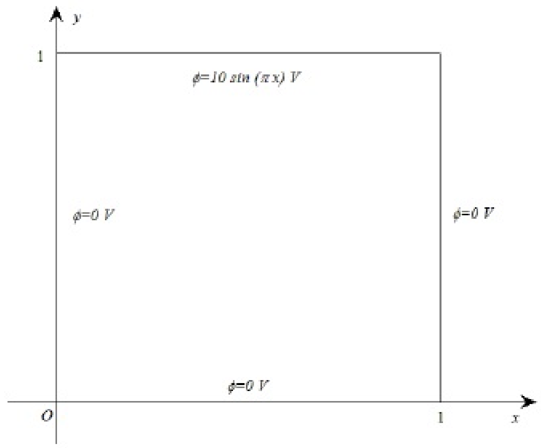

Fig. 1. The cross-section of long straight metal slot.

Table 1 : Comparison of the normalized error and CPU time

Comparison parameter	Penalty	MLPG-FE	TM	BSW
Normalized error	1.000	0.984	0.975	1.107
Normalized CPU time	1.000	1.002	1.105	0.783

3.2 The Axisymmetric Coaxial Waveguides

We consider the axisymmetric coaxial waveguides [10] illustrated in Fig. 2, where two coaxial waveguides having different inner radii are joined. This geometry is rotationally symmetric with respect to the z-axis, so in the ρz-plane the electric potential satisfies the following equation

$$-\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\epsilon,\rho\frac{\partial\phi}{\partial\rho}\right)-\frac{\partial}{\partial z}\left(\epsilon,\rho\frac{\partial\phi}{\partial z}\right)=\frac{\rho_c}{\epsilon_0}. \tag{23}$$

where ρ_c is the charge density, ϵ_r is the relative permittivity, and ϵ_0 is the vacuum permittivity. Further, since the perturbation is confined near the join, the potential at some distance away from the join should be the same as in the unperturbed case. Therefore, the potential far enough away from the join is independent of z, or in other words, it satisfies the condition

$$\frac{\partial\phi}{\partial z}=0. \tag{24}$$

This can be used the boundary condition to terminate the computational domain. The finite element method, the MLPG with the penalty method and the transformation method are separately used to solve the axisymmetric coaxial waveguides. 821 nodes are set in the computational domain uniformly. The equipotential contours are plotted in Fig. 3. The numerical solutions of the electric potential of the nodes on Line AA' are listed in Table 2. It can be seen that the presented transformation method is more efficient for the imposition of the essential boundary condition than the penalty method.

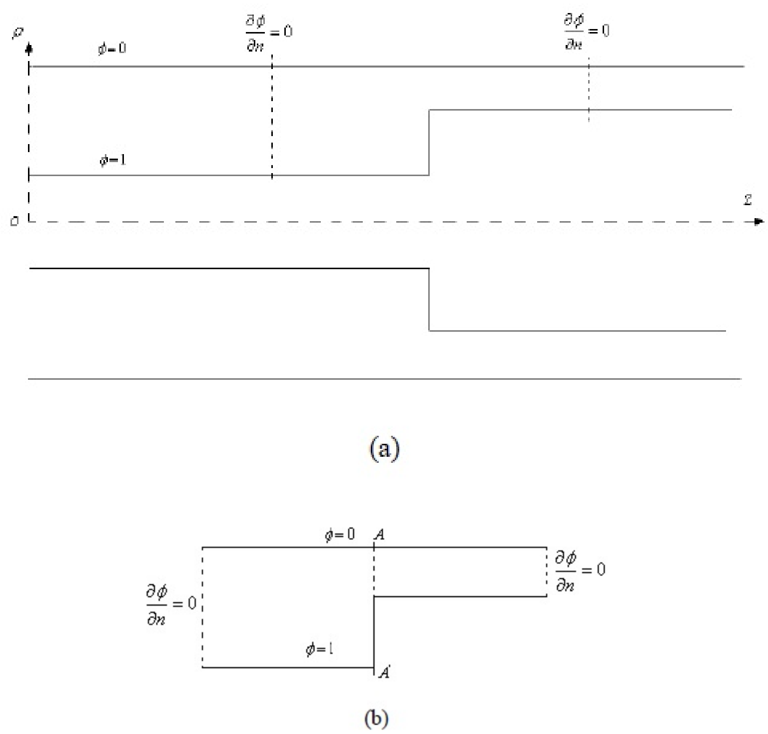

Fig. 2. Join between two coaxial waveguides. (a) Its cross section in the ρz – plane. (b) The computational domain.

