International J. of Math. Sci. & Engg. Appls. (IJMSEA) ISSN 0973-9424, Vol. 9 No. III (September, 2015), pp. 141-152

SEMIRING-VALUED GRAPHS

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Abstract

In algebraic graph theory, the algebraic methods are applied to problems about graphs. The theory of semirings can also be applied to solve certain social network problems. Motivated by this, in this paper, we introduce a new notion called *S*-valued graphs, combining the algebric structure of semiring with that of graphs.

1. Introduction

Algebraic graph theory [1] can be viewed as an extension of graph theory in which algebraic methods are applied to problems about the graphs. The ultimate aim is to translate the properties of graphs into algebraic properties and then using the results and methods of algebra to reduce theorems about graphs.

Key Words : Semirings, Graphs, S-valued graphs, regular S-graphs. AMS Subject Classification : 05C25, 16Y60.

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Semiring theory stands with a foot in each of two mathematical domains. On one hand, semirings are abstract mathematical structures and that study is part of abstract algebra arising initially from the work of Dedekind [2], Krull [4], Macaulay [5] and others, on the theory of ideals of a commutative ring and then the more general work of Vandiver [6]. On the other hand, the moderen interest in semiring arises primarly from the fields of applied mathematics such as optimization theory, the theory of discrete-dynamical systems, automata theory and formal language theory as well as from the allied areas of theoretical computer science, theoretical physics and graph theory. Even though the concept of semiring was first introduced by H. S. Vandiver in 1934, the developments of the theory in semirings and ordered semirings have been taking place since 1950.

Jonathan S. Golan [3] has introduced the notion of S-valued graph where he considers a function $g: V \times V \to S$ such that $g(v_1, v_2) \neq 0$. But nothing more has been dealt. This motivated us to study graphs whose vertices and edges are assigned values from the semiring S. Golan considers the S-valued graph by assigning values to edges only. However we assign values to every vertex of the graph and the weights of an edge is assigned in relation to the weights of the vertices incident with the edges. Since every semiring possesses a canonical pre-order, for any edge $e = (v_i, v_j)$, we can assign the weight of e as the minimum weights of v_i and v_j . Such a graph we call it as a S-valued graph. If S is replaced by I = [0, 1], then the S-valued graph is nothing but a fuzzy graph. Thus the notion of S-valued graph can be considered as a generalization of both the graph theory and the fuzzy graph theory.

In this paper, we introduce the notion of S-valued graph where S is any semiring with a canonical pre-order. We give several examples to illustrate the relation between a S-valued graph and an ordinary (crisp) graph.

2. Preliminaries

In this section, we recall some basic definitions that are needed for our work.

Definition 2.1: A semiring $(S, +, \cdot)$ is an algebraic system with a non-empty set S together with two binary operations + and \cdot such that

- 1. (S, +, 0) is a monoid.
- 2. (S, \cdot) is a semigroup.

- 3. For all $a, b, c \in S$, $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$.
- 4. $0 \cdot x = x \cdot 0 = 0 \forall x \in S$.

Definition 2.2: Let $(S, +, \cdot)$ be a semiring. \leq is said to be a Canonical Pre-order if for $a, b \in S$, $a \leq b$ if and only if there exists $c \in S$ such that a + c = b.

Example 2.3: Let $S = \{0, a, b, c\}$. S is a semiring with the binary operations '+' and '.' defined by the following Cayley tables.

+	0	a	b	c	•	0	a	b	c
0	0	a	b	c	0	0	0	0	0
a	a	a	b	c	a	0	a	a	a
b	b	b	b	c	b	0	a	a	a
с	c	c	c	b	c	0	a	a	a

In S, we define a canonical pre-order \leq :

$$0 \leq 0, \ 0 \leq a, \ 0 \leq b, \ 0 \leq c, \ a \leq a, \ b \leq b, \ c \leq c, \ a \leq b, \ a \leq c, \ b \leq c, \ c \leq b$$

Definition 2.4: A graph consists of a set of objects $V = \{v_1, v_2, \dots\}$ called vertices and another set $E = \{e_1, e_2, \dots\}$ called edges, such that each edge e_k is identified with an unordered pair (v_i, v_j) of vertices. The vertices (v_i, v_j) associated with the edge e_k are called the end vertices of e_k .

