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PSEUDO INJECTIVE AND PSEUDO QP-INJECTIVE S-SYSTEMS OVER MONOIDS

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Abstract

The concept of quasi injective S-systems over monoids is generalized to pseudo injective by Yan and QP-injective S-systems by the authors. A new kind of generalization, namely pseudo quasi principally injective systems over monoids is studied. On the way, we complete an early result by Yan on pseudo injective S-systems and obtained analogous properties to that notion of pseudo injectivity on the module theory. Several properties of these kinds of generalizations are discussed. Conditions under which pseudo injective S-systems being quasi-injective are considered. Also, we obtain characterizations of pseudo quasi principally injective S-systems. The relationship between the classes of pseudo injective S-systems with quasi injective S-systems and pseudo quasi principally injective S-systems with quasi principally injective S-systems are considered . As a consequence, conditions to versus these classes are shown.

1. Introduction and Preliminaries

In [6], we introduced a generalization of quasi injective S-systems which was quasi principally injective S-systems (QP-injective) and obtained some results. Among these

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Key Words : *Pseudo Injective S-systems, Pseudo quasi principally injective S-systems, Quasi-projective S-systems, Quasi principally injective S-systems, Pseudo MP-injective S-systems.*

results, we exhibited some conditions to versus this generalization with other generalization relevant to quasi injective S-systems which also introduced by us for example principally quasi injective S-systems (PQ-injective) [and hence versus with quasi injective S-systems]. More generally, in this work, we continue to find another weak form of quasi injectivity called pseudo injective S-systems and pseudo QP-injective to study behavior of quasi injective through the property, "an S-system M_s is quasi injective if and only if it is invariant in injective envelope of itself"' which is satisfy in pseudo injective S-system for every S-monomorphism from injective envelope of S-system M_s to itself. Thus, we can connect quasi injective by their generalization pseudo injective by adding this property as a condition for pseudo injective S-systems to be quasi injective in Theorem 2.17, when we define cog-reversible S-system . On the other hand, characterizations and properties of quasi injective S-systems and QP-injective S-system also satisfy by pseudo QP-injective S-systems for example Proposition 3.5, Proposition 3.10, Theorem 3.14 and Theorem 3.15. Note that, we will use terminology and notations from [6] freely.

Let M_s, N_s be right S-systems. An S-system E is called **injective** if for every S-monomorphism $f : M_s \to N_s$ and every S-homomorphism $g : M_s \to E$, there is an S-homomorphism $h : N_s \to E$ such that $hf = g$ [8]. A right S-system N_s is called **M_s-injective** if for each S-monomorphism f from S-system B_s into S-system M_s and every S-homomorphism $g : B_s \to N_s$, there is an S-homomorphism $h : M_s \to N_s$ such that $hf = g$. Thus N_s is **injective** if and only if N_s is M_s-injective for all S-system M_s [11]. The concept of injectivity was generalized to quasi injective S-system by A. M. Lopez [1], such that an S-system N_s is **quasi injective** if and only if N_s is N_s-injective. More generally, Yan gave generalized quasi injective S-system to **pseudo injective**, such that an S-system M_s is called pseudo-injective if each S-monomorphism of a subsystem of M_s into M_s extends to an S-endomorphism of M_s [10]. An S-system M_s is called principal injective S-system (simply, **C-injective**) if for any S-system B_s, any principal (cyclic) subsystem C of B_s, any homomorphism f from C into M_s can be extended to one from B_s to M_s [7]. An S-system M_s is called **principally quasi injective** (this means PQ-injective) if every S-homomorphism from a principal subsystem of M_s to M_s extends to an S-endomorphism of M_s [5]. An S-system N_s is called **M-principally injective** if for every S-homomorphism from M-cyclic subsystem of M_s into N_s can be extended to an S-homomorphism from M_s into N_s (for short N_s is MP**-injective**) [6]. An S-system M_s is called **quasi-principally injective** if it is MP-injective, that is every S-homomorphism from M-cyclic subsystem of M_s to M_s can be extended to S-endomorphism of M_s (M_s is QP-injective) [6].

This paper is subdivided into two parts. The first part is devoted to pseudo injective S-systems. We obtained a characterization of this class analogous to that of pseudo injective module. Certain class of subsystems which inherit this property are considered. In the second part, a characterizations of pseudo quasi principally injective S-systems over monoids are investigated. The relationship between the classes of pseudo quasi principally injective S-systems with quasi principally injective S-systems are considered. As a consequence , conditions to versus these classes are shown.

2. Pseudo Injective S-systems

In [10], Yan gave definition of pseudo injective S-systems and studied the properties of linear equation S-system on this class. By the following, we give a general case :

Definition 2.1 : Let M_s, N_s be S-systems. N_s is M-pseudo injective if for every S-subsystem A of M_s, each S-monomorphism $f : A \to N_s$ can be extended to an S-homomorphism $g : M_s \to N_s$. An S-system N_s is called pseudo injective if it is N-pseudo injective.

Remarks and Examples 2.2 :

(1) Every quasi injective S-system is pseudo injective. But the converse is not true in general, for example, let S be a monoid such that $S = \{a, b, c, 0, e\}$, with a, b be left zero of S and $ca = cb = cc = a$ and 0, e be zero, identity elements of S respectively. Then consider S as an S-system over itself. It is clear that every subset of S is subsystem of S_s. Since the only S-monomorphism from subsystems is the inclusion map can be trivially extended to identity map of S_s, so S_s is S-pseudo injective system, but when we take $N = \{a, b\}$ be subsystem of S_s and f be S-homomorphism defined by $f(x) = \left\{ \begin{array}{ccc} b & \text{if} & x = a \\ a & \text{if} & x = b \end{array} \right\}$, then this S-homomorphism cannot be extended to S-homomorphism $g : S_s \to S_s$. If not, that is there exists S-homomorphism $g : S_s \to S_s$ such that $g(x) = f(x), \;\; \forall\, x \in N$, which is the trivial S-homomorphism (or zero map) since other extension is not S-homomorphism. Then, $b = f(a) = g(a) = a(0)$ which implies that $b = a(0)$ and

this is a contradiction.