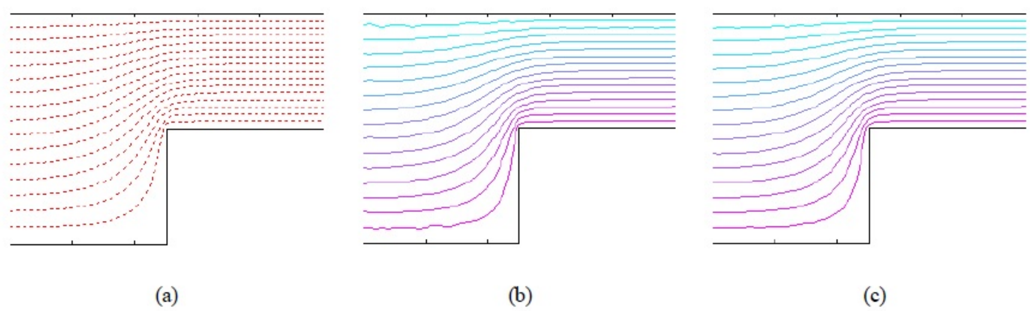

Fig. 3. Equipotential lines near the join between two coaxial waveguides. (a) The finite element method. (b) The penalty method. (c) The transformation method.

Table 2 : The electric potential solution of the nodes on Line AA'

Node	FEM	MLPG with penalty	MLPG with MTM
26	1.00000	0.99103	1.00000
78	1.00000	1.00016	1.00000
130	1.00000	1.00052	1.00000
182	1.00000	1.00043	1.00000
234	1.00000	1.00028	1.00000
286	1.00000	0.98841	1.00000
388	0.73827	0.73278	0.73229
490	0.54347	0.53495	0.53464
592	0.35922	0.35151	0.35168
694	0.17888	0.17327	0.17462
796	0.00000	0.00043	0.00000

3.3 The Transformer Model

Consider the electromagnetic model of the end region of a transformer [11]. The transformation method is recommended to compute the end fields of a power transformer in Fig.4. The corresponding boundary value problem is described as

$$\begin{gathered} \epsilon\frac{\partial^2\phi}{\partial x^2}+\epsilon\frac{\partial^2\phi}{\partial y^2}=0 \\ \phi|_{\Gamma_1}=0,\quad \phi|_{\Gamma_3}=1,\quad \left.\frac{\partial\phi}{\partial n}\right|_{\Gamma_2}=0. \end{gathered} \tag{23}$$

The reliability of the proposed TM method is implicated in Fig.5, which shows the accurate comparison of the numerical results for the traditional finite element method and the transformation method, respectively. It can be seen the transformation method is a good alternative to enforce essential boundary conditions in the MLPG method.

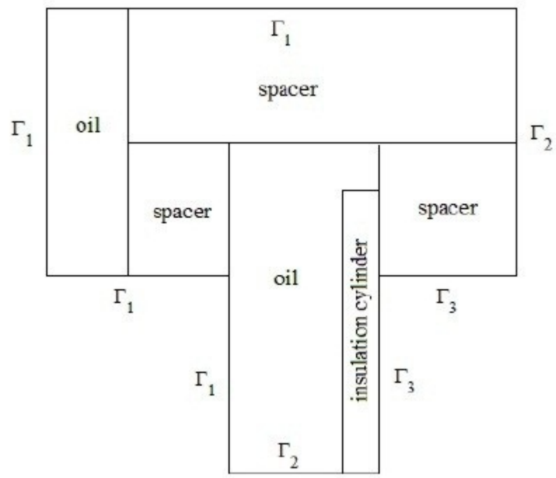

Fig. 4 Schematic diagram of the end region of a power transformer.

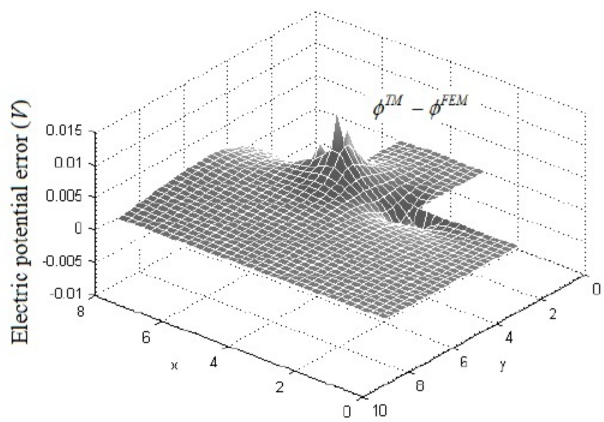

Fig.- 5.Comparison of the electric potential in the end of the transformer by using the MLPG with the transformation method (TM) and the finite element method (FEM).

4. Conclusion

The transformation method and the boundary singular weight method in the MLPG method are developed to impose the essential boundary condition. Comparison research with existing approaches has been performed to compute electromagnetic models. Compared with the existing techniques, the proposed approaches can directly enforce essential boundary conditions exactly. In addition, the transformation method is accurate and robust, and the boundary singular weight method has saved computational time to a great extent. Furthermore, the development of the techniques for the enforcement of the essential boundary condition is hopeful to be extended to solve a wide range of electromagnetic problems, especially such as models where the geometrical shape or disposition changes with time, inverse shape optimizations and so on.

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