Definition 2.5: An edge having the same vertex as both of its end vertices is called a loop. Edges in a graph associated with a given pair of vertices- that is, more than one edge associated with a given pair of vertices are called parallel edges. A graph that has neither self-loop nor parallel edges is called a simple graph.

Definition 2.6 : A vertex u of a graph G is said to be incident with an edge e if u is an end point of e. Any two vertices of a graph which are incident with a common edge are called adjacent vertices.

Definition 2.7: A graph is said to be finite if both of its vertex set and edge set are finite. Empty graph can be defined as a graph in which $V = \phi$ and $E = \phi$. A graph without any edges is called a Null graph. Every vertex in a null graph is an isolated vertex.

Definition 2.8: The degree of a vertex in a graph is defined to be the number of edges incident with that vertex. A graph in which all vertices are of equal degree is called a regular graph.

Definition 2.9: A graph H is said to be a subgraph of a graph G, if all the vertices and all the edges of H are in G, and each edge of H has the same end vertices as in G.

3. Semiring-Valued Graphs

Definition 3.1: Let $G = (V, E \subset V \times V)$ be a given graph with both $V, E \neq \phi$. For any semiring $(S, +, \cdot)$, a semiring-valued graph (or a *S*-valued graph), G^S , is defined to be the graph $G^S = (V, E, \sigma, \psi)$ where $\sigma : V \to S$ and $\psi : E \to S$ is defined to be

$$\psi(x,y) = \begin{cases} \min \left\{ \sigma(x), \sigma(y) \right\} & if \ \sigma(x) \preceq \sigma(y) \text{ or } \sigma(y) \preceq \sigma(x) \\ 0 & otherwise \end{cases}$$

for every unordered pair (x, y) of $E \subset V \times V$. We call σ , a *S*-vertex set and ψ , a *S*-edge set of *S*-valued graph G^S .

Henceforth, we call a S-valued graph simply as a S- graph.

Remark 3.2 : The vertices and edges of G^S are the vertices and edges as in its underlying graph. Since every semiring possess a canonical pre-order, σ and ψ are welldefind. In most general case, both vertices and edges of a S-graph have values in the semiring S, called S-values. If $\sigma(x) = a$ for every $x \in V$, and for some $a \in S$ then S-edges of G^S alone have S-values. If $\psi(x, y) = b$ for every $(x, y) \in E$, and for some $b \in S$ then S-vertices of G^S alone have S-values.

Example 3.3 : Consider the semiring with the canonical pre-order \leq given in Example 2.3. Let G be the underlying graph given by :

G = (V, E) where $V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ and

$$E = \{(v_1, v_2), (v_1, v_4), (v_1, v_6), (v_2, v_3), (v_2, v_5), (v_2, v_7), (v_3, v_6), (v_3, v_5), (v_4, v_7)\}.$$

Corresponding to the graph G, we define the S–graph G^S as follows: Define $\sigma: V \to S$ by

$$\sigma(v_1) = \sigma(v_3) = \sigma(v_5) = \sigma(v_7) = a; \sigma(v_2) = \sigma(v_4) = \sigma(v_6) = b.$$

Therefore the S-vertex set of $G^S = \{a, b\}$. Now define $\psi : E \to S$ as follows

$$\psi(v_1, v_2) = \min \{ \sigma(v_1), \sigma(v_2) \} = \min \{ a, b \} = a(:: a \leq b)$$

Similarly, $\psi(v_1, v_2) = \psi(v_1, v_6) = \psi(v_2, v_3) = \psi(v_2, v_5) = \psi(v_2, v_7) = a$ $\psi(v_3, v_5) = \psi(v_3, v_6) = \psi(v_4, v_7) = a.$ Then the S- edge set of $G^S=\{a\}$. The two graphs are shown below.



Remark 3.4 : Since σ can be defined on V in many ways, the S-graph G^S is not unique for a given underlying graph G.