(2) Let M_s, N_s, W_s be S-systems. If N_s is M_s-pseudo injective and $M_s \cong W_s$, then it is easy to see that N_s is W_s-pseudo injective system. Also, every isomorphic S-system to M_s-pseudo injective system is M_s-pseudo injective system.

Proposition 2.3 : Let M_s and N_s be S-systems. Then :

(1) If N_s is M_s-pseudo injective, then any S-monomorphism $f : N_s \to M_s$ splits.

(2) N_s is injective S-system if and only if N_s is M_s-pseudo injective for all M_s.

Proof : (1) It is clear that N_s is isomorphic to $f(N)$, so $f(N)$ is M_s-pseudo injectivity. (2) By (1), if N_s is M_s-pseudo injective for all M_s, then every S-monomorphism $f : N_s \to M_s$ splits for all S-systems M_s, hence N_s is injective.

Proposition 2.4 : Every M-pseudo injective S-system is A-pseudo injective for any subsystem A of M_s.

Proof : Let X be a subsystem of A in M_s and f be S-monomorphism from X into N. Then, since N is M_s-pseudo injective, so there exists S-homomorphism $g : M_s \to N$ which extends f. Consider the diagram (1), where $i_1(i_2)$ be the inclusion map of $X(A)$ in $A(M_s)$. Then, we have $goi_2oi_1 = f$. Now, put $g'(= g|_A) : A \to N_s$ be S-homomorphism which extends f also. Hence, N_s is A-pseudo injective.

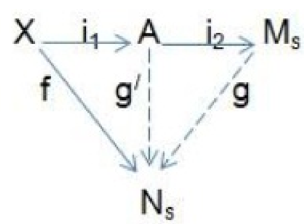

Diagram (1)

Lemma 2.5 : Every retract of M_s-pseudo injective S-system is M_s-pseudo injective.

Proof : Assume that N_s is M_s-pseudo injective S-system, and A be a retract of N_s, so there a subsystem W of N_s and S-epimorphism $\alpha : N_s \to W$ such that $A \cong W$ and $\alpha|_w = i_w$. This means $\alpha(w) = w,\ \forall\, w \in W$. Thus, we have S-epimorphism $\alpha : N_s \to A$ such that $\alpha(a) = a, \quad \forall\ a \in A$. Let X be a subsystem of M_s and $f : X \to A$ be

S-monomorphism. Define $g : X \to N_s$ by $g(x) = (f(x), 0)$, $\forall\ x \in X$. This means g is S-monomorphism [in fact, if $g(x_1) = g(x_2)$], this implies $(f(x_1), 0) = (f(x_2), 0)$, so $f(x_1) = f(x_2)$. Since f is S-monomorphism, so $x_1 = x_2$, thus g is S-monomorphism. Since N_s is M_s-pseudo injective, so there exists S-homomorphism $g' : M_s \to N_s$ such that $g' o i_x = g$. Let j and π be the injection and projection map of A into N_s (and N_s onto A). Now, define $h(= \pi o g') : M_s \to A$ be S-homomorphism such that $h o i_x = \pi o g' i_x = \pi o g = f$, so $h o i_x = f$. This means h extends f and A is M_s-pseudo injective.

Before the next proposition, we need the following lemma :

Lemma 2.6 [2] : Let M_s and N_s be S-systems and $\varphi \in Hom(M_s, N_s)$. If A_s is intersection large in N_s, then $\varphi^{-1}(A_s)$ is intersection large in M_s. (In particular, if N is intersection large in M, then for each $m \in M_s$, $[N, m] = \{s \in S | ms \in N\}$ is intersection large right ideal in S_s).

Proposition 2.7 : If an S-system N_s is M_s-pseudo injective with $\psi_M = I_M$, then $\alpha(M_) \subseteq N_s$ for every S-monomorphism $\alpha : E(M_s) \to E(N_s)$.. In particular, if H_s is pseudo injective with $\psi_H = I_H$, then $\alpha(H_s) \subseteq H_s$ for every S-monomorphism $\alpha \in End(E(H_s))$.

Proof : Let N_s be M_s-pseudo injective and α be S-monomorphism from $E(M)$ into $E(N)$. Define $X = \{m \in M_s | \alpha(m) \in N_s\}$. Since N_s is M_s-pseudo injective, so $\alpha|_X$ can be extended to $\beta : M_s \to N_s$. Since $E(N)$ is $E(M)$-injective, so $E(N)$ is M_s-injective by Proposition 2.4. This means, there exists S-homomorphism $h : M_s \to E(N)$ which extend $\alpha|_X$. The proof is complete, when $\beta(M) = h(M)$. Assume that $\beta(m_0) \neq h(m_0)$ for some $m_0 \in M_s$. Since N is is essential in $E(N)$ and $\Theta \neq h(m_0) \in E(N)$, so there exists $s \in S$, such that $\Theta \neq h(m_0)s \in N$ Thus $h(m_0 s) \in N$ implies that $m_0 s \in X$. On the other hand, $\beta(m_0)s = \beta(m_0 s) \in N$. Note that, since N_s is $\cap$-large in $E(N)$, so $[N, h(m_0)]$ is $\cap$-large right ideal in S_s by Lemma 2.6. Thus, for $h(m_0)\psi_M \beta(m_0)$, and since $\psi_M = I_M$, we have $h(m_0) = \beta(m_0)$ and this is a contradiction. Hence, $h(M) = \beta(M) \subseteq N_s$. Since $h(M) = \alpha(M)$, then this implies that $\alpha(M) = \beta(M) \subseteq N_s$.

Let $T = Hom_s(M, M)$. Recall that M_s is $a(T, S)$-bisystem if it is a right S-system and a left T-system such that $f(ms) = (fm)s$ for $f \in T, m \in M_s$ and $s \in S$.

