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PRINCIPAL k-LEFT IDEALS IN THE MATRIX SEMIRING $M_n(\mathfrak{B})$

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Abstract

Let $\mathfrak{B} = \{0, 1\}$. Here the two operations of '+' and '.' on $\mathfrak{B}$ are defined as follows : $0+0 = 0$, $0+1 = 1$, $1+0 = 1$, $1+1 = 1$, $0.0 = 0$, $0.1 = 0$, $1.0 = 0$, $1.1 = 1$. Then $(\mathfrak{B}, +, .)$ is a semiring called the Boolean semiring. The set of all $n \times n$ matrices over the Boolean semiring $\mathfrak{B}$ form a semiring under the operations of matrix addition and matrix multiplication and is denoted by $M_n(\mathfrak{B})$, where n is a positive integer. In this paper we study the principal left ideals generated by the matrices in the semiring $M_n(\mathfrak{B})$, $(n > 1)$ and characterise the principal k-left ideals of $M_n(\mathfrak{B})$.

1. Introduction

The concept of semiring was introduced by H.S.Vandiver [5] in 1934. A semiring is a nonempty set S on which the operations of '+' and '.' have been defined such that the following conditions are satisfied :

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1. $(S,+)$ is a commutative monoid with identity element 0.
2. $(S,.)$ is a monoid with identity element 1.
3. Multiplication '.' distributes over addition '+'from either side.
4. $0.s = 0 = s.0$, for all $s \in S$.

Here after S denotes the semiring $(S,+,.)$ and sa denotes $s.a$ for $s, a \in S$.Here $1 \neq 0$, to avoid the trivial case. Again 0 is the only absorbing zero. For, if $z \in S$ with $zs = z = sz$, for all $s \in S$ then $0 = 0z = z$. A subset $A \neq \phi$ of a semiring S is called a left (right) ideal of S if $a + b \in A$ and $sa \in A(as \in A)$, for all $a, b \in A$ and $s \in S$. An ideal in a semiring S is a nonempty subset which is both a left and right ideal. A principal left ideal generated by $a \in S$ is the left ideal $Sa = \{sa : s \in S\}$. A principal right ideal generated by $a \in S$ is the right ideal $aS = \{as : s \in S\}$. Throughtout this paper we use the notation $< a >_L$ to denote the principal left ideal generated by $a \in S$ and $< a >_R$ to denote the principal right ideal generated by $a \in S$. Any two elements of a semiring S are said to be $\mathscr{L}$-equivalent if they generate the same principal left ideal of S. $\mathscr{R}$-equivalence is defined dually. In 1958, Henriksen [1] defined a more restricted class of ideals in a semiring which he called a k-ideal. An ideal A of a semiring S is called a k-deal if $a, a + b \in A$ implies that $b \in A$,for all $a, b \in S$. This ideal is also called a subtractive ideal. Both $\{0\}$ and S are k-deals of the semiring S. For the terminology regarding semirings used in this paper refer [2] and [3].

1.1Boolean Semiring

We denote the Boolean semiring by $(\mathfrak{B},+,.)$ where $\mathfrak{B} = \{0,1\}$ and the operations of + and . are defined as follows:

$$0 + 0 = 0, 0 + 1 = 1, 1 + 0 = 1, 1 + 1 = 1, 0.0 = 0, 0.1 = 0, 1.0 = 0, 1.1 = 1.$$

Throughout this paper $\mathfrak{B}$ refers to the Boolean semiring $(\mathfrak{B},+,.)$

1.2 Boolean Vector Space

We have the following from K. H. Kim [4]. Let V_n denote the set of all n-tuples $(a_1, a_2, ..., a_n)$ over the Boolean semiring $\mathfrak{B}$. An element of V_n is called a Boolean vector of dimension n. The system V_n together with the operations of componentwise addition and scalar multiplication is called the Boolean vector space of dimension n. Let $V^n = \{v^T : v \in V_n\}$, where v^T means the column vector. Let e_i be the n-tuple with 1 as the i^{th} coordinate and 0 otherwise. Further, we define $e^i = {e_i}^T$. A subset B of a vectorspace V is said to be a basis of Vif B is linearly independent and B generates V. Here $\{e_1, e_2, \ldots, e_n\}$ is the basis of V_n and $\{{e_1}^T, {e_2}^T, ..., {e_n}^T\}$ is the basis of V^n.

