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## IMPLICIT ITERATION PROCESS FOR A FINITE FAMILY OF NONEXPANSIVE NONSELF-MAPPINGS IN BANACH SPACES

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### Abstract

The purpose of this paper is to study the weak and strong convergence of an implicit iteration process to a common fixed point for a finite family of nonexpansive nonself-mappings in Banach spaces. The results presented in this paper extend and improve the corresponding results of [Xu and Ori, Numer. Funct. Anal. optim. 22(2001) 767-773; Zhou and Chang, Numer. Fund. Anal. Optim. 23(2002) 911-921; Chidume and Shahzad, Nonlinear. Anal. 62(2005), 1149-1156].

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## 1. Introduction

Let  $C$  be a nonempty subset of a normed linear space  $X$ . Let  $T$  be a self-mapping of  $C$ . Then  $T$  is said to be *nonexpansive* if  $\|Tx - Ty\| \leq \|x - y\|$  for all  $x, y \in C$ . Fixed point iteration process for nonexpansive mappings in Banach spaces including Mann and Ishikawa iteration process have been studied extensively by many authors; see [5, 9, 10, 14]. The convergence problem of an implicit iteration process have been studied by Browder [1, 2], Xu and Yin [17], Takahashi and Kim [13], and Jung and Kim [6] respectively. In 2001, Xu and Ori [16] introduced the following implicit iteration process for a finite family of nonexpansive mappings  $\{T_i : i \in I\}$  (here  $I = \{1, 2, \dots, N\}$ ) with  $\{\alpha_n\}$  a real sequence in  $(0, 1)$ , and an initial point  $x_0 \in C$ :

$$\begin{aligned} x_1 &= \alpha_1 x_0 + (1 - \alpha_1) T_1 x_1 \\ x_2 &= \alpha_2 x_1 + (1 - \alpha_2) T_2 x_2 \\ &\vdots \\ x_N &= \alpha_N x_{N-1} + (1 - \alpha_N) T_N x_N \\ x_{N+1} &= \alpha_{N+1} x_N + (1 - \alpha_{N+1}) T_1 x_{N+1} \\ &\vdots \end{aligned}$$

which can be written in the following compact form:

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_n x_n, \quad n \geq 1 \quad (1.1.1)$$

where  $T_n = T_{n \pmod N}$  (here the *mod*  $N$  function takes values in  $I$ ). Xu and Ori also proved the weak convergence of these process to a common fixed point of the finite family defined in a Hilbert space. In [18], Zhou and Chang studied the weak and strong convergence of this implicit process to a common fixed point for a finite family of nonexpansive mappings in a uniformly convex Banach space. Recently, Chidume and Shahzad [3] proved that Xu and Ori's iteration process converges strongly if one of the mappings is semi-compact. More precisely, they proved the following result.

**Theorem CS** . Let  $X$  be a uniformly convex Banach space and  $C$  be a nonempty closed convex subset of  $X$ . Let  $\{T_i : i \in I\}$  be  $N$  non-expansive self-mappings of  $C$  such that  $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$ . Suppose that  $\{T_i : i \in I\}$  satisfies condition (B). Let  $\{\alpha_n\}_{n=1}^\infty \subset [\delta, 1 - \delta]$  for some  $\delta \in (0, 1)$ . For arbitrary  $x_0 \in C$ , defined the sequence  $\{x_n\}$  by (1.1.1). Then  $\{x_n\}$  converges strongly to a common fixed point of the mappings  $\{T_i : i \in I\}$ .

In the above result, the family  $\{T_i : i \in I\}$  remain self-mappings of a nonempty closed convex subset  $C$  of a real uniformly convex Banach space. If, however, the domain  $C$  of  $T$  is a proper subset of  $X$ (and this is the case in several applications), and each  $T_n$  map  $C$  into  $X$  then, the implicit iteration (1.1.1) may fail converges to a common fixed point of the mappings  $\{T_i : i \in I\}$ .