Definition 3.5 : Let $G^S = (V, E, \sigma, \psi)$ be the *S*-graph corresponding to a given underlying graph G = (V, E). An *S*-graph $H^S = (P, L, \tau, \gamma)$ is called a *S*-subgraph of G^S if H = (P, L) is a subgraph of *G* with $P \subset V$, $L \subset E$, $\tau \subset \sigma$ and $\gamma \subset \psi$. That is,

$$\tau \subset \sigma \Rightarrow \tau(x) \preceq \sigma(x), \ x \in P \ and \ \gamma \subset \psi \Rightarrow \gamma(x,y) \preceq \psi(x,y), \ (x,y) \in L \subset P \times P.$$

Definition 3.6: Let $G^S = (V, E, \sigma, \psi)$ be an *S*-graph and $H^S = (P, L, \tau, \gamma)$ be its *S*-subgraph. H^S is called an *S*- subgraph of G^S induced by *P* if $P \subset V$, $L \subset E, \tau(x) = \sigma(x)$, for every $x \in P$ and $\gamma(x, y) = \psi(x, y)$ for every $(x, y) \in L$.

Remark 3.7: In the *S*-graph G^S , the value of σ on a vertex v will be denoted by $v(\sigma(v))$ and the value of ψ at an edge (v_i, v_j) will be denoted by $\psi(v_i, v_j)$ over the edge (v_i, v_j) .

Example 3.8: G



Consider the semiring with canonical pre-order \leq as given in Example 2.3. Let G = (V, E) where $V = \{v_1, v_2, v_3, v_4, v_5\}$ and

 $E = \{(v_1, v_2), (v_1, v_5), (v_2, v_3), (v_2, v_5), (v_3, v_4), (v_4, v_5)\}.$

Consider the S-graph G^S corresponding to G. Let $P = \{v_2, v_3, v_4\}$ and $L = \{(v_2, v_3), (v_3, v_4)\}$. Let $\tau(v_2) = \tau(v_3) = a$, and $\tau(v_4) = b$. Therefore, $\tau(v_2) = a \preceq \sigma(v_2) = a$, $\tau(v_3) = a \preceq \sigma(v_3) = b$ and $\tau(v_4) = b \preceq \sigma(v_4) = b$. Hence $\tau \subset \sigma$. Now define $\gamma : E \to S$ as follows

$$\gamma(v_2, v_3) = \min \{\tau(v_2), \tau(v_3)\} = \min \{a, a\} = a \quad (\because a \preceq a)$$

Similarly, $\gamma(v_2, v_3) = a \preceq \psi(v_2, v_3) = a$; $\gamma(v_3, v_4) = a \preceq b = \psi(v_3, v_4)$ Therefore $\gamma \subset \psi$.

Thus the S- subgraph $H^S = (P, L, \tau, \gamma)$ is given by



For the S-graph G^S be given in this example, define $\gamma(x) = \sigma(x), \forall x \in P$ and $\tau(x, y) = \psi(x, y), \forall * / ll(x, y) \in L$. Thus the S-subgraph $H^S(P)$ is the subgraph of G^S induced by $P = \{v_1, v_2, v_3, v_4\}$.

Remark 3.9: From the above example we observe that H = (P, L) is a subgraph of G = (V, E) but the S-subgraph H^S need not have the same S-values as in G^S .

Definition 3.10: Let $G^S = (V, E, \sigma, \psi)$ be an S- graph and $H^S = (P, L, \tau, \gamma)$ be its S-subgraph. H^S is called a spanning S-subgraph of G^S if $P = V, L \subset E, \tau(x) = \sigma(x)$, for every $x \in P$ and $\gamma(x, y) = \psi(x, y)$, for every $(x, y) \in L$.

Example 3.11: Consider the *S*-graph G^S given in Example 3.8. Let P = V; $L = \{v_1v_2, v_2v_3, v_3v_4\} \subset E$. Define $\tau(v_1) = \sigma(v_1)$; $\tau(v_2) = \sigma(v_2)$; $\tau(v_3) = \sigma(v_3)$; $\tau(v_4) = \sigma(v_4)$; and $\gamma(v_1, v_2) = \psi(v_1, v_2), \gamma(v_2, v_3) = \psi(v_2, v_3), \gamma(v_3, v_4) = \psi(v_3, v_4)$. Then the *S*- subgraph H^S is,



So far, corresponding to a given crisp graph G, and a given semiring S we have obtained a S- graph G^S . In the following we discuss the method to obtain a crisp graph from a given S- graph.

Definition 3.12: Let $G^S = (V, E, \sigma, \psi)$ be an S-graph where $(S, +, \cdot)$ is a semiring with \preceq . For any $t \in S$, $G_t = (\sigma^t, \psi^t)$ is a crisp graph with the vertex set $\sigma^t = \{x \in V/t \preceq \sigma(x)\}$ and the edge set $\psi^t = \{(x, y) \in E/t \preceq \psi(x, y)\}$.