The following corollary from above proposition and lemma in [1] which is :

Lemma 2.8 [1] : Let $H = Hom_s(E(M), E(M))$. If M_s is an (H, S)-bisubsystem of

$E(M)$, then M_s is quasi injective.

Definition 2.9 : Let M_s be an S-system. A congruence ρ on M_s is called large on M_s, if for every congruence α on M_s with $\alpha \neq I_M$ (the trivial congruence) we have $\alpha \cap \rho \neq I_M$.

Examples 2.10 :

(1) The universal congruence on S-system M_s is clearly large on M_s.

(2) Let the semigroup $S = \{a, b, c, d, e\}$ with the multiplication given by : $a^2 = b$, $ab = ac = ad = ba = ca = da = a$, $b^2 = c^2 = bc = cb = bd = db = cd = dc = b$, $d^2 = d$ and e is the identity element. Then, consider the congruence ρ on S defined by : $\rho = \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c), (d, d), (e, e)\}$. It is a matter of calculations, ρ is the non-trivial congruence on S_s. So ρ is a large congruence on S_s.

(3) Let the semigroup $S = \{a, b, c, d, e\}$ defined by : a is zero element of S, $b^2 = bc = cb = db = bd = be = eb = b$, $c^2 = cd = dc = ce = ec = c$, $d^2 = d$, $de = ed = e^2 = e$ and σ the congruence on S defined by : $\sigma = \{(a, a), (b, b), (c, c), (c, e), (d, d), (e, c), (e, e)\}$. Then, it is easy to check that σ is not large congruence on S.

As the intersection of congruence's on M_s is again congruence on M_s, let X be a subset of $M_s \times M_s$. Denote by $\rho(X)$ the smallest congruence on M_s containing X. A congruence ρ is called finitely generated, if there is a finite subset X such that $\rho = \rho(X)$. A congruence ρ is called monocyclic, if it is finitely generated by one element $(x, y) \in M \times M$ and is denoted by $\rho(x, y)$.

In the following we describe the monocyclic congruence $\rho(x, y)$ interms of its elements.

Proposition 2.11 : Let M_s be an S-system with $x, y \in M_s$. Then

$$\rho(x, y) = \bigcup_{n=1}^{\infty} (x, y)S^n = \bigcup_{n=1}^{\infty} (xS^n, yS^n) = \{(x, y)s_1 s_2 \cdots s_n | n \geq 1, s_i \in S\}.$$

Proof : Write $A = \bigcup_{n=1}^{\infty} (x, y)S^n = \{(x, y)s_1 s_2 \cdots s_n | n \geq 1, s_i \in S\}$. It is clear that A contains (x, y). If $(x, y)s_1 s_2 \cdots s_n \in A$ and $s \in S$, then $(x, y)s_1 s_2 \cdots s_n s \in A$, this implies that A is a congruence on M_s. If ω is a congruence on M_s containing (x, y) , then $A \subseteq \omega$. Thus A is the smallest congruence on M_s containing (x, y) and so $A = \rho(x, y)$.

It is clear that $\{(x,y)s_1s_2\cdots s_n|n\geq 1, s_i\in S\}=\{(x,y)s|s\in S\}$, so we shall consider $\rho(x,y)=\{(x,y)s|s\in S\}$.

Proposition 2.12 : Let M_s be an S-system. A congruence ρ on M_s is large if and only if $\rho(x,y)\cap\rho\neq I_M$ for each $x,y\in M_s$ with $x\neq y$.

Proof : As $x\neq y$, then $\rho(x,y)\neq I_M$ and hence $\rho(x,y)\cap\rho\neq I_M$ for all $x,y\in M_s$. Conversely , let $\alpha\neq I_M$ be a congruence on M_s, then $x,y\in M_s$ with $x\neq y$. By the condition $I_M\neq\rho(x,y)\cap\rho\subseteq\alpha\cap\rho$.

Proposition 2.13 : Let M_s be an S-system. A congruence ρ on M_s is large if and only if for each $x,y\in M_s$ with $x\neq y$, there exists an element $s\in S$ such that $xs\neq ys$ and $(x,y)s\in\rho$.

Proof : Assume that ρ is large on M_s. If $x,y\in M_s$ with $x\neq y$, then $\rho(x,y)\neq I_M$ and hence by Proposition 2.12, $\rho(x,y)\cap\rho\neq I_M$, so there exists $s\in S$ such that $xs=ys$ and $(x,y)s\in\rho$. Conversely, for each $x,y\in M$ with $x\neq y$, by the condition, there is an element $s\in S$ such that $xs\neq ys$ and $(x,y)s\in\rho$, this means that $\rho(x,y)\cap\rho\neq I_M$. Again Proposition 2.12 implies that ρ is large on M_s.

Proposition 2.14 : Let $\alpha: M_s\to N_s$ be an S-homomorphism. If ρ is a large congruence on N_s, then $\alpha^{-1}(\rho)$ is a large congruence on M_s, where $\alpha^{-1}(\rho)=\{(x,y)\in M\times M|(\alpha(x),\alpha(y))\in\rho\}$.

Proof : It is that $\alpha^{-1}(\rho)$ is an equivalence relation on M_s. By the definition of S-homomorphism, we have $\alpha^{-1}(\rho)$ is a congruence on M_s. For any $x,y\in M_s$ with $x\neq y$, then either $\alpha(x)=\alpha(y)$ or $\alpha(x)\neq\alpha(y)$. If $\alpha(x)=\alpha(y)$, then $(\alpha(x),\alpha(y))\in\rho$ and hence $(x,y)\in\alpha^{-1}(\rho)$, this implies that $\rho(x,y)\cap\alpha^{-1}(\rho)\neq I_M$ and hence by proposition 2.12, $\alpha^{-1}(\rho)$ is large on M_s. If $\alpha(x)\neq\alpha(y)$, then by Proposition 2.13 there exists $s\in S$ such that $\alpha(x)s\neq\alpha(y)s$ and $(\alpha(x)s,\alpha(y)s)\in\rho$, so $xs\neq ys$ and $(x,y)s\in\alpha^{-1}(\rho)$ and hence $\alpha^{-1}(\rho)$ is large on M_s.

