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LIU'S LAWS AND p-BCL^+ ALGEBRAS

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Abstract

We are completely devoted to study of comparatively new algebraic object about BCL^+ algebras, introduced by the author. For BCL^+ algebras naturally the question arises−satisfying the dual Liu's laws we introduce the notion of pseudocomplement and give new propositions. This paper is also concerned with lattices on BCL^+ algebras. Some new results are established for pseudocomplemented BCL^+ algebras (simply p-BCL^+ algebras). We show that the Liu's completeness laws are very interesting−not just in BCL^+ algebras, but also in lattice theory and set theory. Even in cosmology.

1. Introduction

In 2011-2012, the author [1, 2] introduced BCL-algebras and BCL^+ algebras. Later on, more papers have appeared in which BCL-algebras and their related concept by D. Al-Kadi and R. Hosny [3]; soft BCL-algebras were treated by D. Al-Kadi [4]. For the general development of BCL^+ algebras, the author [5-9] gives a characterization of a partial order and a topology in BCL^+ algebras; discuss some distributions of BCL^+

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algebras; introduce filtrations, deductive systems and funnels in BCL^+ algebras.
The article will be introduced the concept of a pseudocomplement in BCL^+ algebras, especially with the laws of set theory, so I show that the dual Liu's laws [10] extends to pseudocomplement BCL^+ algebras, or p-BCL^+ algebras for short, and introduce the Liu's completeness laws. The results also show that the lattice on p-BCL^+ algebras forms an algebraic lattice which is distributive. Finally, it looks into further research about the deductive system of BCL^+ algebras, and we prove that if L be a distributive p-BCL^+ algebra, then $H(L)$ be a filtration if it is a deductive system. For Liu's law, the author became one of the first to apply it to the new cosmology [11, 12]. Although the dual Liu's laws and the Liu's completeness laws are the most important set identities, the idea in this paper, but can be applied to other algebraic logic, like BL-algebras in [13], Heyting algebras in [14], Hilbert algebras in [15], etc.

2. Preliminaries

In this Section 2, we first review some relevant concepts, as follows.

Definition 2.1 (Liu [2]) : A BCL^+ algebrais a triple $(Y; *, 1)$, where Y is a nonempty set, $*$ is a binary operation on Y, $1 \in Y$ is an element such that the following three axioms hold for any $x, y, z \in Y$:

(BCL^+1) $x * x = 1$.

(BCL^+2) $x * y = 1$ and $y * x = 1$ imply $x = y$.

(BCL^+3) $((x * y) * z) * ((x * z) * y) = (z * y) * x$.

Definition 2.2 (Liu [6]) : Let $(X; \circ, *, 1)$ be a BCL^+_D algebra with two binary operations $\circ$ and $*$ that satisfies the following properties for any $x, y, z \in X$:

(BCL^+_D1) An algebra $D(X) = (X; \circ)$ is a distributive algebra.

(BCL^+_D2) An algebra $P(X) = (X; *, 1)$ is a algebra such that $g(x, y, z) = (x * y) * z$.

(BCL^+_D3) $x * (y \circ z) = (x * y) \circ (x * z)$ (right weakly distribution rule).

(BCL^+_D4) $(y \circ z) * x = (y * x) \circ (z * x)$ (left weakly distribution rule).

Definition 2.3 (Liu [5]) : Let $(Y; *, 1)$ be a $BCL^+_{\leq}$ algebra, for all $x, y, z \in Y$, then

$(BCL^+_{\leq}1)$ $x \leq x$.

$(BCL^+_{\leq}2)$ If $x \leq y$ and $y \leq x$, then $x = y$.

$(BCL^+_{\leq}3)$ $((x * y) * z) * ((x * z) * y) \leq (z * y) * x$.

Theorem 2.1 (Liu [2]) : Assume that $(Y; *, 1)$ is a BCL^+ algebra. Then the following hold for all $x, y, z \in Y$:

(i) $(x * (x * y)) * y = 1$.

(ii) $x * 1 = x$ implies $x = 1$.

(iii) $((x * y) * (x * z)) * (z * y) = 1$.

(iv) $(BCL^{+}2)$ $\quad x * y = 1$ and $y * x = 1$ imply $x = y$.

Law 2.1 (Liu [10]) : Let U be a set, then the following identities hold for all $x, y \in U$:

(DL1) $x \cap (\overline{x \cap y}) = x \cap \overline{y}$ (difference-set law).

(DL2) $x \cup (\overline{x \cup y}) = x \cup \overline{y}$ (Liu's law).

Note that (DL1) and (DL2), collectively called Dual Liu's Laws in the axioms of set theory.

Next, we can make this notion of deductive system more precise in BCL^{+} algebras, as follows.

Definition 2.4 (Liu [8]) : If D be a nonempty subset of a BCL^{+} algebra $(Y; *, 1)$. Then we say that D be a deductive system if

(D1) $1 \in D$.