Example 3.13 : Let $(S = \{0, a, b, c\}, +, \cdot)$ be a semiring with the following Cayley tables:

+	0	a	b	c	•	0	a	b	c
0	0	a	b	c	0	0	0	0	0
a	a	b	c	c	a	0	a	b	c
b	b	c	c	c	b	0	b	c	c
c	c	c	c	c	c	0	c	c	c

Let \leq be a canonical pre-order in S, given by

$$0 \leq 0, \ 0 \leq a, \ 0 \leq b, \ 0 \leq c, \ a \leq a, \ a \leq b, \ a \leq c, \ b \leq b, \ b \leq c, \ c \leq c$$

Consider the S- graph G^S , where $\sigma: V \to S$ is defined by

$$\sigma(v_1) = \sigma(v_4) = a; \sigma(v_2) = \sigma(v_5) = b \text{ and } \sigma(v_3) = c$$

and $\psi: E \to S$ by $\psi(v_2, v_3) = \psi(v_2, v_5) = \psi(v_3, v_5) = b$ and

$$\psi(v_1, v_2) = \psi(v_1, v_3) = \psi(v_1, v_4) = \psi(v_1, v_5) = \psi(v_2, v_4) = \psi(v_3, v_4) = a$$



Suppose t = b. Then

$$\sigma^{b} = \{ v \in V/b \le \sigma(v) \} \Rightarrow \sigma^{b} = \{ v_{2}, v_{3}, v_{5} \}$$
$$\psi^{b} = \{ (x, y) \in V \times V / t \le \psi(x, y) \} \Rightarrow \psi^{b} = \{ (v_{2}, v_{3}), (v_{2}, v_{5}), (v_{3}, v_{5}) \}$$

Therefore the required crisp graph G_b , corresponding to G^S is



Remark 3.14: Clearly $\sigma^t \subset V$ and $\psi^t \subset E$. Suppose there is no other element in S which is related to t. Then $\sigma^t = \{x \in V \mid t \preceq \sigma(x) = t\}$ and $\psi^t = \{(x, y) \in E \mid t \preceq \psi(x, y) = t\}$. We call $G_t = (\sigma^t, \psi^t)$ as a level graph of the S-graph G^S . Two cases arise.

Case 1: If there is no edge whose incident vertices having S-values as t then ψ^t is empty. We call G_t , a null graph.

Case 2: If there is no vertex x that assigns S-value as t, then both σ^t and ψ^t are empty. In this case, we call G_t , an empty graph. We rule out this possibility. **Theorem 3.15**: Let G^S be a S- graph. If a S-graph H^S is a S-subgraph of G^S then H_a is a subgraph of G_a , for any $a \in S$.

Proof: Let $G^S = (V, E, \sigma, \psi)$ be a given S graph. Let $H^S = (P, L, \tau, \gamma)$ be a S subgraph of G^S .

Therefore $P \subset V, L \subset E, \tau \subset \sigma$ and $\gamma \subset \psi$ where $\tau \subset \sigma \Rightarrow \tau(v) \preceq \sigma(v)$, for all $v \in P$, and $\gamma \subset \psi \Rightarrow \gamma(v_i, v_j) \preceq \psi(v_i, v_j)$ for all $(v_i, v_j) \in L$. Let $a \in S$. Let $H_a = (\tau^a, \gamma^a)$ where $\tau^a = \{v \in P \mid a \preceq \tau(v)\}; \text{ and } \gamma^a = \{(v_i, v_j) \in L \mid a \preceq \gamma(v_i, v_j)\}$

And $G_a = (\sigma^a, \psi^a)$ where

$$\sigma^a = \{ v \in V/a \preceq \sigma(v) \}; \text{ and } \psi^a = \{ (v_i, v_j) \in E/a \preceq \psi(v_i, v_j) \}$$

Claim : H_a is a subgraph of G_a .