The purpose of this paper is to construct an implicit iteration scheme for ap- proximating a common fixed point for a finite family of nonexpansive nonself- maps (when such a common fixed point exists)and to prove some strong and weak convergence theorems for such family in a uniformly convexly Banach space. This result presented in this paper extend and improve the corresponding ones announced by Xu and Ori [16], Zhou and Chang [18], Chidume and Shahzad [3], and many others.

## 2. Preliminaries

In this section, we recall the well-known concepts and results.

A mapping  $T : C \longrightarrow C$  is called *demiclosed* with respect to  $y \in X$  if for each sequence  $\{x_n\}$  in  $C$  and each  $x \in X$ ,  $x_n \rightharpoonup x$  and  $Tx_n \longrightarrow y$  imply that  $x \in C$  and  $Tx = y$ . A Banach space  $X$  is said to satisfy *Opial's condition* [8] if for any sequence  $\{x_n\}$  in  $X$ ,  $x_n \rightharpoonup x$  implies that

$$\limsup_{n \rightarrow \infty} \|x_n - x\| < \limsup_{n \rightarrow \infty} \|x_n - y\|$$

for all  $y \in X$  with  $x \neq y$ . A family  $\{T_i : i \in I\}$  of  $N$  nonself-mappings

of  $C$  with  $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$  is said to satisfy *condition (B)* on  $C$  [3] if there is a nondecreasing function  $f : [0, \infty) \rightarrow [0, \infty)$  with  $f(0) = 0$  and  $f(r) > 0$  for all  $r \in (0, \infty)$  such that

$$\max_{1 \leq i \leq N} \{\|x - T_i x\|\} \geq f(d(x, F))$$

for all  $x \in C$ . A mapping  $T : C \rightarrow C$  is called *semi-compact* (or *hemicompact*) if any sequence  $\{x_n\}$  in  $C$  satisfying  $\|x_n - Tx_n\| \rightarrow 0$  as  $n \rightarrow \infty$  has a convergent subsequence. A subset  $C$  of  $X$  is called *retract* of  $X$  if there exists a continuous mapping  $P : X \rightarrow C$  such that  $Px = x$  for all  $x \in C$ . It is well known that every closed convex subset of a uniformly convex Banach space is a retract. A mapping  $P : X \rightarrow C$  is called *retraction* if  $P^2 = P$ . It follows that if a mapping  $P$  is a retraction, then  $Py = y$  for all  $y$  in the range of  $P$ . In what follows, we shall make use the following lemmas.

**Lemma 2.1** [Tan and Xu[14]] Let  $\{s_n\}, \{t_n\}$  be two nonnegative sequences satisfying

$$s_{n+1} \leq s_n + t_n \text{ for all } n \geq 1.$$

If  $\sum_{n=1}^{\infty} t_n < \infty$ , then  $\lim_{n \rightarrow \infty} s_n$  exists. Moreover, if there exists a subsequence  $\{s_{n_j}\}$  of  $\{s_n\}$  such that  $s_{n_j} \rightarrow 0$  as  $j \rightarrow \infty$ , then  $s_n \rightarrow 0$  as  $n \rightarrow \infty$ .

**Lemma 2.2** [Browder[1]] Let  $X$  be a uniformly convex Banach space,  $C$  a nonempty closed convex subset of  $X$  and  $T : C \rightarrow X$  a nonexpansive mapping. Then  $I - T$  is demi-closed at zero.

**Lemma 2.3** [[4], Lemma 1.4] Let  $X$  be a uniformly convex Banach space and  $B_r(0)$  be a closed ball of  $X$ . Then there exists a continuous strictly increasing convex function  $g : [0, \infty) \rightarrow [0, \infty)$  with  $g(0) = 0$  such that

$$\|\lambda x + \mu y + \gamma z\|^2 \leq \lambda \|x\|^2 + \mu \|y\|^2 + \gamma \|z\|^2 - \lambda \mu g(\|x - y\|)$$

for all  $x, y \in B_r(0) = \{x \in X : \|x\| \leq r\}$  and  $\lambda, \mu, \gamma \in [0, 1]$  with  $\lambda + \mu + \gamma = 1$ .