Theorem 2.15 : Let M_s be a nonsingular S-system with $\ell_M(s)=\Theta$ for each $s\in S$. Then a congruence ρ on M_s is large if and only if M_s/ρ is singular.

Proof : Assume that M_s/ρ is singular. For each $x,y\in M_s$ with $x\neq y$, then there exists a large ideal I of S such that $(\overline{x},\overline{y})I=\Theta$, that is $(xI,yI)\subseteq\rho$. Since M_s is nonsingular, then both xI and yI are nonzero and distinct, but it is easy to show that (xI,yI) is a congruence on M_s and $I_M\neq(xI,yI)\subseteq(xI,yI)\cap\rho\subseteq(xS,yS)\cap\rho\subseteq\rho(x,y)\cap\rho$. By

Proposition 2.12, we have ρ is a large on M_s. Conversely, let $(\overline{x},\overline{y}) \in M_s/\rho \times M_s/\rho$, then $(x,y) \in M_s \times M_s$. Define $f,g : S_s \to M_s$ by $f(s) = xs$ and $g(s) = ys$ for $s \in S$. Then f and g are S-homomorphism. Proposition 2.14 implies that $f^{-1}(\rho)$ and $g^{-1}(\rho)$ are large congruences on S and hence $f^{-1}(\rho) \cap g^{-1}(\rho)$ is a large congruence on S, where $f^{-1}(\rho) = \{(s,t) \in S \times S | (xs, xt) \in \rho\}$. Define $I = \{s \in S | (xs, ys) \in \rho\}$. Then I is a right ideal of S. Consider J is a nonzero right ideal of S. Let $u\ (\neq \Theta) \in J$. The condition $\ell_M(S) = \Theta$ implies that $xu \neq yu$, since ρ is large on M_s, then by Proposition 2.13, there exists an element s in S such that $xus \neq yus$ and $(xus, yus) \in \rho$ and hence $us(\neq \Theta) \in I \cap J$. This shows that I is a large right ideal of S. Thus $(x,y)I \subseteq \rho = ker(\rho^{\#})$ where $\rho^{\#}$ is the natural epimorphism of M_s onto $(M/\rho)s$ and $(\overline{x},\overline{y})I = \Theta$. So $(\overline{x},\overline{y}) \in \psi_{M/\rho}$ and hence $M_s/\rho \times M_s/\rho = \psi_{M/\rho'}$ this implies that M_s/ρ is singular.

Definition 2.16 : An S-system M_s is called cog-reversible if each congruence ρ on M_s with $\rho \neq I_M$ is large on M_s.

For example Z_Z and Q_Z are cog-reversible Z-systems. As every congruence ρ on Z_z (and Q_Z) with $\rho \neq I_Z$ (and $\rho \neq I_Q$) is large on Z_z (and Q_Z).

Theorem 2.17 : Let M_s be a cog-reversible nonsingular S-system with $\ell_M(s) = \Theta$ for each $s \in S$. Then M_s is pseudo injective if and only if M_s is quasi injective.

Proof : Let A be a subsystem of an S-system M_s and f be a nonzero S-homomorphism from A into M_s. If f is S-monomorphism, then there is nothing to prove. So assume f is not S-monomorphism. Since $E(M)$ is injective, then $E(M)$ is an M (respectively $E(M)$)-injective. Thus there is S-homomorphism $h : M_s \to E(M)$ such that $ho\omega_A = \omega_M of$, where ω_A (respectively ω_M) is the inclusion mapping of A (respectively M_s) into M_s (respectively $E(M)$). Again there is an S-homomorphism $g : E(M) \to E(M)$ such that $go\omega_M = h$. Then either $ker(h) = I_M$ or $ker(h) \neq I_M$. If $ker(h) = I_M$, then h is S-monomorphism. Largeness of M_s in $E(M)$ implies that g is S-monomorphism, so $g(M_s) \subseteq M_s$ by Proposition 2.7. Thus, $h(M_s) \subseteq M_s$ which is extension of f, since $h(A) = ho\omega_A(A) = \omega_M of(A) = f(A)$. If $ker(h) \neq I_M$, then $ker(h)$ is large on M_s, so Theorem 2.16 implies that $M_s/ker(h)$ is singular. But $M_s/ker(h) \cong h(M) \subseteq M_s$, so $M_s/ker(h)$ is nonsingular. This two cases implies that $ker(h) = M \times M$. This implies that h (and hence f) is zero map.

Recall that an S-systems A_s and B_s are called mutually (pseudo) injective if A_s is B_s-(pseudo) injective and B_s is A_s-(pseudo) injective.

Proposition 2.18 : Let A_s and B_s be mutually pseudo injective S-systems, with $\psi_A = i_A$ and $\psi_B = i_B$. If $E(A_s) \cong E(B_s)$, then every S-isomorphism $\alpha : E(A_s) \to E(B_s)$ reduces to an S-isomorphism $\alpha' : A_s \to B_s$. In particular, $A_s \cong B_s$, consequently, A_s and B_s are pseudo injective S-systems.

Proof : Let $f : E(A_s) \to E(B_s)$ be an S-isomorphism. Since $\psi_A = i_A$, so by proposition 2.7 $f(A_s) \subseteq B_s$, similarly, since $f^{-1} : E(B_s) \to E(A_s)$ be an S-isomorphism and $\psi_B = i_B$, so by Proposition 2.7 $f^{-1}(B_s) \subseteq A$. Thus, $B_s = (ff^{-1})(B_s) = f(f^{-1}(B_s)) \subseteq f(A_s) \subseteq B_s$. Hence $f(A_s) = B_s$. Therefore, $f|_A : A_s \to B_s$ is an S-isomorphism, so $A_s \cong B_s$. Moreover, as A_s is B_s-pseudo injective and $B_s \cong A_s$, we have A_s is A_s-pseudo injective. This means A_s is pseudo injective.

