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ON A TYPE OF M-PROJECTIVE CURVATURE TENSOR ON KENMOTSU MANIFOLDS

M. SAROJA DEVI & JAY PRAKASH SINGH
[1,2] Department of Mathematics and Computer Science,
Mizoram University, Tanhril, Aizawl-796004, India
E-mail: [1] devi_saroja@rediffmail.com, [2] jpsmaths@gmail.com

Abstract

The object of the present paper is to study the properties of M-projective curvature tensor on Kenmotsu manifolds. It has been shown that globally ϕ- M-projectively symmetric Kenmotsu manifold is an Einstein manifold.

1. Introduction

The study of odd dimensional manifolds with contact and almost contact structures was initiated by Boothby and Wong [5] in 1958 rather from topological point of view. Sasaki and Hatakeyama [7] re-investigated them using tensor calculus in 1961. In [8], Tanno classified connected almost contact metric manifolds whose automorphism groups possesses the maximum dimension. For such a manifold M^n, the sectional

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curvature of plane sections containing ξ is a constant, say c. If $c > 0$, M^n is homogeneous Sasakian manifold of constant sectional curvature. If $c = 0$, M^n is the product of a line or a circle with a Kaehler manifold of constant holomorphic sectional curvature. If $c < 0$, M^n is a warped product space $\mathbf{R} \times_f \mathbf{C}^n$. In 1971, Kenmotsu studied a class of contact Riemannian manifolds satisfying some special conditions [4]. We call it Kenmotsu manifold.

The M-projective curvature tensor is defined by [3]

$$\begin{aligned} W^*(X,Y)Z &= R(X,Y)Z - \tfrac{1}{2(n-1)}\{S(Y,Z)X \\ &- S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\}, \end{aligned} \tag{1.1}$$

where R, S and Q are the Riemannian curvature tensor of type (1,3), the Ricci tensor of type (0,2) and the Ricci operator defined by $g(QX,Y) = S(X,Y)$ respectively.

Definition 1.1 : An n-dimensional Kenmotsu manifold is said to be ξ-M-projectively flat if $W^*(X,Y)\xi = 0$, where $X, Y \in TM^n$.

Definition 1.2 : An n-dimensional Kenmotsu manifold is said to be ϕ-M-projectively flat if $W^*(\phi X, \phi Y, \phi Z, \phi U) = 0$, where $X, Y, Z, U \in TM^n$.

The paper is organized as follows. After preliminaries in section 2, in section 3 we consider globally ϕ-M-projectively symmetric Kenmotsu manifolds. Section 4 deals with 3-dimensional locally ϕ-M-projectively symmetric Kenmotsu manifolds. In section 5, we prove that an n-dimensional Kenmotsu manifold is ξ-M-projectively flat if and only if it is an Einstein manifold. In section 6, we show that an n-dimensional ϕ-M-projectively flat Kenmotsu manifold is an η-Einstein manifold. In section 7, we prove that a Kenmotsu manifold of harmonic M-projective curvature tensor with killing vector ξ is an η-Einstein manifold. Finally, an example of 3-dimensional Kenmotsu manifold is given.

2. Preliminaries

Let $(M^n, \phi, \xi, \eta, g)$ be an n-dimensional (where n=2m+1) almost contact metric manifold, where ϕ is a (1,1)- tensor field, ξ is the structure vector field , η is a 1-form and g is the Riemannian metric. It is well known that the (ϕ, ξ, η, g) structure satisfies the conditions [2]

$$\phi^2(X) = -X + \eta(X)\xi, \tag{2.1}$$

$$g(X, \xi) = \eta(X), \tag{2.2}$$

$$\phi\xi = 0, \quad \eta\phi = 0, \quad \eta(\xi) = 1, \tag{2.3}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.4}$$

for any vector fields X and Y on M^n.
If moreover

$$(D_X\phi)(Y) = -g(X, \phi Y)\xi - \eta(Y)\phi X, \tag{2.5}$$

$$D_X\xi = X - \eta(X)\xi, \tag{2.6}$$

where D is the Riemannian connection, then $(M^n, \phi, \xi, \eta, g)$ is called a Kenmotsu manifold. It is well known [4] that

