CHARACTERIZATION OF PRIME FILTERS IN \((\mathbb{Z}^+, \leq_C)\)

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Abstract  
A convolution is a mapping \(C\) of the set \(\mathbb{Z}^+\) of positive integers into the set \(\mathcal{P}(\mathbb{Z}^+)\) of all subsets of \(\mathbb{Z}^+\) such that, for any \(n \in \mathbb{Z}^+\), each member of \(C(n)\) is a divisor of \(n\). If \(D(n)\) is the set of all divisors of \(n\), for any \(n\), then \(D\) is called the Dirichlet’s convolution [2]. If \(U(n)\) is the set of all Unitary (square free) divisors of \(n\), for any \(n\), then \(U\) is called unitary (square free) convolution. Corresponding to any general convolution \(C\), we can define a binary relation \(\leq_C\) on \(\mathbb{Z}^+\) by ‘\(m \leq_C n\) if and only if \(m \in C(n)\)’. In this paper, we present a characterization for the prime filters in \((\mathbb{Z}^+, \leq_C)\), where \(\leq_C\) is the binary relation induced by the convolution \(C\).

1. Introduction  
A convolution is a mapping \(C\) of the set \(\mathbb{Z}^+\) of positive integers into the set \(\mathcal{P}(\mathbb{Z}^+)\) of subsets of \(\mathbb{Z}^+\) such that, for any \(n \in \mathbb{Z}^+,\ Cn\) is a nonempty set of divisors of \(n\). If \(C(n)\) is the set of all divisors of \(n\), for each \(n \in \mathbb{Z}^+\), then \(C\) is the classical Dirichlet convolution [2]. If \(C(n) = \{\lfloor d/n \rfloor \text{ and } (d, \lfloor d/n \rfloor) = 1\}\), then \(C\) is the Unitary convolution

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As another example if \( C(n) = \{d/d|n \text{ and } m^k \text{ does not divide } d \text{ for any } m \in \mathbb{Z}^+\} \) then \( C \) is the \( k \)-free convolution.

\[
C(n) = \{d/d|n \text{ and } (d, \frac{n}{d}) = 1\}.
\]

Corresponding to any convolution \( C \), we can define a binary relation \( \leq_C \) in a natural way by

\[
(m \leq_C n) \text{ if and only if } m \in C(n).
\]

\( \leq_C \) is a partial order on \( \mathbb{Z}^+ \) and is called partial order induced by the convolution \( C \) ([6], [7]). In this paper, we discuss filters in \( (N, \leq^p_C) \) and characterization of prime filters of \( (\mathbb{Z}^+, \leq_C) \) in terms of those of \( (N, \leq^p_C) \).

2. Preliminaries

Let us recall that a partial order on a non-empty set \( X \) is defined as a binary relation \( \leq \) on \( X \) which is reflexive \( (a \leq a) \), transitive \( (a \leq b, b \leq c \implies a \leq c) \) and antisymmetric \( (a \leq b, b \leq a \implies a = b) \) and that a pair \( (X, \leq) \) is called a partially ordered set (poset) if \( X \) is a non-empty set and \( \leq \) is a partial order on \( X \). For any \( A \subseteq X \) and \( x \in X \), \( x \) is called a lower (upper) bound of \( A \) if \( x \leq a \) (respectively \( a \leq x \)) for all \( a \in A \). We have the usual notations of the greatest lower bound (glb) and least upper bound (lub) of \( A \) in \( X \). If \( A \) is a finite subset \( \{a_1, a_2, \ldots, a_n\} \), the glb of \( A \) (lub of \( A \)) is denoted by \( a_1 \land a_2 \land \cdots \land a_n \) or \( \bigwedge_{i=1}^n a_i \) (respectively by \( a_1 \lor a_2 \lor \cdots \lor a_n \) or \( \bigvee_{i=1}^n a_i \)). A partially ordered set \( (X, \leq) \) is called a meet semi lattice if \( a \land b \) (=glb\{\(a, b\}\}) exists for all \( a \) and \( b \in X \). \( (X, \leq) \) is called a join semi lattice if \( a \lor b \) (=lub\{\(a, b\}\}) exists for all \( a \) and \( b \in X \). A poset \( (X, \leq) \) is called a lattice if it is both a meet and join semi lattice. Equivalently, lattice can also be defined as an algebraic system \( (X, \land, \lor) \), where \( \land \) and \( \lor \) are binary operations which are associative, commutative and idempotent and satisfying the absorption laws, namely \( a \land (a \lor b) = a = a \lor (a \land b) \) for all \( a, b \in X \); in this case the partial order \( \leq \) on \( X \) is such that \( a \land b \) and \( a \lor b \) are respectively the glb and lub of \( \{a, b\} \). The algebraic operations \( \land \) and \( \lor \) and the partial order \( \leq \) are related by

\[
a = a \land b \iff a \leq b \iff a \lor b = b.
\]
Throughout the paper, \( \mathbb{Z}^+ \) and \( \mathcal{N} \) denote the set of positive integers and the set of non-negative integers respectively.