1.3 Matrices over a Boolean Semiring

The set of all $n \times n$ matrices over the Boolean semiring $\mathfrak{B}$ form a semiring under the operations of matrix addition and matrix multiplication and is denoted by $M_n(\mathfrak{B})$, where n is a positive integer. In this paper, by a matrix we mean a matrix in $M_n(\mathfrak{B})$, where $n > 1$, unless otherwise stated.

Let $A = [a_{ij}] \in M_n(\mathfrak{B})$. Then the element a_{ij} is called the $(i, j)^{th}$ entry of A. The i^{th} row vector of A is $(a_{i1}, a_{i2}, \ldots, a_{in})$ and the j^{th} column vector of A is $(a_{1j}, a_{2j}, \ldots, a_{nj})^T$. The zero matrix O_n is the matrix, all of whose entries are zero. The identity matrix I_n is the matrix $[\delta_{ij}]$, such that $\delta_{ij} = 1$, if $i = j$ and $\delta_{ij} = 0$, if $i \neq j$. The matrix E_{ij} is the matrix whose $(i, j)^{th}$ entry is 1 and 0 otherwise. A matrix is called a permutation matrix if every row and every column contains exactly one 1. A matrix is said to be a partial permutation matrix if every row and every column of it contains atmost one 1. The row space of a matrix A is the span of the set of all rows of A. A column space is dually defined. Let $R(A)\,(C(A))$ denote the row (column) space of a matrix A. For any matrix $A \in M_n(\mathfrak{B})$, $R(A)$ is a subspace of V_n and $C(A)$ is a subspace of V^n. The following result from [4] is used in this paper.

Proposition 1.1 [4] : Two matrices $A, B \in M_n(\mathfrak{B})$ are $\mathscr{L}$-equivalent if and only if they have the same row space.

2. Principal Left Ideals of $M_n(\mathfrak{B})$

Here we determine the principal left ideals and the principal k-left ideals of $M_n(\mathfrak{B})$.

Proposition 2.1 : The row space of each matrix in the principal left ideal generated by a matrix $B \in M_n(\mathfrak{B})$ is contained in the row space of B.

Proof : Suppose that $B = [b_{ij}]$. We know that the matrices in the principal left ideal generated by the matrix B will be products of the form AB, where $A \in M_n(\mathfrak{B})$. If

$A = [a_{ij}]$, then the $(i,j)^{th}$ element of AB is $\sum_{k=1}^{n} a_{ik}b_{kj}$. Hence the i^{th} row vector of AB

$$= (a_{i1}b_{11} + a_{i2}b_{21} + \cdots + a_{in}b_{n1}, a_{i1}b_{12} + a_{i2}b_{22} + \cdots + a_{in}b_{n2}, \cdots, a_{i1}b_{1n} + a_{i2}b_{2n} + \cdots + a_{in}b_{nn})$$

$$= (a_{i1}b_{11}, a_{i1}b_{12}, \cdots, a_{i1}b_{1n}) + (a_{i2}b_{21}, a_{i2}b_{22}, \ldots, a_{i2}b_{2n}) + \cdots + (a_{in}b_{n1}, a_{in}b_{n2}, \ldots, a_{in}b_{nn})$$

$$= a_{i1}(b_{11}, b_{12}, \ldots, b_{1n}) + a_{i2}(b_{21}, b_{22}, \ldots, b_{2n}) + \cdots + a_{in}(b_{n1}, b_{n2}, \ldots, b_{nn})$$

$$= a_{i1}b_1 + a_{i2}b_2 + \cdots + a_{in}b_n,$$

where $b_k = (b_{k1}, b_{k2}, \ldots, b_{kn})$ is the k^{th} row vector of B.