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.7}$$

$$S(X, \xi) = -(n-1)\eta(X), \tag{2.8}$$

$$(D_X\eta)Y = g(X, Y) - \eta(X)\eta(Y), \tag{2.9}$$

$$S(\phi X, \phi Y) = S(X, Y) + (n-1)\eta(X)\eta(Y). \tag{2.10}$$

A Kenmotsu manifold M^n is said to be η-Einstein if the Ricci tensor S is of the form [1]

$$S(X, Y) = \lambda_1 g(X, Y) + \lambda_2 \eta(X)\eta(Y),$$

for any vetor fields X and Y, where λ_1 and λ_2 are functions on M^n. If $\lambda_2 = 0$, then η-Einstein manifold becomes Einstein manifold.
From [10], we know that for a 3-dimensional Kenmotsu manifold

$$\begin{aligned} R(X, Y)Z &= \tfrac{(r+4)}{2}[g(Y, Z)X - g(X, Z)Y] \\ &- \tfrac{(r+6)}{2}[g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi \\ &+ \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y], \end{aligned} \tag{2.11}$$

$$S(X, Y) = \frac{1}{2}[(r+2)g(X, Y) - (r+6)\eta(X)\eta(Y)], \tag{2.12}$$

where r is the scalar curvature of the manifold.

3. Globally ϕ-M-Projectively Symmetric Kenmotsu Manifolds

Definition 3.1 : A Kenmotsu manifold M^n is said to be globally ϕ-M-projectively symmetric if M-projective curvature tensor W^* satisfies

$$\phi^2((D_U W^*)(X,Y)Z) = 0, \tag{3.1}$$

for all vector fields $X, Y, Z, U \in TM^n$.

Let us suppose that M^n is a globally ϕ-M-projectively symmetric Kenmotsu manifold. Then the equation (3.1) is satisfied.

Now using (2.1) in the equation (3.1), we get

$$-(D_U W^*)(X,Y)Z + \eta((D_U W^*)(X,Y)Z)\xi = 0. \tag{3.2}$$

From (1.1) it follows that

$$\begin{aligned} 0 &= -g((D_U R)(X,Y)Z,V) + \frac{1}{2(n-1)}\{(D_U S)(Y,Z)g(X,V) \\ &- (D_U S)(X,Z)g(Y,V) + (D_U S)(X,V)g(Y,Z) \\ &- (D_U S)(Y,V)g(X,Z)\} + \eta((D_U R)(X,Y)Z)\eta(V) \\ &- \frac{1}{2(n-1)}\{(D_U S)(Y,Z)\eta(X) - (D_U S)(X,Z)\eta(Y) \\ &+ g(Y,Z)(D_U S)(X,\xi) - g(X,Z)(D_U S)(Y,\xi)\}\eta(V). \end{aligned} \tag{3.3}$$

Putting $X = V = e_i$ in the equation (3.3), where $\{e_i\}, (i = 1, 2, ...n)$ is an orthonormal basis of the tangent space at each point of the manifold, and taking summation over i, we get

$$\begin{aligned} 0 &= -(D_U S)(Y,Z) + \frac{1}{2(n-1)}n(D_U S)(Y,Z) \\ &- \frac{1}{2(n-1)}(D_U S)(Y,Z) + \frac{1}{2(n-1)}dr(U)g(Y,Z) \\ &- \frac{1}{2(n-1)}(D_U S)(Y,Z) + \eta((D_U R)(e_i,Y)Z)\eta(e_i) \\ &- \frac{1}{2(n-1)}\{(D_U S)(Y,Z) - (D_U S)(Z,\xi)\eta(Y) \\ &+ g(Y,Z)(D_U S)(\xi,\xi) - (D_U S)(Y,\xi)\eta(Z)\}. \end{aligned}$$

Putting $Z = \xi$ in the above expression we obtain,

$$\begin{aligned} &- \tfrac{n}{2(n-1)}(D_U S)(Y,\xi) + \tfrac{dr(U)}{2(n-1)}\eta(Y) \\ &+ \eta((D_U R)(e_i,Y)\xi)\eta(e_i) = 0. \end{aligned} \tag{3.4}$$

We know that

$$
\begin{aligned}
g((D_U R)(e_i, Y)\xi, \xi) &= g(D_U R(e_i, Y)\xi, \xi) - g(R(D_U e_i, Y)\xi, \xi) \\
&- g(R(e_i, D_U Y)\xi, \xi) - g(R(e_i, Y)D_U \xi, \xi)
\end{aligned}
\tag{3.5}
$$

at $p \in M^n$. Since $\{e_i\}$ is an orthonormal basis, $D_X e_i = 0$ at p. Using (2.7) we find

$$
\begin{aligned}
g(R(e_i, D_U Y)\xi, \xi) &= g(\eta(e_i)D_U Y - \eta(D_U Y)e_i, \xi) \\
&= \eta(e_i)g(D_U Y, \xi) - \eta(D_U Y)g(e_i, \xi) \\
&= 0.
\end{aligned}
\tag{3.6}
$$