**Definition 1**: A mapping \( \mathcal{C} : \mathbb{Z}^+ \to \mathcal{P}(\mathbb{Z}^+) \) is called a convolution if the following are satisfied for any \( n \in \mathbb{Z}^+ \).

1. \( \mathcal{C}(n) \) is a set of positive divisors of \( n \)
2. \( n \in \mathcal{C}(n) \)
3. \( \mathcal{C}(n) = \bigcup_{m \in \mathcal{C}(n)} \mathcal{C}(m) \).

**Definition 2**: For any convolution \( \mathcal{C} \) and \( m \) and \( n \in \mathbb{Z}^+ \), we define

\[
   m \leq n \text{ if and only if } m \in \mathcal{C}(n)
\]

Then \( \leq_{\mathcal{C}} \) is a partial order on \( \mathbb{Z}^+ \) and is called the partial order induced by \( \mathcal{C} \) on \( \mathbb{Z}^+ \).

In fact, for any mapping \( \mathcal{C} : \mathbb{Z}^+ \to \mathcal{P}(\mathbb{Z}^+) \) such that each member of \( \mathcal{C}(n) \) is a divisor of \( n \), \( \leq_{\mathcal{C}} \) is a partial order on \( \mathbb{Z}^+ \) if and only if \( \mathcal{C} \) is a convolution [7], as defined above.

**Definition 3**: For any subset \( A \) of \( \mathbb{Z}^+ \) and for any prime number \( p \), let

\[
   A^p = \{ \theta(n)(p) \mid n \in A \}
\]

Then \( A^p \) is a subset of \( \mathcal{N} \) for each \( p \in \mathbb{P} \).

We have the following two theorems on filters in \( (\mathbb{Z}^+, \leq_{\mathcal{C}}) \) and \( (\mathcal{N}, \leq^p_{\mathcal{C}}) \).

**Theorem 1**: Let \( F \) be a filter of \( (\mathbb{Z}^+, \leq_{\mathcal{C}}) \). Then \( F^p \) is a filter of \( (\mathcal{N}, \leq^p_{\mathcal{C}}) \) for each \( p \in \mathbb{P} \) and \( F = \{ n \in \mathbb{Z}^+ \mid \theta(n)(p) \in F^p \text{ for all } p \in \mathbb{P} \} \) [3].

**Theorem 2**: Let \( F \) be the set of all filters of \( (\mathbb{Z}^+, \leq_{\mathcal{C}}) \) and \( F^p \) be that of \( (\mathcal{N}, \leq^p_{\mathcal{C}}) \) for each \( p \in \mathbb{P} \). Let

\[
   \sum_{p \in \mathbb{P}} F^p = \{ f : P \to \bigcup_{p \in \mathbb{P}} F^p \text{ and } f(p) = \mathcal{N} \text{ for all but finite number of } p's \}
\]

Then \( \sum_{p \in \mathbb{P}} F^p \) is a partially ordered set with respect to the partial order defined by

\[
   f \leq g \text{ if and only if } f(p) \subseteq g(p) \text{ for all } p \in \mathbb{P}
\]

and \( F \) is order isomorphic with \( \sum_{p \in \mathbb{P}} F^p \) [3].

**3. Prime Filters in \( (\mathbb{Z}^+, \leq_{\mathcal{C}}) \)**

**Definition 4**: Let \((S, \wedge)\) be a meet semi lattice. A proper filter \( F \) of \( S \) is called a prime filter if, for any \( a \) and \( b \) in \( S \),
$a \lor b$ exists in $S$ and $a \lor b \in F \implies a \in F$ or $b \in F$.

Note that the concept of prime filter is not just the dual of a prime ideal in a meet semi lattice. Recall that a proper ideal $I$ is prime if and only if, for any ideals $J$ and $K$,

$$J \cap K \subseteq I \implies J \subseteq I \text{ or } K \subseteq I.$$ 

However, we have the following.