### 3. Main Results

Let  $C$  be a nonempty closed convex subset of a real uniformly convex Banach space  $X$ , which is also a nonexpansive retract of  $X$ . Let  $\{T_i : i \in I\}$ , be a finite family of nonexpansive nonself-mappings from  $C$  to  $X$ . For arbitrary  $x_0 \in C$ , the sequence  $\{x_n\}$  is generated as follows:

$$\begin{aligned} x_1 &= P(\alpha_1 x_0 + \beta_1 T_1 x_1 + \gamma_1 u_1), \\ x_2 &= P(\alpha_2 x_1 + \beta_2 T_2 x_2 + \gamma_2 u_2) \\ &\vdots \\ x_N &= P(\alpha_N x_{N-1} + \beta_N T_N x_N + \gamma_N u_N) \\ x_{N+1} &= P(\alpha_{N+1} x_N + \beta_{N+1} T_1 x_{N+1} + \gamma_{N+1} u_{N+1}) \\ &\vdots \end{aligned}$$

The scheme is expressed in a compact form as:

$$x_n = P(\alpha_n x_{n-1} + \beta_n T_n x_n + \gamma_n u_n) \quad \forall n \geq 1, \quad (3.3.1)$$

where  $T_n = T_{n \bmod N}$ ,  $P$  is a nonexpansive retraction of  $X$  onto  $C$ ,  $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$  are appropriate real sequences in  $[0, 1]$  such that  $\alpha_n + \beta_n + \gamma_n = 1, \forall n \geq 1, \sum_{n=1}^{\infty} r_n < \infty$  and  $\{u_n\}$  is a bounded sequence in  $C$ . Then the implicit iterations scheme (3.3.1) is studied.

**Remark 3.1 :** If each  $T_n$  is a self-mapping of  $C$  and  $\gamma_n \equiv 0$  for all  $n \geq 1$ , then (3.3.1) reduces to an implicit iteration scheme (1.1.1).

**Theorem 3.2 :** Let  $X$  be a uniformly convex Banach space and  $C$  be a nonempty closed convex subset of  $X$  which is also a nonexpansive retract of  $X$ . Let  $\{T_i : i \in I\}$  be  $N$  nonexpansive nonself-mappings from  $C$  into  $X$  with  $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$ . Let  $\{\alpha_n\}, \{\beta_n\}$  and  $\{\gamma_n\}$  be sequences in  $[0, 1]$ . For a given  $x_0 \in C$ , define the sequence  $\{x_n\}$  by (3.3.1). Then  $\lim_{n \rightarrow \infty} \|x_n - x^*\|$  exists for each  $x^* \in F$ .

bf Proof : Let  $x^* \in F$ . we have

$$\|x_n - x^*\| = \|P(\alpha_n x_{n-1} + \beta_n T_n x_n + \gamma_n u_n) - x^*\|$$

$$\begin{aligned}
&\leq \|\alpha_n x_{n-1} + \beta_n T_n x_n + \gamma_n u_n - x^*\| \\
&\leq \alpha_n \|x_{n-1} - x^*\| + \beta_n \|T_n x_n - x^*\| + \gamma_n \|u_n - x^*\| \\
&\leq \alpha_n \|x_{n-1} - x^*\| + \beta_n \|x_n - x^*\| + \gamma_n \|u_n - x^*\| \\
&\leq \alpha_n \|x_{n-1} - x^*\| + (1 - \alpha_n) \|x_n - x^*\| + \gamma_n \|u_n - x^*\|.
\end{aligned}$$

Thus

$$\|x_n - x^*\| \leq \|x_{n-1} - x^*\| + \frac{\gamma_n}{\alpha_n} \|u_n - x^*\|. \quad (3.3.2)$$

It follows from Lemma 2.1 that  $\lim_{n \rightarrow \infty} \|x_n - p\|$  exists. This completes the proof.  $\square$

**Theorem 3.3 :** Let  $X$  be a uniformly convex Banach space and  $C$  be a nonempty closed convex subset of  $X$  which is also a nonexpansive retract of  $X$ . Let  $\{T_i : i \in I\}$  be  $N$  nonexpansive nonself-mappings from  $C$  into  $X$  with  $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$ . Let  $\{\alpha_n\}, \{\beta_n\} \subset [\alpha, \beta]$  for some  $\alpha, \beta \in (0, 1)$  and  $\{\gamma_n\}$  be sequence in  $[0, 1]$ . For a given  $x_0 \in C$ , define the sequence  $\{x_n\}$  by (3.3.1). Then  $\lim_{n \rightarrow \infty} \|x_n - T_l x_n\| = 0, \forall l = 1, 2, \dots, N$ .