(D2) $x \in D$ and $x * y \in D$ imply $y \in D$.

Lemma 2.1 (Liu [8]) : Let D be a deductive system of a BCL^{+} algebra $(Y; *, 1)$, and suppose $a \leq x$ whenever $a \in D$. Then $x \in D$.

Theorem 2.2 (Liu [8]) : Suppose that B is a subalgebra of Y. Then B is a filtration and $x \in B$ if and only if $1 * x \in B$.

Theorem 2.3 (Liu [8]) : Let H be a nonempty subset of a BCL^{+} algebra $(Y; *, 1)$. Then H is a filtration if and only if it is a deductive system.

3. The Pseudocomplement of BCL^{+} Algebra

We now wants to treat the pseudocomlemented algebras (simply, p-algebras) as a lattice with a smallest element 0 together with a mapping $\neg : L \to L$ such that $x \wedge y = 0$ if and only if $y \leq \neg x$; further definition having a binary operation $\to$ where $x \to y$ is the pseudocomlement of x relative to y. In particular, $x \to x = 1 \in L$. For a lattice to be complemented it must have a greatest element 1.

Note that Y is a bounded BCL^{+} algebra (unless otherwise specified) with a smallest element 0 relative to the natural ordering. We begin with a general construction, as follows.

Definition 3.1 : Let Y be a BCL^+ algebra. The pseudocomplement of the complemented element x, denoted by $\neg x$, we define $\neg x := x \rightarrow 0$ and define the double complement $\neg\neg x := x$. Let $Y = \{0, 1\}$, x has the values 1 and 0, then $\neg 1 = 0$ and $\neg 0 = 1$.

Definition 3.2 : Let Y be a BCL^+ algebra, and suppose $x \in Y$. We say that $a \wedge x \leq a$ implies $x \leq a \rightarrow a$ for any element a. If $a \in Y$, then $a \rightarrow a$ is the greatest element of Y.

Our key results are the following theorems and Laws.

Theorem 3.1 : Let $L = (L; \vee, \wedge, \rightarrow, 1)$ be a lattice and let $\rightarrow$ be a binary operation $\rightarrow: L \rightarrow L$, then L is a p-BCL^+ algebra if and only if $x, y \in L$. We have

(L1) $x \rightarrow x = 1$.

(L2) $x \wedge (x \rightarrow y) = x \rightarrow y$.

Proof : Part (L1), the identity follows immediately from Definition 3.2 above.

To prove part (L2), it suffices to show that L is distributive. Observe that $x \wedge x = x, 1 \wedge x = x$, let $\wedge$ and $\rightarrow$ satisfy Definition 2.2 (BCL_D^+3) define $*, \circ : L \rightarrow L$. In the homomorphism of algebra case, we may assume that the interchanging $*$ and $\wedge$ and interchanging $\circ$ and $\rightarrow$. Now

$$\begin{aligned} x \wedge (x \rightarrow y) &= (x \wedge x) \rightarrow (x \wedge y) \\ &= x \rightarrow (x \wedge y) \\ &= (x \rightarrow x) \wedge (x \rightarrow y) \\ &= 1 \wedge (x \rightarrow y) \\ &= x \rightarrow y. \end{aligned}$$

The proof is now complete. □

Theorem 3.2 : Let $L = (L; \vee, \wedge, *, 1)$ be a lattice. If $* : L \rightarrow L$. Then L is a BCL^+ algebra if and only if $x, y \in L$ such that

$$x * y = x * (x \wedge y).$$

Proof : By Definition 2.2 (BCL_D^+3) and define $\circ : L \rightarrow L$. In fact, we may assume that the interchanging $\circ$ and $\wedge$ in the homomorphism case in Definition 2.2 (BCL_D^+3).

Then

$$\begin{aligned} x * (x \wedge y) &= (x * x) \wedge (x * y) \\ &= 1 \wedge (x * y) \\ &= x * y. \end{aligned}$$

The proof is now complete. □

Theorem 3.3 : A lattice L be a p-BCL^+ algebra. If there exists negation such that for every $x \in L$, then $\neg x \wedge x = \neg x$.

Proof : From this, we infer from Theorem 3.1, and by Definition 3.1 we have

$$\begin{aligned} \neg x \wedge x &= (x \to 0) \wedge x \\ &= (x \wedge x) \to (0 \wedge x) \\ &= x \to 0 \\ &= \neg x. \end{aligned}$$

The proof is now complete. □

Theorem 3.4 : Let Y be a distributive p-BCL^+ algebra. If there exists negation such that for every $x \in Y$, then $\neg x * x = 1$.