For more properties of pseudo injective S-systems, we have :

Theorem 2.19 : Let M_1 and M_2 be S-systems. If $M_1 \oplus M_2$ is pseudo injective, then M_1 and M_2 are mutually injective.

Proof : Let A be a subsystem of M_2 ,and $f : A \to M_1$ be an S-homomorphism. Define $\alpha : A \to M_1 \oplus M_2$ by $\alpha(a) = (f(a), a), \quad \forall\, a \in A$, then α is S-monomorphism. By proposition 2.4, $M_1 \oplus M_2$ is M_2-pseudo injective, so there exists S-homomorphism $\beta : M_2 \to M_1 \oplus M_2$ such that $\beta o i = \alpha$. Now, let j_1 and π_1 be the injection and projection map of M_1 into $M_1 \oplus M_2$ and $M_1 \oplus M_2$ onto M_1. Then, define $\sigma\ (= \pi_1\beta) : M_2 - M_1$ be S-homomophism extends f, this means $\sigma i = \pi_1 \beta i = \pi_1 j_1 f = I_{M_1} f = f$, which implies $\sigma i = f$.

Corollary 2.20 : If $\oplus_{i\in I} M_i$ is pseudo injective, then M_j is M_K-injective for all distinct $j, k \in I$.

Before the next corollary, we need the following proposition :

Proposition 2.21 : Let M_s be an S-system and $\{N_i | i \in I\}$ be a family of S-systems. Then $\prod_{i\in I} N_i$ is M-injective if and only if N_i is M-injective for every $i \in I$.

Proof : Assume that $N_s = \prod_{i\in I} N_i$ is M_s-injective. Let X be a subsystem of M_s and f be S-homomorphism from X into N_i. Since N_s is M_s-injective S-system then there exists S-homomorphism $g : M_s \to N_s$ such that $goi_X = jof$, where i_X is the inclusion map of X into M_s and j is the injection map of N_i into N_s. Define $h : M_s \to N_i$ such that $h = \pi_i o g$, where π_i is the projection map of N_s into N_i, then $hoi_X = \pi_i ogoi_X = \pi_i ojof = f$. That is for all $a \in X, h(a) = h(i_X(a)) = \pi_i(g(a)) = \pi_i(g(i_X(a))) = \pi_i(j(f(a))) = (\pi_j oj)(f(a)) = f(a)$. Conversely, assume that N_i is M_s-injective for each $i \in I$ and f is S-homomorphism from S-subsystem X of M_s into N_s. Since N_i is M_s-injective , then there exists S-homomorphism $\beta_i : M_s \to N_i$, such that $\beta_i oi_X = \pi_i of$, where π_i is the natural projection of N_s into N_i. So there exists S-homomorphism $\beta : M_s \to N_s$ such that $\beta_i = \pi_i o\beta$. We claim that $\beta oi_X = f$. For this since $\beta_i oi_X = \pi_i o\beta oi_X$, then $\pi_i of = \pi_i o\beta oi_X$, so we obtain $f = \beta oi$. Therefore N_s is M_s-injective.

Corollary 2.22 : For any integer $n \geq 2$. Let M_s be a cog-reversible nonsingular S-system with $\ell_M(s) = \Theta$ for each $s \in S$. Then M_s^n is pseudo injective if and only if M_s is quasi injective.

Proof : If M_n is pseudo injective, then by Proposition 2.4 M_i is M_s-pseudo injective. So by Theorem 2.17, each M_i is quasi injective. Then, by Proposition 2-21, we have M_s is quasi injective. Conversely, if M_s is quasi injective, then by Proposition 2.21, M^n is quasi injective and in particular is pseudo injective.

Proposition 2.23 : Let $M_s = \oplus_{i\in I} M_i$ be a direct sum of a cog-reversible nonsingular S-systems M_i. An S-system M_s is quasi injective if and only if it is pseudo injective.

Proof : Let M_s be pseudo injective S-system. Then, by Corollary 2.20, each M_j is M_i-injective, for all distinct $i, j \in I$. Now, by Lemma 2.5 each M_j is M_i-pseudo injective, so by Theorem 2.17, each M_j is quasi injective. Therefore, by Proposition 2.21 M_s is quasi injective. The other direction is obvious.

Recall that an S-system M_s satisfy C_2**-condition**. If a subsystem N is a retract of M_s and $H \cong N$, where H is a subsystem of M_s, then H is a retract of M_s.

Theorem 2.24 : Every pseudo injective system satisfies C_2-condition.

Proof : Let M_s be pseudo injective S-system and A be a retract subsystem of M_s with $A \cong B$. Let f be an S-isomorphism from subsystem B of M_s into A, then f is S-monomorphism from B into M_s. Since M_s is pseudo injective and A be a retract of M_s, so A is M_s-pseudo injective by Lemma 2.5. Since $A \cong B$, so by (2.2)(2), B is M_s-pseudo injective. Then, by Proposition 2.3(1), f is split. Hence, B is a retract of M_s and so M_s satisfies C_2-condition.

3. Pseudo Quasi Principally Injective Systems

Definition 3.1 : An S-system N_s is called **pseudo M_s-principally-injective**, if for every S-monomorphism from M_s-cyclic subsystem of M_s to N_s can be extended to S-homomorphism from M_s to N_s (if this is the case, we write N_s is **pseudo MP-injective**). An S-system M_s is called **pseudo quasi principally injective** if it is pseudo MP-injective system (if this is the case, we write M_s is **pseudo QP-injective**).

Remarks and Example 3.2 :

(1) Every QP-injective system is pseudo QP-injective. But the converse is not true in general for example, see (2.2) (1).

(2) Retract of pseudo QP-injective S-system is pseudo MP-injective.