Theorem 3: Let $F$ be a proper filter of a meet semi lattice $(S, \land)$ satisfying the property that, for any filters $G$ and $H$ of $S$,

$$G \cap H \subseteq F \implies G \subseteq F \text{ or } H \subseteq F.$$ 

Then $F$ is a prime filter.

Proof: Let $a$ and $b \in S$ such that $a \lor b$ exists and $a \lor b \in F$. Then, consider the principal filters $[a]$ and $[b]$. We have

$$[a] \cap [b] = [a \lor b] \subseteq F.$$ 

and, from the hypothesis, $[a] \subseteq F$ or $[b] \subseteq F$ so that $a \in F$ or $b \in F$.

Thus $F$ is a prime filter.

The converse of the above theorem is not true in general. For, consider the following.

Example 1: Consider the semi lattice $(S, \land)$ whose Hasse diagram is given below.

Let $F = [x] = \{x\}$. If $a$ and $b \in S$ such that $a \lor b$ exists and $a \lor b \in F$, then $a \lor b = x$ and hence one of $a$ and $b$ must be $x$ (Note that $x \lor y, y \lor z, x \lor z$ do not exist in $S$). Therefore $F$ is a prime filter. But,

$$[y] \cap [z] = \emptyset \subseteq F \text{ and } [y] \nsubseteq F \text{ and } [z] \nsubseteq F.$$
Even though the converse of theorem 3. is not true in a meet semi lattice, this is true in the case of a lattice.

**Theorem 4**: Let \((L, \wedge, \vee)\) be a lattice and \(F\) a proper filter of \(L\). Then \(F\) is a prime filter if and only if, for any filters \(G\) and \(H\) in \(L\),

\[
G \cap H \subseteq F \implies G \subseteq F \text{ or } H \subseteq F.
\]

**Proof**: Suppose that \(F\) is a prime filter and \(G\) and \(H\) are filters of \(L\) such that \(G \not\subseteq F\) and \(H \not\subseteq F\). Then, we can choose elements \(a \in G\) and \(a \in H\) such that \(a \notin F\) and \(b \notin F\). Since \(F\) is prime, we have \(a \lor b \notin F\).

But \(a \lor b \in G\) and \(a \lor b \in H\) and hence \(a \lor b \in G \cap H\). Therefore \(G \cap H \not\subseteq F\). The converse is proved in Theorem 3.

From the above theorem, it follows that a proper filter \(F\) of a lattice \(L\) is prime if and only if \(F\) is a prime element in the lattice of filters of \(L\).

**Definition 5**: Let \((S, \wedge)\) be a meet semi lattice with smallest element 0 and let \(0 \neq x \in S\). \(x\) is said to be join-irreducible and \(y\) and \(z \in S\) and \(x = y \lor z \implies x = y\) or \(x = z\).

**Theorem 5**: Let \(x\) be any element in a meet semi lattice \((S, \wedge)\). If \([x]\) is a prime filter of \(S\), then \(x\) is join-irreducible.

**Proof**: Suppose that \(x\) is not join-irreducible. Then there exist elements \(y\) and \(z\) such that

\[
y < x, \ z < x \text{ and } y \lor z \text{ exists and equals to } x.
\]

Now, \(y \lor z \in [x]\) and \(y \notin [x]\) and \(z \notin [x]\) and hence \([x]\) is not a prime filter. The converse of the theorem is not true, even in the case of lattices. For, consider the example given below.

**Example 2**: Let \((L, \wedge, \lor)\) be the lattice whose Hasse diagram is given below.
Here \( x \) is join-irreducible (since 0 is the only element which is strictly less than \( x \)). But \( y \lor z = 1 \in [x] \) and \( y \notin [x] \) and \( z \notin [x] \) and hence \([x]\) is not a prime filter.

However, in the case of distributive lattices, we have the following theorem.

**Theorem 6:** Let \((L, \land, \lor)\) be a distributive lattice and \( x \in L \). Then \([x]\) is a prime filter if and only if \( x \) is join-irreducible.

**Proof:** Note that \([x] = L\) if and only if \( x \) is the smallest element of \( L \). Suppose that \( x \) is join-irreducible. Then \( x \neq 0 \) and hence \([x]\) is a proper filter. If \( y \lor z \in [x] \), then \( x \leq y \lor z \) and therefore

\[
x = x \land (y \lor z) = (x \land y) \lor (x \land z).
\]

Since \( x \) is join-irreducible, \( x = x \land y \) or \( x = x \land z \) and therefore \( y \in [x] \) or \( z \in [x] \). Thus \([x]\) is a prime filter. The converse is proved in Theorem 5.