Using (3.6) in (3.5) we have

$$
g((D_U R)(e_i, Y)\xi, \xi) = g(D_U R(e_i, Y)\xi, \xi) - g(R(e_i, Y)D_U \xi, \xi). \tag{3.7}
$$

Since

$$
g(R(e_i, Y)\xi, \xi) = -g(R(\xi, \xi)Y, e_i) = 0,
$$

we get

$$
g(D_U R(e_i, Y)\xi, \xi) + g(R(e_i, Y)\xi, D_U \xi) = 0. \tag{3.8}
$$

In consequence of (3.8), the equation (3.7) becomes

$$
g((D_U R)(e_i, Y)\xi, \xi) = -g(R(e_i, Y)\xi, D_U \xi) - g(R(e_i, Y)D_U \xi, \xi).
$$

Using (2.6) in the above equation, we find

$$
\begin{aligned}
g((D_U R)(e_i, Y)\xi, \xi) &= -g(R(e_i, Y)\xi, U) + \eta(U)g(R(e_i, Y)\xi, \xi) \\
&- g(R(e_i, Y)U, \xi) + \eta(U)g(R(e_i, Y)\xi, \xi) \\
&= 0,
\end{aligned}
$$

i.e.,

$$
g((D_U R)(e_i, Y)\xi, \xi) = 0. \tag{3.9}
$$

By using (3.9) in the equation (3.4) we get

$$
(D_U S)(Y, \xi) = \frac{1}{n} dr(U)\eta(Y). \tag{3.10}
$$

Putting $Y = \xi$ in (3.10), we get $dr(U) = 0$.
This implies that r is constant.
So from (3.10), we obtain

$$(D_U S)(Y, \xi) = 0,$$

which implies that

$$S(Y, U) = (1 - n)g(Y, U). \tag{3.11}$$

Hence we can state the following:

Theorem 3.1 : If a Kenmotsu manifold is globally ϕ-M-projectively symmetric, then the manifold is an Einstein manifold.

Next suppose $S(X, Y) = \lambda g(X, Y)$,
that is, the manifold is an Einstein manifold.
Then from (1.1) we have

$$(D_U W^*)(X, Y)Z = (D_U R)(X, Y)Z.$$

Applying ϕ^2 on both sides of the above equation we have

$$\phi^2((D_U W^*)(X, Y)Z) = \phi^2((D_U R)(X, Y)Z).$$

Hence we can state:

Theorem 3.2 : A globally ϕ-M-projectively symmetric Kenmotsu manifold is globally ϕ-symmetric.

Remakr 3.1 : Since a globally ϕ -symmetric Kenmotsu manifold is always a globally ϕ-M-projectively symmetric manifold, from Theorem 3.2, we conclude that on a Kenmotsu manifold, globally ϕ-symmetry and globally ϕ-M-projectively symmetry are equivalent.

4. 3-Dimensional Locally ϕ-M-Projectively Symmetric Kenmotsu Manifolds

Definition 4.1 : A Kenmotsu manifold M^n is said to be locally ϕ-M-projectively symmetric if M-projective curvature tensor W^* satisfies

$$\phi^2((D_U W^*)(X, Y)Z) = 0, \tag{4.1}$$

where X, Y, Z and U are horizontal vectors.

Using (2.11) and (2.12) in (1.1), in a 3-dimensional Kenmotsu manifold the M-projective curvature tensor is

$$\begin{aligned} W^*(X,Y)Z &= (\tfrac{r+6}{4})[g(Y,Z)X - g(X,Z)Y] \\ &- (\tfrac{3r+18}{8})[g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\ &+ \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y]. \end{aligned} \tag{4.2}$$

Taking the covariant differentiation to the both sides of the equation (4.2), we have

$$\begin{aligned} (D_U W^*)(X,Y)Z &= \frac{dr(U)}{4}[g(Y,Z)X - g(X,Z)Y] \\ &- \frac{3}{8}dr(U)[g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\ &+ \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y] \\ \\ &- (\frac{3r+18}{8})[g(Y,Z)(D_U\eta)(X)\xi + g(Y,Z)\eta(X)D_U\xi \\ &- g(X,Z)(D_U\eta)(Y)\xi - g(X,Z)\eta(Y)D_U\xi + (D_U\eta)(Y)\eta(Z)X \\ &+ \eta(Y)(D_U\eta)(Z)X - (D_U\eta)(X)\eta(Z)Y \\ &- \eta(X)(D_U\eta)(Z)Y]. \end{aligned} \tag{4.3}$$