**Proof :** Let  $x^* \in F$ . Then, by Theorem 3.2, we obtain  $\lim_{n \rightarrow \infty} \|x_n - x^*\|$  exists and so  $\{x_n\}$  is a bounded sequence. Then there exists  $R > 0$  such that  $x_n \in B_R(0), \forall n \geq 1$ . We claim that  $\lim_{n \rightarrow \infty} \|T_n x_n - x_{n-1}\| = 0$ . It follows from (3.3.2) that

$$\begin{aligned}
\|x_n - x^*\|^2 &= \|P(\alpha_n x_{n-1} + \beta_n T_n x_n + \gamma_n u_n) - x^*\|^2 \\
&= \|\alpha_n (x_{n-1} - x^*) + \beta_n (T_n x_n - x^*) + \gamma_n (u_n - x^*)\|^2 \\
&\leq \alpha_n \|x_{n-1} - x^*\|^2 + \beta_n \|T_n x_n - x^*\|^2 + \gamma_n \|u_n - x^*\|^2 \\
&\quad - \alpha_n \beta_n g(\|x_{n-1} - T_n x_n\|) \\
&\leq \alpha_n \|x_{n-1} - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|u_n - x^*\|^2 \\
&\quad - \alpha_n \beta_n g(\|x_{n-1} - T_n x_n\|) \\
&\leq \alpha_n \|x_{n-1} - x^*\|^2 + \beta_n \left[ \|x_{n-1} - x^*\| + \frac{\gamma_n}{\alpha_n} \|u_n - x^*\| \right]^2 \\
&\quad + \gamma_n \|u_n - x^*\|^2 - \alpha_n \beta_n g(\|x_{n-1} - T_n x_n\|) \\
&= \alpha_n \|x_{n-1} - x^*\|^2 + \beta_n \|x_{n-1} - x^*\|^2 + 2\beta_n \frac{\gamma_n}{\alpha_n} \|x_{n-1}
\end{aligned}$$

$$\begin{aligned}
 & -x^* \|\|u_n - x^*\| + \beta_n \left(\frac{\gamma_n}{\alpha_n}\right)^2 \|u_n - x^*\|^2 \\
 & + \gamma_n \|u_n - x^*\|^2 - \alpha_n \beta_n g(\|x_{n-1} - T_n x_n\|).
 \end{aligned}$$

Thus

$$\begin{aligned}
 \alpha_n \beta_n g(\|x_{n-1} - T_n x_n\|) & \leq (\alpha_n + \beta_n) \|x_{n-1} - x^*\|^2 - \|x_n - x^*\|^2 \\
 & + 2\beta_n \frac{\gamma_n}{\alpha_n} M^2 + \beta_n \left(\frac{\gamma_n}{\alpha_n}\right)^2 M^2 + \gamma_n M^2,
 \end{aligned}$$

where  $M = \max\{R + \|x^*\|, \sup_{n \geq 1} \|u_n - x^*\|\}$ . Since  $\{\alpha_n\}$  and  $\{\beta_n\}$  are sequences in  $[\alpha, \beta]$ , we have

$$\begin{aligned}
 \alpha^2 g(\|x_{n-1} - T_n x_n\|) & \leq \|x_{n-1} - x^*\|^2 - \|x_n - x^*\|^2 + 2\frac{\beta}{\alpha} M^2 \gamma_n \\
 & + \frac{\beta}{\alpha^2} M^2 \gamma_n^2 + M^2 \gamma_n = \|x_{n-1} - x^*\|^2 - \|x_n - x^*\|^2 \\
 & + (2\frac{\beta}{\alpha} M^2 + M^2 + \frac{\beta}{\alpha^2} M^2) \gamma_n.
 \end{aligned}$$