Now, we shall determine all the prime filters of the meet semi lattice \((\mathbb{Z}^+, \leq_C)\) where \( C \) is a multiplicative convolution which is closed under finite intersections.

We have the following theorem on irreducible elements in meet semi lattice \((\mathbb{Z}^+, \leq_C)\).

**Theorem 7:** Let \( C \) be a multiplicative convolution such that \((\mathbb{Z}^+, \leq_C)\) is meet semi lattice and \( x \in \mathbb{Z}^+ \). Then \( x \) is join-irreducible in \((\mathbb{Z}^+, \leq_C)\) if and only if \( x = p^a \) for some prime number \( p \) and a join-irreducible element \( a \) in \((\mathbb{N}, \leq_C)\) \([4] [5]\).

**Theorem 8:** Let \( F \) be a prime filter of \((\mathbb{Z}^+, \leq_C)\). Then \( F = [p^a] \) for some prime number \( p \) and a positive integer \( a \) which is join-irreducible in \((\mathbb{N}, \leq_C)\).

**Proof:** By hypothesis, \( F \) is a prime filter. That is, there exists \( x \in \mathbb{Z}^+ \) such that \( F = [x] \). By Theorem 5, \( x \) is join-irreducible. Also, by Theorem 7, \( x = p^a \) for some prime number \( p \) and a join-irreducible element \( a \) in \((\mathbb{N}, \leq_C)\). Thus \( F = [p^a] \).

The converse of the above theorem is not true, even when \((\mathbb{Z}^+, \leq_C)\) is a lattice. For, consider the following.

**Example 3:** For any prime number \( p \) and \( a \in \mathbb{N} \), define

\[
C(p^a) = \begin{cases} 
\{1, p^a\} & \text{if } a < 4 \\
\{1, p, p^2, \ldots, p^a\} & \text{if } a \geq 4
\end{cases}
\]

and extend \( C \) to \( \mathbb{Z}^+ \) multiplicatively; that is

\[
C(\prod_{i=1}^r p_i^{a_i}) = \prod_{i=1}^r C(p_i^{a_i})
\]
for any distinct primes $p_1, p_2, \ldots, p_r$ and $a_1, a_2, \ldots, a_r \in \mathcal{N}$. Then $C$ is a multiplicative convolution such that $(\mathbb{Z}^+, \leq_C)$ is a lattice.

The Hasse diagram for $(\mathcal{N}, \leq_C)$ is given below, for any prime number $p$.

Clearly 1 is join-irreducible in $(\mathcal{N}, \leq_C)$. But $[2^1]$ is not a prime filter, since $2^2 \vee 2^3 = 2^4 \in [2^1)$, $2^2 \not\in [2^1)$ and $2^3 \not\in [2^1)$

However, we have the following

**Theorem 9**: Suppose that $(\mathbb{Z}^+, \leq_C)$ is a distributive lattice and $F$ a filter of $(\mathbb{Z}^+, \leq_C)$. Then $F$ is a prime filter if and only if $F = \{ p^a \}$ where $p$ is a prime number and $a$ is join irreducible in $(\mathcal{N}, \leq_C)$.

**Proof**: This follows from Theorem 6 and Theorem 7.

In the following we get another characterization of prime filters of $(\mathbb{Z}^+, \leq_C)$ in terms of those of $(\mathcal{N}, \leq_C)$.

For any filter $F$ of $(\mathbb{Z}^+, \leq_C)$ and for any $p \in P$, we define

$$F^p = \{ \theta(n)(p) \mid n \in F \}$$

where $\theta(n)(p)$ is the largest $a \in \mathcal{N}$ such that $p^a$ divides $n$.

**Theorem 10**: A filter $(\mathbb{Z}^+, \leq_C)$ is prime if and only if there exists unique $p \in P$ such that $F^p$ is a prime filter of $(\mathcal{N}, \leq_C)$ and $F^q = \mathcal{N}$ for all $q \neq p$ in $P$ and, in this case,

$$F = \{ n \in \mathbb{Z}^+ \mid \theta(n)(p) \in F^p \}.$$

**Proof**: Suppose that $F$ is a prime filter of $(\mathbb{Z}^+, \leq_C)$. Then, by Theorem 8, $F = \{ p^a \}$ for some $p \in P$ and for some $a \in \mathcal{N}$. For this $p$, we prove that $F^p$ is a prime filter of $(\mathcal{N}, \leq_C)$. Also,
\[ F^p = \{ \theta(n)(p) \mid n \in F \} \]
\[ = \{ \theta(n)(p) \mid p^a \leq_C n \} \]
\[ = \{ b \in N \mid a \leq_C b \} = [a] \text{ in } (N, \leq_P) \]

Since \( a > 0 \), \([a]\) and hence \( F^p \) is a proper filter of \((N, \leq_P)\).