Now assume that X, Y and Z are horizontal vector fields. So the equation (4.3) becomes

$$\begin{aligned} (D_U W^*)(X,Y)Z &= \frac{dr(U)}{4}[g(Y,Z)X - g(X,Z)Y] \\ &- (\frac{3r+18}{8})[g(Y,Z)(D_U\eta)(X)\xi \\ &- g(X,Z)(D_U\eta)(Y)\xi]. \end{aligned} \tag{4.4}$$

Applying ϕ^2 on both sides of (4.4) and making use of (2.1), we obtain

$$\phi^2((D_U W^*)(X,Y)Z) = -\frac{dr(U)}{4}[g(Y,Z)X - g(X,Z)Y]. \tag{4.5}$$

Hence we can state the following:

Theorem 4.1 : A 3-dimensional Kenmotsu manifold is locally ϕ-M-projectively symmetric if and only if the scalar curvature r is constant.

5. ξ-M-Projectively Flat Kenmotsu Manifolds

From (1.1), we obtain

$$\begin{aligned} R(X,Y)\xi &= \tfrac{1}{2(n-1)}\{S(Y,\xi)X - S(X,\xi)Y \\ &+ g(Y,\xi)QX - g(X,\xi)QY\}. \end{aligned} \tag{5.1}$$

Using (2.7), (2.8) in (5.1), we get

$$\eta(Y)QX - \eta(X)QY + (n-1)\{\eta(Y)X - \eta(X)Y\} = 0. \tag{5.2}$$

Putting $Y = \xi$ in (5.2) and using (2.3), we have

$$QX = -(n-1)X. \tag{5.3}$$

Taking inner product with U of (5.3) yields

$$S(X,U) = -(n-1)g(X,U). \tag{5.4}$$

From relation (5.4), we conclude that the manifold is an Einstein manifold.

Conversely , we assume that an n-dimensional Kenmotsu manifold satisfies (5.4). Then we easily obtain from (1.1) that

$$W^*(X,Y)\xi = 0.$$

In view of the above discussions we state the following:

Theorem 5.1 : An n-dimensional Kenmotsu manifold is ξ-M-projectively flat if and only if it is an Einstein manifold.

6. ϕ-M-Projectively Flat Kenmotsu Manifolds

In an n-dimensional almost contact metric manifold, if $\{e_1, ...e_{n-1}, \xi\}$ is a local orthonormal basis of the tangent space of the manifold, then $\{\phi e_1, ...\phi e_{n-1}, \xi\}$ is also a local orthonormal basis . It is easy to verify that

$$\sum_{i=1}^{n-1} g(R(\phi e_i, \phi Y)\phi Z, \phi e_i) = S(\phi Y, \phi Z) + g(\phi Y, \phi Z), \tag{6.1}$$

$$\sum_{i=1}^{n-1} S(\phi e_i, \phi e_i) = r - n + 1, \tag{6.2}$$

$$\sum_{i=1}^{n-1} g(\phi e_i, \phi Z)S(\phi Y, \phi e_i) = S(\phi Y, \phi Z), \tag{6.3}$$

$$\sum_{i=1}^{n-1} g(\phi e_i, \phi e_i) = n - 1, \tag{6.4}$$

and

$$\sum_{i=1}^{n-1} g(\phi e_i, \phi Z)g(\phi Y, \phi e_i) = g(\phi Y, \phi Z). \tag{6.5}$$

Then we have from (1.1) that

$$\begin{aligned} {}'R(\phi X, \phi Y, \phi Z, \phi U) &= \frac{1}{2(n-1)}\{S(\phi Y, \phi Z)g(\phi X, \phi U) \\ &- S(\phi X, \phi Z)g(\phi Y, \phi U) + g(\phi Y, \phi Z)S(\phi X, \phi U) \\ &- g(\phi X, \phi Z)S(\phi Y, \phi U)\}. \end{aligned} \tag{6.6}$$