That is , the i^{th} row of AB is of the form $\sum_{k=1}^{n} a_{ik}b_k$, where b_k represents the k^{th} row vector of B. In other words the i^{th} row of the product AB is a linear combination of the rows of B where the scalars for the i^{th} row are a_{ik}, for $k = 1, 2, \ldots, n$ which is either 0 or 1. Hence we see that the rows of AB are finite sums of the row vectors of B. Now $R(B)$ is the span of the row vectors of B. That is, $R(B)$ is the set of all finite sums of the rows of B. Thus the row space of each matrix in the principal left ideal generated by a matrix $B \in M_n(\mathfrak{B})$ is contained in the row space of B. □

Remark 2.2 : Let $< B >_L = \{B_1, B_2, \ldots, B_m\}$. Then each row of B_j, for $j = 1, 2, \ldots, m$ is a finite sum of the rows of B.

Theorem 2.3 : If $< B >_L = \{B_1, B_2, \ldots, B_m\}$ then $R(B_1) \cup R(B_2) \cup \ldots \cup R(B_m) = R(B)$.

Proof : Let $B = [b_{ij}] \in M_n(\mathfrak{B})$ and $< B >_L = \{B_1, B_2, \ldots, B_m\}$. Then by Proposition 2.1, $R(B_1) \subseteq R(B), R(B_2) \subseteq R(B), \ldots, R(B_m) \subseteq R(B)$. Hence $R(B_1) \cup R(B_2) \cup \ldots \cup R(B_m) \subseteq R(B)$.

On the other hand, let $x \in R(B)$. Then $x = a_{i1}b_1 + a_{i2}b_2 + \cdots + a_{in}b_n$, where $a_{ik} = 0$ or 1, for $k = 1, 2, \ldots, n$ and $b_k = (b_{k1}, b_{k2}, \ldots, b_{kn})$ is the k^{th} row vector of B. Now consider the matrix $A = [c_{rs}]$, where $c_{rs} = a_{ik}$, if $r = i, s = k$ and 0 otherwise. Then $A \in M_n(\mathfrak{B})$ and

$$AB = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \ldots & a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{i1} & b_{i2} & \ldots & b_{in} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

Hence the i^{th} row vector of AB

$$= (a_{i1}b_{11} + a_{i2}b_{21} + \cdots + a_{in}b_{n1}, a_{i1}b_{12} + a_{i2}b_{22} + \cdots + a_{in}b_{n2}, \cdots, a_{i1}b_{1n} + a_{i2}b_{2n} + \cdots + a_{in}b_{nn})$$
$$= (a_{i1}b_{11}, a_{i1}b_{12}, \ldots, a_{i1}b_{1n}) + (a_{i2}b_{21}, a_{i2}b_{22}, \ldots, a_{i2}b_{2n}) + \cdots + (a_{in}b_{n1}, a_{in}b_{n2}, \ldots, a_{in}b_{nn})$$
$$= a_{i1}(b_{11}, b_{12}, \ldots, b_{1n}) + a_{i2}(b_{21}, b_{22}, \ldots, b_{2n}) + \cdots + a_{in}(b_{n1}, b_{n2}, \ldots, b_{nn})$$
$$= a_{i1}b_1 + a_{i2}b_2 + \cdots + a_{in}b_n$$
$$= x.$$

This shows that $x \in R(AB)$, where $AB = B_j$, for some $j = 1, 2, \ldots, m$. Therefore $R(B) \subseteq R(B_1) \cup R(B_2) \cup \ldots \cup R(B_m)$. Thus we have $R(B_1) \cup R(B_2) \cup \ldots \cup R(B_m) = R(B)$. □

Remark 2.4 : The principal left ideal generated by O_n is $\{0\}$ and it is a k-ideal.

Theorem 2.5 : The row space of a matrix $A \in M_n(\mathfrak{B})$ is V_n if and only if A is a permutation matrix.