Observe that, for any \( m \in \mathbb{Z}^+ \) such that \( p \) does not divide \( m \), we have \( p^a \wedge m = 1 \) and, since \( p^a \in F \) and \( F \) is a proper filter of \((\mathbb{Z}^+, \leq_C)\), we get that \( m \notin F \). Let \( b \) and \( c \in N \) such that \( b \vee c \) exists in \((N, \leq_P)\) and \( b \vee c \in F^p = [a] \). Then \( b \vee c = \theta(n)(p) \) for some \( n \in F \). Let us write \( n = p^{b \vee c}.m \), where \( m \in \mathbb{Z}^+ \) such that \( (p, m) = 1 \). Since \( b \vee c \) exists in \((N, \leq_P)\), it follows that \( p^b \vee p^c \) exists in \((\mathbb{Z}^+, \leq_C)\) and is equal to \( p^{b \vee c} \). Also, since \((p, m) = 1\), \((p^{b \vee c}, m)\) is also 1 and hence \( p^{b \vee c}.m = n \). Therefore
\[ p^b \vee p^c \vee m = n \in F \]

Since \((p, m) = 1\), \( p \) does not divide \( m \) and hence \( m \notin F \). Since \( F \) is prime, \( p^b \in F \) or \( p^c \in F \) and therefore \( b \in F^p \) or \( c \in F^p \). Thus \( F^p \) is prime.

Also, for any \( p \neq q \in P \),
\[ b \in N \implies p^a \leq_C p^a.q^b \]
\[ \implies p^a.q^b \in [p^a] = F \]
\[ \implies b = \theta(p^a.q^b)(q) \in F^q \]

and hence \( F^q = N \) for all \( p \neq q \in P \). The uniqueness of \( p \) is trivial. Further, by Theorem 1,
\[ F = \{ n \in \mathbb{Z}^+ \mid \theta(n)(q) \in F^q \text{ for all } q \in P \} = \{ n \in \mathbb{Z}^+ \mid \theta(n)(p) \in F^p \} \]

since \( F^q = N \) for all \( q \neq p \).

Conversely suppose that there exists \( p \in P \) such that \( F^p \) is a prime filter of \((N, \leq_P)\) and \( F^q = N \) for all \( p \neq q \in N \). Let \( m \) and \( n \in \mathbb{Z}^+ \) such that \( m \vee n \) exists in \((\mathbb{Z}^+, \leq_C)\) and \( m \vee n \in F \). Then \( \theta(m)(p) \vee \theta(n)(p) \) exists in \((N, \leq_P)\) and is equal to \( \theta(m \vee n)(p) \in F^p \).

Since \( F^p \) is a prime filter, either \( \theta(m)(p) \in F \) or \( \theta(n)(p) \in F \). Since \( F^q = N \) for all \( q \neq p \in N \), we get that
\[ \theta(m)(q) \in F^q \text{ for all } q \in P \]
or \[ \theta(n)(q) \in F^q \text{ for all } q \in P. \]
Since $F = \{ k \in \mathbb{Z}^+ \mid \theta(k)(q) \in F^q \text{ for all } q \in P \}$, by Theorem 1, we have $m \in F$ or $n \in F$. Thus $F$ is a prime filter of $(\mathbb{Z}^+, \leq_C)$.

**Theorem 11**: Let $(S, \land)$ be any meet semi lattice. Then every proper filter of $(S, \land)$ is prime if and only if, for any $x$ and $y$ in $S$,

$$x \lor y \text{ exists in } S \iff x \text{ and } y \text{ are comparable.}$$

**Proof**: Suppose that every proper filter of $(S, \land)$ is prime. Let $x$ and $y \in S$. If $x$ and $y$ are comparable, then clearly $x \lor y$ exists in $S$. On the other hand, suppose $x \lor y$ exists and $x \lor y = z$. If $[z] = S$, then $x$ and $y \in [z]$ and hence $x = z = y$. If $[z] \neq S$, then by hypothesis, $[z]$ is a prime filter and $x \lor y \in [z]$ and hence $x \in [z]$ or $y \in [z]$ so that $x = z$ or $y = z$. Therefore $x = x \lor y$ or $y = x \lor y$, which imply that $x$ and $y$ are comparable. The converse is trivial.

**References**