Hence

$$\begin{aligned}
 \alpha^2 \sum_{n=1}^{\infty} g(\|x_{n-1} - T_n x_n\|) & \leq \|x_0 - x^*\|^2 + (2\frac{\beta}{\alpha} M^2 + M^2 \\
 & + \frac{\beta}{\alpha^2} M^2) \sum_{n=1}^{\infty} \gamma_n < \infty.
 \end{aligned}$$

This implies that

$$\lim_{n \rightarrow \infty} g(\|x_{n-1} - T_n x_n\|) = 0.$$

Since  $g$  is strictly increasing continuous and  $g(0) = 0$ . Hence

$$\lim_{n \rightarrow \infty} \|x_{n-1} - T_n x_n\| = 0.$$

Since  $\|x_n - x_{n-1}\| \leq \beta_n \|T_n x_n - x_{n-1}\| + \gamma_n \|u_n - x_{n-1}\|$ , it follows that

$$\lim_{n \rightarrow \infty} \|x_n - x_{n-1}\| = 0.$$

Thus  $\lim_{n \rightarrow \infty} \|x_n - x_{n+l}\| = 0$  for all  $l = 1, 2, \dots, N$ . Now we observe that

$$\|x_n - T_n x_n\| \leq \|x_n - x_{n-1}\| + \|x_{n-1} - T_n x_n\|.$$

This implies that  $\lim_{n \rightarrow \infty} \|x_n - T_n x_n\| = 0$ . For each  $l \in \{1, 2, \dots, N\}$ ,

$$\begin{aligned} \|x_n - T_{n+l} x_n\| &\leq \|x_n - x_{n+l}\| + \|x_{n+l} - T_{n+l} x_{n+l}\| + \|T_{n+l} x_{n+l} - T_{n+l} x_n\| \\ &\leq 2\|x_n - x_{n+l}\| + \|x_{n+l} - T_{n+l} x_{n+l}\| \end{aligned}$$

which in taking the limit  $n \rightarrow \infty$  yields  $\lim_{n \rightarrow \infty} \|x_n - T_{n+l} x_n\| = 0$  for all  $l = 1, 2, \dots, N$ . Consequently, we have  $\lim_{n \rightarrow \infty} \|x_n - T_l x_n\| = 0$ , for all  $l = 1, 2, \dots, N$ .  $\square$

Now, we state and prove our main theorems.

**Theorem 3.4 :** Let  $X$  be a uniformly convex Banach space and  $C$  be a nonempty closed convex subset of  $X$  which is also a nonexpansive retract of  $X$ . Let  $\{T_i : i \in I\}$  be a finite family of nonexpansive nonself-mappings from  $C$  to  $X$  satisfies condition (B) and with  $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$ . Let the sequence  $\{\alpha_n\}, \{\beta_n\} \subset [\alpha, \beta]$  for some  $\alpha, \beta \in (0, 1)$  and  $\{\gamma_n\}$  be sequence in  $[0, 1]$ . For a given  $x_0 \in C$ , define the sequence  $\{x_n\}$  by (3.3.1). Then  $\{x_n\}$  converges strongly to a common fixed point of the mappings  $\{T_i : i \in I\}$ .

**Proof :** For any  $x^* \in F$ . Then as in the proof of from (3.3.2) that

$$\|x_n - x^*\| \leq \|x_{n-1} - x^*\| + \frac{\gamma_n}{\alpha_n} \|u_n - x^*\| \leq \|x_{n-1} - x^*\| + \frac{\gamma_n}{\alpha} K,$$

for all  $n \geq 1$  and  $K = \sup_{n \geq 1} \|u_n - x^*\|$ . This implies that

$$d(x_n, F) \leq d(x_{n-1}, F) + \frac{\gamma_n}{\alpha} K.$$

It follows from Lemma 2.1 that  $\lim_{n \rightarrow \infty} d(x_n, F)$  exists. Since  $\{T_i : i \in I\}$  satisfies condition (B) we have