Proof : By Definition 3.1 we have

$$\begin{aligned} \neg x * x &= (x \to 0) * x \\ &= (x * x) \to (0 * x) \\ &= 1 \to (0 * x) \\ &= (1 \to 0) * (1 \to x) \\ &= 0 * ((x \to x) \to x) \\ &= 0 * ((x \to x) \to (x \to x)) \\ &= 0 * (x \to x) \\ &= (0 * x) \to (0 * x) \\ &= 1. \end{aligned}$$

The proof is now complete. □

As an application of Law 2.1, we obtain the following striking properties.

Proposition 3.1 : Let L be a lattice, define a map $\rightharpoondown: L \to L$, for all $x, y \in L$ with the following properties:

(DY1) $x \wedge (\rightharpoondown (x \wedge y)) = x \wedge (\rightharpoondown y)$.

(DY2) $x \vee (\rightharpoondown (x \vee y)) = x \vee (\rightharpoondown y)$.

We say that L be a p-BCL^+ algebra, (DY1) and (DY2) collectively called Dual Liu's Laws.

Proposition 3.2 : Let L be a p-BCL^+ algebra, for all $x, y \in L$. Then

(A1) $x \leq \rightharpoondown\rightharpoondown x$.

(A2) $x \leq y$ implies $\rightharpoondown y \leq \rightharpoondown x$.

Theorem 3.5 : Let L be a lattice and let $x, y \in L$. Then we see that Proposition 3.1 (DY1) and (DY2) are equivalent.

Proof : (DY1) implies (DY2) is obvious. We may assume that (DY1) is a tenable identity, let $x, y \in L$. We have

$$\begin{aligned}
x \vee (\rightharpoondown (x \vee y)) &= x \vee (\rightharpoondown x \wedge (\rightharpoondown y)) \\
&= (x \vee (\rightharpoondown x)) \wedge (x \vee (\rightharpoondown y)) \\
&= 1 \wedge (\rightharpoondown (\rightharpoondown x \wedge (\rightharpoondown\rightharpoondown y))) \\
&= \rightharpoondown (\rightharpoondown x \wedge (\rightharpoondown (\rightharpoondown x \wedge (\rightharpoondown y)))) \\
&= x \vee (\rightharpoondown x \wedge (\rightharpoondown y)) \\
&= (x \vee (\rightharpoondown x)) \wedge (x \vee (\rightharpoondown y)) \\
&= 1 \wedge (x \vee (\rightharpoondown y)) \\
&= x \vee (\rightharpoondown y).
\end{aligned}$$

That is it, condition (DY2) clearly holds. □

Proposition 3.3 : Let L be a complete lattice, for all $\{x_i, y_i | i \in I\} \subseteq L$ with the following properties:

(DY3) $\bigwedge_{i\in I} x_i \wedge (\rightharpoondown \bigvee_{i\in I} y_i) = \bigwedge_{i\in I} (x_i \wedge (\rightharpoondown y_i))$.

(DY4) $\bigvee_{i\in I} x_i \wedge (\rightharpoondown \bigwedge_{i\in I} y_i) = \bigvee_{i\in I} (x_i \wedge (\rightharpoondown y_i))$.

(DY5) $\bigwedge_{i\in I} x_i \vee (\rightharpoondown \bigvee_{i\in I} y_i) = \bigwedge_{i\in I} (x_i \vee (\rightharpoondown y_i))$.

(DY6) $\bigvee_{i\in I} x_i \vee (\rightharpoondown \bigwedge_{i\in I} y_i) = \bigvee_{i\in I} (x_i \vee (\rightharpoondown y_i))$.

We say that L be a p-BCL^+ algebra, (DY3)- (DY6) are called Liu's completeness Laws.

Theorem 3.6 : Let L be a p-BCL^+ algebra and let $x, y \in L$. Then

$$x * (\neg (x \wedge y)) = x \wedge (\neg y).$$

Proof : By Theorem 3.2 and Proposition 3.1 (DY1), we can write

$$\begin{aligned} x * (\neg (x \wedge y)) &= x * (x \wedge (\neg (x \wedge y))) \\ &= x * (x \wedge (\neg y)) \\ &= x * (\neg y). \end{aligned}$$

The proof is now complete. □

Theorem 3.7 : Let L be a distributive p-BCL^+ algebra. Then $H(L)$ be a filtration if it is a deductive system.

Proof : Note that an element g is dense if $\neg g = 0$ or, equivalently, we have $\neg\neg g = 1$. Also, $H(L)$ to denote the set of dense element of L.

Next we may assume $g \leq x$ and $g \in H(L)$, then $\neg x \leq \neg g$ and $\neg g = 0$, we obtain $\neg x = 0$, and so $x \in H(L)$. Let $x, y \in H(L)$, then $\neg y = 0$, we have

$$\begin{aligned} \neg\neg (x * y) &= \neg\neg x * \neg\neg y \\ &= \neg 0 * \neg 0 \\ &= 1 \in H(L). \end{aligned}$$

Therefore, $x * y \in H(L)$ and this shows that $H(L)$ be a filtration. □

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