Let $\{e_1, e_2, ..., e_{n-1}, \xi\}$ be a local orthonormal basis of the tangent space of the manifold. Then $\{\phi e_1, \phi e_2, ..., \phi e_{n-1}, \xi\}$ is also a local orthonormal basis of the tangent space. Putting $X = U = e_i$ in (6.6) and summing up from 1 to (n-1) we have,

$$\begin{aligned} \sum_{i=1}^{n-1}\{{}'R(\phi e_i, \phi Y, \phi Z, \phi e_i)\} &= \frac{1}{2(n-1)}\sum_{i=1}^{n-1}[S(\phi Y, \phi Z)g(\phi e_i, \phi e_i) \\ &- S(\phi e_i, \phi Z)g(\phi Y, \phi e_i) \\ &+ g(\phi Y, \phi Z)S(\phi e_i, \phi e_i) \\ &- g(\phi e_i, \phi Z)S(\phi Y, \phi e_i)]. \end{aligned} \tag{6.7}$$

Using (6.1), (6.2), (6.3) and (6.4) in (6.7), we obtain

$$S(\phi Y, \phi Z) = (\frac{r - 3n + 3}{n + 1})g(\phi Y, \phi Z). \tag{6.8}$$

Replacing Y and Z by ϕY and ϕZ in (6.8) and using (2.1) we have

$$S(Y, Z) = (\frac{r - 3n + 3}{n + 1})g(Y, Z) + (\frac{-r - n^2 + 3n - 2}{n + 1})\eta(Y)\eta(Z). \tag{6.9}$$

Putting $Y = Z = e_i$ in (6.9) and taking summation over $i, 1 \le i \le n$ we get by using (6.4) that

$$r = -(2n^2 - 3n + 1). \tag{6.10}$$

In view of the above discussions we have the following:

Proposition 6.1 : An n-dimensional ϕ-M-projectively flat Kenmotsu manifold is an η- Einstein manifold with constant curvature.

7. Harmonic M-Projective Curvature Tensor on Kenmotsu Manifolds

Let us assume that ξ is a killing vector, then S and r remain invariant under it, i.e.,

$$L_\xi S = 0 \tag{7.1}$$

and

$$L_\xi r = 0, \tag{7.2}$$

where L denotes Lie derivation.

Definition 7.1 : The Riemannian curvature tensor R is harmonic if

$$(divR)(X, Y, Z) = 0. \tag{7.3}$$

Definition 7.2 : A Riemannian manifold M^n is of harmonic M-projective curvature tensor if

$$(divW^*)(X, Y, Z) = 0. \tag{7.4}$$

In a Kenmotsu manifold it is known [6] that

$$\begin{aligned}(divW^*)(X, Y, Z) &= \tfrac{1}{2(n-1)}[(2n-3)\{(D_X S)(Y, Z) - (D_Y S)(X, Z)\} \\ &- \tfrac{1}{2}\{dr(X)g(Y, Z) - dr(Y)g(X, Z)\}].\end{aligned} \tag{7.5}$$

Theorem 7.1 : If a Kenmotsu manifold is of harmonic M-projective curvature tensor and ξ is killing vector, then the manifold is an η -Einstein manifold.

Proof : Let M^n be a Kenmotsu manifold that satisfies $divW^* = 0$.

Then from the equation (7.5) we have

$$(D_X S)(Y, Z) - (D_Y S)(X, Z) = \frac{1}{2(2n-3)}[dr(X)g(Y, Z) - dr(Y)g(X, Z)]. \tag{7.6}$$

From (7.1), it follows that

$$(D_\xi S)(Y, Z) = -S(D_Y \xi, Z) - S(Y, D_Z \xi) \tag{7.7}$$

and from (7.2), we get $dr(\xi) = 0$. Putting $X = \xi$ in (7.6), we obtain

$$\begin{aligned}(D_\xi S)(Y, Z) - (D_Y S)(\xi, Z) &= \tfrac{1}{2(2n-3)}[g(Y, Z)dr(\xi) \\ &- \eta(Z)dr(Y)].\end{aligned} \tag{7.8}$$

Making use of (7.7) in (7.8), we have

$$\begin{aligned} -S(D_Y\xi, Z) - S(Y, D_Z\xi) - (D_Y S)(\xi, Z) &= \tfrac{1}{2(2n-3)}[g(Y,Z)dr(\xi) \\ &- \eta(Z)dr(Y)]. \end{aligned} \tag{7.9}$$

In consequence of $dr(\xi) = 0$, the above equation assume the form

$$-S(Y, D_Z\xi) - D_Y S(\xi, Z) + S(\xi, D_Y Z) = -\frac{1}{2(2n-3)}\eta(Z)dr(Y). \tag{7.10}$$