Proof : Let $A \in M_n(\mathfrak{B})$. First assume that $R(A) = V_n$. Then $\{e_1, e_2, \ldots, e_n\}$ is the basis for $V_n = R(A)$. Thus $\{e_1, e_2, \ldots, e_n\}$ form the n row vectors of A. Therefore A is a permutation matrix. Conversely assume that A is a permutation matrix. By definition we know that the rows of A are precisely $e_1, e_2, \ldots, e_n$ in some order. Since these vectors form the basis for V_n, the row space of A is V_n. Thus the result. □

Theorem 2.6 : The principal left ideal generated by a matrix B is $M_n(\mathfrak{B})$ if and only if B is a permutation matrix.

Suppose that the principal left ideal generated by B is $M_n(\mathfrak{B})$. Then by Theorem 2.3, the row space of B is V_n. Hence by Theorem 2.5, B is a permutation matrix.

Conversely suppose that B is a permutation matrix. Then by Theorem 2.5, the row space of B is V_n. Consider the identity matrix $I_n \in M_n(\mathfrak{B})$. Since it is also a permutation matrix, by Theorem 2.5, the row space of I_n is V_n. By Proposition 1.1, we know that matrices with the same row space are $\mathscr{L}$-equivalent. That is, they generate the same principal left ideal. We know that the principal left ideal generated by I_n is $M_n(\mathfrak{B})$ itself. Therefore the principal left ideal generated by B is $M_n(\mathfrak{B})$. □

Remark 2.7 : The principal left ideal generated by a permutation matrix is a k-ideal, since $M_n(\mathfrak{B})$ is a k-ideal.

Theorem 2.8 : Each matrix in the principal left ideal generated by a matrix B having exactly one column zero, has that corresponding column zero.

Proof : Let $B = [b_{kl}] \in M_n(\mathfrak{B})$ be a matrix in which the j^{th} column is the only zero column. Then $b_{kj} = 0, \forall k = 1, 2, \ldots, n$. We know that the matrices in the principal left ideal generated by B will be products of the form AB, where $A = [a_{ik}]$ is any matrix in $M_n(\mathfrak{B})$. Then for $i = 1, 2, \ldots, n$ the $(i, j)^{th}$ element of $AB = \sum_{k=1}^{n} a_{ik}b_{kj} = 0$, since $b_{kj} = 0, \forall k = 1, 2, \ldots, n$. This means that the $(1, j)^{th}, (2, j)^{th}, \ldots, (n, j)^{th}$ elements are zero. Therefore the j^{th} column of AB is 0. Thus a matrix in which exactly one column is zero will generate a principal left ideal in which each matrix has that corresponding column zero. □

Corollary 2.9 : Each matrix in the principal left ideal generated by a matrix B having more than one column zero, has those corresponding columns zero.

Proof : Follows directly from Theorem 2.8. □

Remark 2.10 : Every permutation matrix is a partial permutation matrix. But the converse need not be true. For example, the matrix $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is a partial permutation matrix but not a permutation matrix.

Theorem 2.11 : The principal left ideal generated by a partial permutation matrix is a k-ideal.

Proof : Let $B \in M_n(\mathfrak{B})$ be a partial permutation matrix. Then from the definition of a partial permutation matrix we know that the rows of B will be any n vectors from the set $\{e_1, e_2, \ldots, e_n, 0\}$, without the repetition of the basis row vectors $e_i, i = 1, 2, \ldots, n$.

Case 1 : Let the rows of B be $\{e_1, e_2, \ldots, e_n\}$ in some order. Then B is a permutation matrix. Hence by Remark 2.7, the principal left ideal generated by B is a k-ideal.

Case 2 : Let B be a partial permutation matrix in which the j^{th} column is the only zero column and the rows of B are the $(n-1)$ basis vectors $e_1, e_2, \ldots, e_{j-1}, e_{j+1}, \ldots, e_n$ and 0 in some order. Now by Theorem 2.3, we see that the rows of the matrices in the principal left ideal generated by B will have all possible finite sums of the rows of B. Since the rows of B contain all the basis vectors $e_1, e_2, \ldots, e_n$ except e_j , the rows of the matrices in the principal left ideal generated by B will be all the rows with the j^{th}

entry zero. Thus the principal left ideal generated by B will have all the matrices with the j^{th} column zero.