$$f(d(x_n, F)) \leq \max_{1 \leq l \leq N} \|x_n - T_l x_n\|.$$

Applying Theorem 3.3 we have  $\lim_{n \rightarrow \infty} f(d(x_n, F)) = 0$ . By the property of  $f$  that  $f(0) = 0$  and  $f(r) > 0$  for all  $r \in (0, \infty)$ , we have  $\lim_{n \rightarrow \infty} d(x_n, F) = 0$ . Then the sequence  $\{x_n\}$  is a Cauchy sequence (see [?]). By the completeness of the space  $X$ , there exists  $p \in C$  such that

$\lim_{n \rightarrow \infty} x_n = p$ . Next, we prove that  $p \in F$ . To this end, we let  $\varepsilon > 0$  be given. Then there exists  $n_1 \in \mathbb{N}$ , such that

$$\|x_n - p\| < \frac{\varepsilon}{4},$$

for each  $n \geq n_1$ . Since  $\lim_{n \rightarrow \infty} d(x_n, F) = 0$ , there exists  $n_2 \geq n_1$  such that  $d(x_{n_2}, F) < \frac{\varepsilon}{4}$ . This implies that there exists  $w^* \in F$  such that  $\|x_{n_2} - w^*\| < \frac{\varepsilon}{4}$ . Then for each  $i \in I$  and  $n \geq n_2$ , we have

$$\|T_i p - p\| \leq \|T_i p - w^*\| + \|w^* - p\| \leq 2\|p - w^*\| \leq 2(\|p - x_{n_2}\| + \|x_{n_2} - w^*\|) < \varepsilon.$$

This implies that  $T_i p = p$  for all  $i \in I$ . Hence  $p \in F$ . This completely the proof.

**Theorem 3.5** : Let  $X$  be a uniformly convex Banach space and  $C$  be a nonempty closed convex subset of  $X$  which is also a nonexpansive retract of  $X$ . Let  $\{T_i : i \in I\}$  be a finite family of  $N$  nonexpansive nonself-mappings from  $C$  to  $X$  with  $F \neq \emptyset$ . Suppose that one of the mapping in  $\{T_i : i \in I\}$  is semi-compact. Let the sequence  $\{\alpha_n\}, \{\beta_n\} \subset [\alpha, \beta]$  for some  $\alpha, \beta \in (0, 1)$  and  $\{\gamma_n\}$  be sequence in  $[0, 1]$ . For given  $x_0 \in C$ , the sequence  $\{x_n\}$  defined by (3.3.1), converges strongly to a common fixed point of the mapping  $\{T_i : i \in I\}$ .

**Proof** : Suppose that  $T_{i_0}$  is semi-compact for some  $i_0 \in I$ . It follows from Theorem 3.3 that  $\lim_{n \rightarrow \infty} \|T_{i_0} x_n - x_n\| = 0$ . By semi-compactness of the mapping  $T_{i_0}$  there exist a subsequence  $\{x_{n_j}\}$  of  $\{x_n\}$  such  $\lim_{j \rightarrow \infty} x_{n_j} = p \in C$  and  $\lim_{j \rightarrow \infty} \|T_{i_0} x_{n_j} - x_{n_j}\| = 0$ . Now by Theorem 3.3, we obtain that  $\lim_{j \rightarrow \infty} \|x_{n_j} - T_l x_{n_j}\| = 0$  for all  $l \in I$ , this implies that  $\|p - T_l p\| = 0$  for all  $l \in I$ . Thus  $p \in F$ . which leads to  $\lim_{n \rightarrow \infty} d(x_n, F) = 0$ , it follows as in the proof of Theorem 3.4, we obtain  $\{x_n\}$  converges strongly to some common fixed point in  $F$ . This completely the proof.

For each  $T_n$  are nonexpansive self-mappings of  $C$ , and  $\gamma_n \equiv 0$  for all  $n \geq 0$ , so we can obtain the following result which is a generalization of Theorem 3.2 and Theorem 3.3. in [?].