Proof : Let M_s be pseudo QP-injective S-system and N be a retract M_s-cyclic subsystem of M_s. Let A be M_s-cyclic subsystem of M_s and $f : A \to N$ be S-monomorphism. Define $\alpha : A \to M_s$ by $\alpha = j_N o f$, where j_N be the injection map of N into M_s, then α is S-monomorphism. Since M_s is pseudo QP-injective, so there exists S-homomorphism $\beta : M_s \to M_s$ such that $\beta o i_A = \alpha$, where i_A be the inclusion map of A into M_s. Now, let π_N be the projection map of M_s onto N. Then, define σ $(= \pi_N \beta) : M_s \to N$. Thus for each $a \in A$ we have that $\sigma o i_A(a) = (\pi_N o \beta o i_A)(a) = \pi_N(\alpha(a)) = \pi_N(j_N o f(a)) = \pi_N(f(a)) = f(a)$. Therefore, an S-homomophism σ is extends f. Thus, N is pseudo MP-injective system.

(3) Let A_s, B_s and M_s be a right S-systems and A_s is pseudo MP-injective, if B_s isomorphic to A_s, then B_s is also pseudo MP-injective.

(4) Let N_s, M_s and A_s be S-systems. If A_s is pseudo MP-injective and $N_s \cong M_s$, then A_s is pseudo NP-injective S-system.

Proposition 3.3 : Let M_s be S-system. M_s is pseudo QP-injective if and only if M_s is pseudo NP-injective for every M_s-cyclic subsystem N of M_s. In particular, if B is a retract of N, then M_s is pseudo BP-injective system.

Proof : Let N be M_s-cyclic subsystem of S-system M_s . Assume that A be N_s-cyclic subsystem of N. Let f be S-monomorphism from A into M_s and $i_1(i_2)$ be the inclusion map of $A(N)$ into $N(M_s)$. Since M_s is pseudo QP-injective, so there exists

S-homomorphism $g : M_s \to M_s$ such that $goi_2oi_2 = f$, this means g is extension of f. Define an S-homomorphism g_1 $(= goi_2) : N \to M_s$, then $g_1oi_1 = goi_2oi_1 = f$. Thus, g_1 is extension of f and M_s is pseudo NP-injective system. Conversely, by taking M_s is M_s-cyclic subsystem of M_s.

Corollary 3.4 : Let M_s be S-system and N_s be pseudo MP-injective system, then N is a retract of M_s if and only if N is M_s-cyclic subsystem of M_s.

Proof : As every retract of an S-system M_s is M_s-cyclic subsystem of M_s [6]. Conversely, by taking f is the identity map of N in the proof of Proposition 3.3.

Before the next Proposition, we need the following concept :

Let N_s and M_s be two S-systems. Recall that N_s is **M_s-projective or projective relative to M_s**, where M_s be S-system, if for every S-system C_s, every S-homomorphism f from S-system N_s into S-system C_s can be lifted with respect to every S-epimorphism g from M_s into C_s, that is there exists S-homomorphism h from N_s into M_s such that $gh = f$ [9]. An S-system N_s is called **projective** if it is projective relative to every right S-system. Also an S-system N_s is called **quasi-projective** if N_s is **N_s-projective** [9]. Note that if N_s is M_s-projective, then every S-epimorphism from S-system M_s into N_s is split. Also, retract of M_s-projective S-system is M_s-projective [3].

Proposition 3.5 : Let M_s be a pseudo QP-injective S-system, and $\alpha \in T = End(M_s)$. The following statements are equivalent :

(1) $\alpha(M)$ is a retract of M_s,

(2) $\alpha(M)$ is a pseudo MP-injective. In additional, if M_s is quasi projective S-system, then (1) and (2) are equivalent to :

(3) $\alpha(M)$ is M-projective.

Proof : $(1 \to 2)$ Follows from (3.2) (2).

$(2 \to 1)$ As $\alpha(M)$ is M_s-cyclic subsystem of M_s, so by Corollary 3.4, $\alpha(M)$ is a retract of M_s.

$(2 \to 3)$ By (2) and Corollary 3.4, we have $\alpha(M)$ is a retract of M_s. Since M_s is quasi projective S-system, so $\alpha(M)$ is M_s-projective.

$(3 \to 2)$ Assume that $\alpha(M)$ is M_s-projective. Let A be M_s-cyclic subsystem of M_s and σ be S-monomorphism from A into $\alpha(M)$. Since $\alpha(M)$ is M_s-cyclic, so there exists

S-epimorphism $\beta : M_s \to \alpha(M)$. Since $\alpha(M)$ is M_s-projective, so β split. This means there S-homomorphism k from $\alpha(M)$ into M_s, such that $\beta ok = I_{\alpha(M)}$. Then, define $f = ko\sigma$. Since f is S-monomorphism (whence $\beta ok = I_{\alpha(M)}$) and M_s is pseudo QP-injective, so there exists S-homomorphism $h : M_s \to M_s$ such that $hoi = f$. Since M_s is quasi projective, so $\beta oh = g$, where g is an S-homomorphism from M_s into $\alpha(M)$. Thus, we have $goi = \beta ohoi = \beta of = \beta oko\sigma = I_{\alpha(M)}o\sigma$. This means $\alpha(M)$ is pseudo MP-injective system.

Corollary 3.6 : Let M_s be a pseudo QP-injective S-system and quasi projective. Then the following statements holds for M_s-cyclic subsystem N of M_s :

(1) N is a retract of M_s .

(2) N is pseudo MP-injective. In additional, if M_s is quasi projective S-system, then (1) and (2) are equivalent to :

(3) N is M_s-projective.

The following proposition explain under which conditions pseudo QP-injective being QP-injective:

Proposition 3.7 : Let M_s be a cog-reversible nonsingular S-system with $\ell_M(s) = \Theta$ for each $s \in S$. If M_s is pseudo QP-injective, then M_s is QP-injective.