Using (2.6) and (2.9) in the above, we have

$$\begin{aligned} &-S(Y,Z) + (n-1)g(Y,Z) - 2(n-1)\eta(Y)\eta(Z) \\ &= -\tfrac{1}{2(2n-3)}\eta(Z)dr(Y). \end{aligned} \tag{7.11}$$

Replacing Z by ϕZ in the above equation, we get

$$S(Y, \phi Z) = (n-1)g(Y, \phi Z). \tag{7.12}$$

Again replacing Y by ϕY and using (2.4) and (2.10) the above equation gives

$$S(Y,Z) = (n-1)g(Y,Z) - 2(n-1)\eta(Y)\eta(Z),$$

i.e., the manifold is an η-Einstein manifold.

8. Example of a Locally ϕ-M-Projectively Symmetric Kenmotsu Manifold in 3-Dimension

Example 8.1 : We consider the 3-dimensional manifold $M^3 = \{(x,y,z) \in \mathbb{R}^3, z \neq 0\}$, where (x,y,z) are standard co-ordinate of $\mathbb{R}^3$.

The vector fields

$e_1 = z\frac{\partial}{\partial x}, e_2 = z\frac{\partial}{\partial y}, e_3 = -z\frac{\partial}{\partial z}$

are linearly independent at each point of M^3.

Let g be the Riemannian meric defined by

$g(e_1, e_3) = g(e_1, e_2) = g(e_2, e_3) = 0,$

$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1.$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in TM^n$.

Let ϕ be the (1,1) tensor field defined by

$\phi(e_1) = -e_2, \phi(e_2) = e_1, \phi(e_3) = 0$.
Then using the linearity of ϕ and g , we have

$$\eta(e_3) = 1,$$

$$\phi^2 Z = -Z + \eta(Z)e_3,$$

$$g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),$$

for any $Z, W \in TM^n$. Then for $e_3 = \xi$, the structure (ϕ, ξ, η, g) defines an almost contact metric structure on M^3.
Let D be the Levi-Civita connection with respect to metric g. Then we have

$$\begin{aligned} [e_1, e_3] &= e_1 e_3 - e_3 e_1 \\ &= z\frac{\partial}{\partial x}(-z\frac{\partial}{\partial z}) - (-z\frac{\partial}{\partial z})(z\frac{\partial}{\partial x}) \\ &= -z^2\frac{\partial^2}{\partial x \partial z} + z^2\frac{\partial^2}{\partial z \partial x} + z\frac{\partial}{\partial x} \\ &= e_1. \end{aligned} \tag{8.1}$$

Similarly, $[e_1, e_2] = 0$ and $[e_2, e_3] = e_2$.
The Riemannian connection D of the metric g is given by

$$\begin{aligned} 2g(D_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &- g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]), \end{aligned} \tag{8.2}$$

which is known as Koszul's formula. Using (7.2) we have

$$\begin{aligned} 2g(D_{e_1} e_3, e_1) &= -2g(e_1, -e_1) \\ &= 2g(e_1, e_1). \end{aligned} \tag{8.3}$$

Again by (8.2), we have

$$2g(D_{e_1} e_3, e_2) = 0 = 2g(e_1, e_2) \tag{8.4}$$

and

$$2g(D_{e_1} e_3, e_3) = 0 = 2g(e_1, e_3). \tag{8.5}$$

From (8.3), (8.4) and (8.5), we obtain

$$2g(D_{e_1}e_3, X) = 2g(e_1, X), \tag{8.6}$$

for all $X \in TM^n$. Thus $D_{e_1}e_3 = e_1$. Therefore, (8.2) further yields

$$\begin{aligned} &D_{e_1}e_3 = e_1, D_{e_1}e_2 = 0, D_{e_1}e_1 = -e_3, \\ &D_{e_2}e_3 = e_2, D_{e_2}e_2 = e_3, D_{e_2}e_1 = 0, \\ &D_{e_3}e_3 = 0, D_{e_3}e_2 = 0, D_{e_3}e_1 = 0. \end{aligned} \tag{8.7}$$

From the above it follows that the manifold satisfies
$D_X\xi = X - \eta(X)\xi$, for $\xi = e_3$.
Hence the manifold is a Kenmotsu manifold.

Remark 8.1 : In [9] the authors have shown that the above example shows that a 3-dimensional Kenmotsu manifold is locally ϕ-concircularly symmetric iff the scalar curvature r is constant. Similarly we can show that the above example supports Theorem 4.1.

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