**Theorem 3.6** [Theorem 3.2, [3]] : Let  $X$  be a uniformly convex Banach space and  $C$  be a nonempty closed convex subset of  $X$ . Let  $\{T_i : i \in I\}$

be  $N$ -nonexpansive self-mappings of  $C$  with  $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$ . Suppose that  $\{T_i : i \in I\}$  satisfies condition (B). Let  $\{\alpha_n\}_{n \geq 1} \subset [\delta, 1 - \delta]$  for some  $\delta \in (0, 1)$ . From arbitrary  $x_0 \in C$ , define the sequence  $\{x_n\}$  by

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_n x_n, \quad \forall n \geq 1.$$

Then  $\{x_n\}$  converges strongly to a common fixed point of the mappings  $\{T_i : i \in I\}$ .

**Theorem 3.7** [Theorem 3.3 [3]] : Let  $X$  be a uniformly convex Banach space and  $C$  be a nonempty closed convex subset of  $X$ . Let  $\{T_i : i \in I\}$  be  $N$ -nonexpansive self-mappings of  $C$  with  $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$ . Suppose that one of the mapping in  $\{T_i : i \in I\}$  is semi-compact. Let  $\{\alpha_n\}_{n \geq 1} \subset [\delta, 1 - \delta]$  for some  $\delta \in (0, 1)$ . From arbitrary  $x_0 \in C$ , define the sequence  $\{x_n\}$  by

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_n x_n, \quad \forall n \geq 1.$$

Then  $\{x_n\}$  converges strongly to a common fixed point of the mappings  $\{T_i : i \in I\}$ .

In the next results, we prove weak convergence of the sequence  $\{x_n\}$  which define by (3.3.1) in uniformly convex Banach space satisfying *Opial's condition*.

**Lemma 3.8** [Suantai[12]] : Let  $X$  be a Banach space which satisfies Opial's condition and let  $\{x_n\}$  be a sequence in  $X$ . Let  $u, v \in X$  be such that  $\lim_{n \rightarrow \infty} \|x_n - u\|$  and  $\lim_{n \rightarrow \infty} \|x_n - v\|$  exists. If  $\{x_{n_k}\}$  and  $\{x_{m_k}\}$  are subsequence of  $\{x_n\}$  which converge weakly to  $u$  and  $v$ , respectively, then  $u = v$ .

**Theorem 3.9** : Let  $X$  be a uniformly convex Banach space which satisfies Opial's condition, and  $C$  a nonempty closed convex subset of  $X$  which is also nonexpansive retract of  $X$ . Let  $\{T_i : i \in I\}$  be a finite family of  $N$ -nonexpansive nonself-mappings from  $C$  to  $X$  such that  $F \neq \emptyset$ . Let  $x_0 \in C$ ,  $\{\alpha_n\}, \{\beta_n\}$  be sequence in  $[\alpha, \beta]$  for some  $\alpha, \beta \in (0, 1)$  and  $\{\gamma_n\} \subset [0, 1]$  and  $\{x_n\}$  be a sequence generated by

(3.3.1). Then  $\{x_n\}$  converges weakly to a common fixed point of the mapping  $\{T_i : i \in I\}$ .

**Proof :** Since  $X$  is uniformly convex and  $\{x_n\}$  is bounded, we may assume that  $x_n \rightarrow q$  weakly as  $n \rightarrow \infty$ , with out loss of generality. By Theorem 3.2, we have  $q \in F(T_i)$  for all  $i \in I$ . Hence  $q \in F$ . Suppose that there exist subsequence  $\{x_{n_k}\}$  and  $\{x_{m_k}\}$  of  $\{x_n\}$  converge weakly to  $u$  and  $v$  respectively. By Lemma 2.2,  $u, v \in F$ . By Theorem 3.2  $\lim_{n \rightarrow \infty} \|x_n - u\|$  and  $\lim_{n \rightarrow \infty} \|x_n - v\|$  exists. It follows from Lemma 3.8 that  $u = v$ . Therefore  $\{x_n\}$  converges weakly to a common fixed point  $p$  in  $F$ .

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