Proof : Let N be M_s-cyclic subsystem of S-system M_s and f be S-homomorphism from N into M_s. If f is one-to-one, then there is nothing to prove. If f is not one-to-one, then by using the proof of Theorem 3.2.17, we get the required. This means that M_s is QP-injective S-system.

Proposition 3.8 : Let M_s be a principal self-generator S-system. Then, every pseudo QP-injective S-systems is pseudo injective.

Proof : Let N be a subsystem of S-system M_s and f be S-monomorphism from N into M_s. Since M_s is principal self-generator, so there exists some $\alpha : M_s \to N$ such that $m =$ that $m = \alpha(m_1)$, $\forall \ m \in M_s$. This means α is S-epimorphism, thus N is M_s-cyclic subsystem of M_s. As M_s is pseudo QP-injective, so f can be extend to S-endomorphism g of M_s such that $goi = f$, where i be the inclusion map of N into M_s. Therefore, M_s is pseudo injective S-system.

Theorem 3.9 : Let M_1 and M_2 be two S-systems. If $M_1 \oplus M_2$ is pseudo QP-injective, then M_i is M_j-principally injective for $i, j = \{1, 2\}$.

Proof : Let $M_1 \oplus M_2$ be pseudo QP-injective. Let A be M_2-cyclic subsystem of M_2 and f be S-homomorphism from A into M_1. let j_1 and π_1 be the injection and projection map of M_1 into $M_1 \oplus M_2$ and $M_1 \oplus M_2$ onto M_1 respectively. Define $\alpha : A \to M_1 \oplus M_2$ by $\alpha(a) = (f(a), a), \ \forall a \in A$. It is clear that α is S-monomorphism. Since $M_1 \oplus M_2$ is pseudo QP-injective, so by Proposition 3.3, $M_1 \oplus M_2$ is pseudo M_2P-injective. Hence, there exists S-homomorphism g from M_2 into $M_1 \oplus M_2$ such that $goi = \alpha$, where i be the inclusion map of A into M_2. Now, put $h = \pi_1 og$ from M_2 into M_1. Thus, $\forall \ a \in A$ we have $h(a) = \pi_1 og(a) = \pi_1 o\alpha(a) = \pi_1(\alpha(a)) = \pi_1(f(a), a) = f(a)$. This means M_1 is M_2P-injective.

Corollary 3.10 : Let $\{M_1\}_{i\in I}$ be a family of S-systems. If $\oplus_{i\in I} M_i$ is pseudo $M_K P$-injective, then M_j is $M_K P$-injective system for all distinct $j, k \in I$.

Proposition 3.11 : For any integer $n \geq 2$, M_S^n is pseudo QP-injective if and only if M_s is QP-injective.

Proof : If M_s^n is pseudo QP-injective, then by Theorem 3.9, we have M_s is MP-injective. This means M_s is QP-injective. Conversely, assume that M_s is QP-injective system, this means M_s is MP-injective system. By Proposition 2.13, M_s^n is QP-injective and hence by (3.2) (1), M_s^n is pseudo QP-injective system.

Proposition 3.12 : Let an S-system M_s be pseudo QP-injective and $T = End(M_s)$. If $Im\alpha$ is essential (large) subsystem of M_s, where $\alpha \in T$, then any S-monomorphism from $\alpha(M)$ into M_s can be extended to an S-monomorphism in T.

Proof : Let $f : \alpha(M) \to M_s$ be S-monomorphism. Since M_s is pseudo QP-injective system, so there exists S-homomorphism $g : M_s \to M_s$ such that $f = g_i$, where $i : \alpha(M) \to M_s$ is the inclusion map. Then, $f\alpha = g_i\alpha = g\alpha$. Now, let $g(\alpha(m_1)) = g(\alpha(m_2))$, where $m_1, m_2 \in M_s$, then $f(\alpha(m_1)) = f(\alpha(m_2))$. Since f is monomorphism, so $\alpha(m_1) = \alpha(m_2)$ and on the other hand $\alpha(M)$ is essential subsystem of M_s, so g is monomorphism.

Corollary 3.13 : If M_s is pseudo QP-injective reversible S-system, then every S-monomorphism from M-cyclic subsystem of M_s into M_s splits.

Proof : By taking M_s is M_s-cyclic and I_M be identity map of M_s. Then, by pseudo QP-injectivity of M_s, for any S-monomorphism $f : M_s \to M_s$, there exists S-homomorphism $h : M_s \to M_s$ such that $hf = I_M$ and this means f is split.

The proof of the following Corollary from Corollary 3.13 and Proposition 2.3 (2).

Corollary 3.14 : Let M_s be reversible and principal self-generator S-system. Then M_s is pseudo QP-injective S-system for all S-system M_s if and only if M_s is injective. The following theorems and lemma give a characterization of pseudo QP-injective S-systems :

Theorem 3.15 : Let M_s be an S-system. Then M_s is pseudo QP-injective system if and only if $ker(\alpha) = ker(\beta)$, implies $T\alpha = T\beta$ for all $\alpha, \beta \in T = End(M_s)$.

Proof : $\Rightarrow$) Let $\alpha, \beta \in T$ with $ker(\alpha) = ker(\beta)$. Define $\varphi : \alpha(M) \to M_s$ by $\varphi(\alpha(m)) = \beta(m)$ for every $m \in M_s$. Let $\alpha(m_1), \alpha(m_2) \in \alpha(M)$ such that $\alpha(m_1) = \alpha(m_2)$. Then $(m1, m2) \in ker(\alpha) = ker(\beta)$, so $\beta(m_1) = \beta(m_2)$. Hence $\varphi(\alpha(m_1)) = \varphi(\alpha(m_2))$ and φ is well-defined, the reverse steps gives that φ is S-monomorphism. For every $m \in M_s$ and $s \in S$, we have $\varphi(\alpha(ms)) = \beta(ms) = \beta(m)s = \varphi(\alpha(m))s$. This shows that φ is an S-homomorphism. Since M_s is pseudo QP-injective system and $\alpha(M)$ is M-cyclic subsystem of M_s, so there exists S-homomorphism $\psi : M_s \to M_s$ such that $\psi i = \varphi$, where i is the inclusion map of $\alpha(m)$ into M_s. Thus, $\beta = \varphi\alpha = \psi i\alpha = \psi\alpha \in T\alpha$. Then, $T\beta \subseteq T\alpha$. Similarly, $T\alpha \subseteq T\beta$, therefore $T\alpha = T\beta$.