Let $v \in \tau^a \Rightarrow a \preceq \tau(v) \preceq \sigma(v) \Rightarrow a \preceq \sigma(v) \Rightarrow v \in \sigma^a$. Therefore

$$\tau^a \subset \sigma^a \tag{3.1}$$

For, $(v_i, v_j) \in \gamma^a, a \preceq \gamma(v_i, v_j) \preceq \psi(v_i, v_j) \Rightarrow (v_i, v_j) \in \psi^a$. Therefore

$$\gamma^a \subset \psi^a \tag{3.2}$$

From (3.1) and (3.2), we conclude that H_a is a subgraph of G_a .

Remark 3.16 : Since for any crisp graph H_a , a subgraph of G_a , the existance of their S-graphs are not unique, H^S need not be a S subgraph of G^S .

Definition 3.17: If $\sigma(x) = a, \forall x \in V$ and some $a \in S$ then the corresponding S-graph G^S is called a vertex regular S-graph (or simply vertex regular).

Definition 3.18: An S- graph G^S is said to be an edge regular S-graph (or simply edge regular) if $\psi(x, y) = a$ for every $(x, y) \in E$ and some $a \in S$.

Example 3.19 : Consider the *S*-graph G_1^S as in Example 3.3, where $\sigma(v_i) = a, \forall v_i \in V$.



Here G_1^S is a vertex regular S- graph and G^S is an edge regular S- graph but not a vertex regular S- graph.

Definition 3.20: An S- graph G^s is said to be S-regular if it is both a vertex regular and an edge regular, S- graph.

Example 3.21 : Consider the S- graph G^S given in Example 3.3.



Here $\sigma(v_i) = a$, for every $v_i \in V$ and $\psi(v_i, v_j) = a$, for every $(v_i, v_j) \in E$.

Then G^S is both vertex and edge regular and hence an S- regular graph.

We present here an example of a S- graph, which is neither a vertex regular nor an edge regular S- graph.

Example 3.22 : Let $(S = \{0, a, b, c\}, +, \cdot)$ be a semiring with the following Cayley tables:

+	0	a	b	c	•	0	a	b	
0	0	a	b	c	0	0	0	0	(
a	a	b	c	c	a	0	a	b	
b	b	c	c	c	b	0	b	c	
c	c	c	c	c	c	0	c	c	

Let \leq be a canonical pre-order in S given by :

 $0 \leq 0, \ 0 \leq a, \ 0 \leq b, \ 0 \leq c, \ a \leq a, \ b \leq b, \ c \leq c, \ a \leq b, \ a \leq c, \ b \leq c$

Let G = (V, E) be the given graph with $V = \{v_1, v_2, v_3, v_4, v_5\}$ and $E = \{(v_1, v_2), (v_1, v_3), (v_2, v_4), (v_3, v_5), (v_3, v_4), (v_4, v_5)\}$. Then



The S-graph G^S given above is neither a vertex regular nor an edge regular S-graph. Lemma 3.23 : Every vertex regular S- graph is an edge regular S- graph. Proof : Let $G^s = (V, E, \sigma, \psi)$ be a vertex regular S- graph. That is $\sigma(v_i) = a$ for every $v_i \in V$ and some $a \in S$.

Let $(v_i, v_j) \in E$ be arbitrary. Then

$$\psi(v_i, v_j) = \min \left\{ \sigma(v_i), \sigma(v_j) \right\} = \min \left\{ a, a \right\} = a \quad (\because a \preceq a)$$

Therefore $\psi(v_i, v_j) = a$ for every $(v_i, v_j) \in E$, proving that G^s is an edge regular S-graph.

Remark 3.24 : Converse of the above lemma is not true. That is every edge regular S-graph is not a vertex regular S-graph as seen from Example 3.19.

Theorem 3.25 : G^s is S-regular graph iff G^s is vertex regular.

Proof : Let G^s be S-regular. By definition of S-regular graph, we get G^s is vertex regular graph.

Conversely, assume that G^s is a vertex regular S- graph.

Then by lemma 3.23 G^s is an edge regular S-graph.

Thus G^s is an S-regular graph.

Remark 3.26 : In the crisp graph theory, regularity is defined using the notion of vertex degree. But in our work the regularity is defined in terms of S-values corresponding to the vertex set defined by σ , and the edge set defined by ψ .

4. Conclusion

Borrowing the idea from Golan [3], we have introduced the notion of a semiring-valued graph in this paper. This generalizes the notions of both the crisp and fuzzy graph theory. In our future work we will analyze the notion of regularity of

S- graphs including the degree of a vertex as in the case of crisp graph theory.

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