Now, to prove that the principal left ideal generated by B is a k-ideal, let $X, X + Y \in < B >_L$ where $X = [x_{pq}]$ and $Y = [y_{pq}]$.Hence $X + Y = [x_{pq} + y_{pq}]$, where $p = 1, 2, \ldots, n, q = 1, 2, \ldots, n$. Since $X, X+Y \in < B >_L$ we have $x_{pj} = 0$, for $p = 1, 2, \ldots, n$ and also $x_{pj} + y_{pj} = 0$, for $p = 1, 2, \ldots, n$.Hence $y_{pj} = 0$, for $p = 1, 2, \ldots, n$. Thus $Y \in < B >_L$. Hence the principal left ideal generated by B is a k-ideal.

Case 3 : Let B be a partial permutation matrix having $r(\neq 0, 1)$ zero columns (say $j_1, j_2, \ldots, j_r$) and the rows of B are the $(n-r)$ basis vectors of the set $\{e_1, e_2, \ldots, e_n\} - \{e_{j_1}, e_{j_2}, \ldots, e_{j_r}\}$ and zero vector in some order without the repetition of basis row vectors. Then as in case 2 we can prove that the principal left ideal generated by B will be a k-ideal. □

Definition 2.12 : A matrix is said to be a row partial permutation matrix if every row contains atmost one 1.

Remark 2.13 : Every partial permutation matrix is a row partial permutation matrix. But the converse need not be true. For example, the matrix $A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ is a row partial permutation matrix but not a partial permutation matrix.

Theorem 2.14 : The principal left ideal generated by a row partial permutation matrix is a k- ideal.

Proof : Let $B \in M_n(\mathfrak{B})$ be a row partial permutation matrix. Then from definition **??**, we know that the rows of B will be any n vectors from the set $\{e_1, e_2, \ldots, e_n, 0\}$.

Case 1 : Let the rows of B be $e_1, e_2, \ldots, e_n$ in some order. Then B is a permutation matrix. Hence by Remark 2.7, the principal left ideal generated by B is a k-ideal.

Case 2 : Let the rows of B be any n vectors from the set $\{e_1, e_2, \ldots, e_n, 0\}$, without the repetition of the basis row vectors $e_i, i = 1, 2, \ldots, n$. Then B is a partial permutation matrix. Hence by Theorem 2.11, the principal left ideal generated by B is a k-ideal.

Case 3 : Let $B \in M_n(\mathfrak{B})$ be a row partial permutation matrix in which the j^{th} column is the only zero column and the rows of B are from the $(n-1)$ basis vectors $e_1, e_2, \ldots, e_{j-1}, e_{j+1}, \ldots, e_n$ with repetition in some order. Now by Theorem 2.3, we know that the rows of the matrices in the principal left ideal generated by B will have all possible finite sums of the rows of B. Since the rows of B are from the basis vectors $e_1, e_2, \ldots, e_n$ except e_j , the rows of the matrices in the principal left ideal generated by B will be all the rows with the j^{th} entry zero. Thus the principal left ideal generated by B will have all the matrices with the j^{th} column zero.

Now, to prove that the principal left ideal generated by B is a k-ideal, let $X, X + Y \in \ < B >_L$ where $X = [x_{pq}]$ and $Y = [y_{pq}]$. Hence $X + Y = [x_{pq} + y_{pq}]$, where $p = 1, 2, \ldots, n, q = 1, 2, \ldots, n$. Since $X, X + Y \in \ < B >_L$ we have $x_{pj} = 0$, for $p = 1, 2, \ldots, n$ and also $x_{pj} + y_{pj} = 0$, for $p = 1, 2, \ldots, n$. Hence $y_{pj} = 0$, for $p = 1, 2, \ldots, n$. Thus $Y \in \ < B >_L$. Hence the principal left ideal generated by B is a k-ideal.