$\Leftarrow$) Let $\alpha \in T$ and $f : \alpha(M) \to M_s$ be S-monomorphism from M-cyclic subsystem $\alpha(M)$ of M_s into S-system M_s. Then ker f = ker i , where i is the inclusion map from $\alpha(M)$ into M_s. Since $f(\alpha(M)) \cong \alpha(M)$, and similarly $i(\alpha(M)) \cong \alpha(M)$, so this means $f, i \in T$. Then by assumption, $Tf = Ti$, so we have $f \in Ti$. Thus, $f = hi$, for some $h \in T$. This shows that M_s is pseudo QP-injective system.

Theorem 3.16 : Let M_s be pseudo QP-injective S-system and $T = End(M_s)$ with $\alpha, \beta \in T$. Then :

(1) If $\alpha(M)$ embeds in $\beta(M)$, then $T\alpha$ is an image of $T\beta$.

(2) If $\alpha(M) \cong \beta(M)$, then $T\alpha \cong T\beta$.

Proof : (1) Let $f : \alpha(M) \to \beta(M)$ be S-monomorphism. Let i_1 (respectively i_2) be the inclusion maps of $\alpha(M)$ [respectively $\beta(M)$] into M_s. Since i_2of is S-monomorphism and M_s is pseudo QP-injective system, so there exists S-homomorphism $\overline{f} : M_s \to M_s$ such that $\overline{f}oi_1 = i_2of$. Define $\sigma : T\beta \to T\alpha$ by $\sigma(\lambda\beta) = \lambda\overline{f}\alpha, \lambda \in T$. If $\lambda_1\beta = \lambda_2\beta$ for $m \in M_s$. $\overline{f}\alpha(m) = (\overline{f}oi_1)(\alpha(m)) = (i_2of)(\alpha(m)) = f(\alpha(m))$ and hence $\lambda\overline{f}\alpha(m) = \lambda f(\alpha(m))$, so σ is well-defined. It is clear that σ is T-homomorphism, in fact, let $\lambda\ \beta \in T\beta$ and $g \in T$, then $\sigma(g(\lambda\beta)) = \sigma\ ((g\lambda)\beta) = g\lambda\overline{f}\alpha = g(\lambda\overline{f}\alpha) = g\ \sigma(\lambda\ \beta)$. We

claim that $ker(\overline{f}\alpha) = ker\alpha$. Let $(x_1, x_2) \in ker(\overline{f}\alpha)$ which implies $\overline{f}\alpha(x_1) = \overline{f}\alpha(x_2)$. This implies $f(\alpha(x_1)) = f(\alpha(x_2))$, since f is S-monomorphism, so $\alpha(x_1) = \alpha(x_2)$. Thus $(x_1, x_2) \in ker\alpha$. Also, it is clear that $ker\alpha \subseteq ker(\overline{f}\alpha)$. Thus, $ker(\overline{f}\alpha) = ker\alpha$. Hence, by Theorem 3.14, we have $T\alpha = T\overline{f}\alpha$, so there exists $\lambda \in T$ such that $\alpha = \lambda\overline{f}\alpha$, then $\alpha = \lambda\overline{f}\alpha = \sigma(\lambda\beta) \in \sigma(T\beta)$. This implies $T\alpha = \sigma(T\beta)$. Then σ is T-epimorphism.
(2) Let $f : \alpha(M) \to \beta(M)$ is S-isomorphism. Let i_1 (respectively i_2) be the inclusion maps of $\alpha(M)$ [respectively $\beta(M)$] into M_s. Since i_2of is S-monomorphism and M_s is pseudo QP-injective system, so i_2of can be extended to $\overline{f} : M_s \to M_s$ such that $\overline{f}oi_1 = i_2of$. Define $\sigma : T\beta \to T\alpha$ by $\sigma(\lambda\beta) = \lambda\overline{f}\alpha$, for every $\lambda \in T$. As in part (1), σ is well-defined and T-epimorphism. Now, let $\sigma(\lambda_1\beta) = \sigma(\lambda_2\beta)$, then$\lambda_1\overline{f}\alpha = \lambda_2\overline{f}\alpha$. Since $\overline{f}\alpha(M) = \overline{f}oi_1(\alpha(M)) = i_2of(\alpha(M)) = f\alpha(M) = \beta(M)$, then $\lambda\overline{f}\alpha(M) = \lambda\beta(M)$, hence $\lambda_1\beta(M) = \lambda_1\overline{f}\alpha(M) = \lambda_2\overline{f}\alpha(M) = \lambda_2\beta(M)$, then $\lambda_1\beta = \lambda_2\beta$. Hence σ is T-monomorphism.

Lemma 3.17 : Let M_s be a pseudo QP-injective system and $T = End(M_s)$. If $\alpha(M)$ is a simple S-system, $\alpha \in T$, then $T\alpha$ is a simple T-system.

Proof : Let$\Theta \neq f\alpha \in T\alpha$. Then $f : \alpha(M) \to f\alpha(M)$ is an S-isomorphism by hypothesis, so let $\sigma : f\alpha(M) \to \alpha(M)$ be the inverse. If $\overline{\sigma} \in T$ extends σ, then for $m \in M_s$, we have $\alpha(m) = \sigma(f\alpha(m)) = \overline{\sigma}(f\alpha(m)) \in Tf\alpha$ and hence $T\alpha = Tf\alpha$.

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