Case 4 : Let B be a row partial permutation matrix having $r(\neq 0, 1)$ zero columns (say $j_1, j_2, \ldots, j_r$) and the rows of B are from the $(n - r)$ basis vectors of the set $\{e_1, e_2, \ldots, e_n\} - \{e_{j_1}, e_{j_2}, \ldots, e_{j_r}\}$ with repetition in some order. Then as in case 3 we can prove that the principal left ideal generated by B is a k-ideal.

Case 5 : Let B be a row partial permutation matrix having $r(\neq 0, 1)$ zero columns (say $j_1, j_2, \ldots, j_r$) and the rows of B are from the $(n - r)$ basis vectors of the set $\{e_1, e_2, \ldots, e_n\} - \{e_{j_1}, e_{j_2}, \ldots, e_{j_r}\}$ and zero vector with repetition in some order. Then as in case 3 we can prove that the principal left ideal generated by B is a k-ideal. □

Till now we have considered matrices having atmost one non-zero element in every row. Now let us look at the case of matrices having rows with more than one non-zero element.

Theorem 2.15 : Let $B = [b_{ij}]$ be a non-zero matrix in $M_n(\mathfrak{B})$. Suppose that B has rows having more than one non-zero elements and it contains basis row vector e_k, whenever $b_{ik} = 1$. Then the principal left ideal generated by B is a k-ideal.

Proof : Consider a non-zero matrix $B = [b_{ij}] \in M_n(\mathfrak{B})$, having rows with more than one non-zero element and having the row e_k, whenever $b_{ik} = 1$. By Remark 2.2, we see that $< B >_L$ will contain those matrices whose rows are finite sums of the rows of B. Let $B_1 \in < B >_L$. Then the rows of B_1 are finite sums of the rows of B. Let $B_2 \in M_n(\mathfrak{B})$ such that $B_1 + B_2 \in < B >_L$. Then the rows of $B_1 + B_2$ are finite sums of the rows of B. Therefore the rows of B_2 has non-zero elements only in those positions where the rows of B has non-zero elements. Hence the rows of B_2 are also finite sums of the rows of B under the stated conditions of B. Therefore $B_2 \in < B >_L$. Thus the principal left ideal generated by B is a k-ideal. □

Definition 2.16 : If the principal left ideal generated by a matrix B is a k-ideal then it is called a principal k-left ideal of $M_n(\mathfrak{B})$. It is denoted by $< B >_{L_k}$.

In the following theorem we characterize the principal k-left ideals of $M_n(\mathfrak{B})$.

Theorem 2.17 : Consider a non-zero matrix $B = [b_{ij}]$ in $M_n(\mathfrak{B})$. Then $< B >_L$ is a k- ideal if and only if B contains basis row vector e_k whenever $b_{ik} = 1$.

Proof : Consider a non-zero matrix $B = [b_{ij}]$ in $M_n(\mathfrak{B})$. Assume that $< B >_L$ is a k-ideal. Suppose that B is a row partial permutation matrix. Then, by Defintion 2.12, B contains basis row vector e_k whenever $b_{ik} = 1$. Hence the theorem is true in this case. Next we consider the case where B is not a row partial permutation matrix. Then B has rows with more than one non-zero element. Without loss of generality assume that for $i = 1, b_{ij} = 1$, for $j = 1, 2$ and $b_{ij} = 0$ otherwise. Without loss of generality also assume that B has e_1 as one of the rows and it does not have any row of the form e_2. Then, by Remark 2.2, the principal left ideal $< B >_L$ will contain matrices with rows which are finite sums of the rows of B. Then the matrices in the principal left ideal will contain rows with more than one non-zero elements (ie., in the 1^{st} and 2^{nd} positions respectively). But this principal left ideal $< B >_L$ will not have matrices with the row e_2. Then the principal left ideal generated by B will not be a k-ideal, which is a contradiction. For, we have $B \in < B >_L$. Also, $B + E_{12} = B \in < B >_L$. But $E_{12} \notin < B >_L$, showing that $< B >_L$ is not a k-ideal. Hence our assumption that B does not contain the basis row vector e_2 is wrong. Thus B contains the basis row vector e_k whenever $b_{ik} = 1$.

Conversely assume that B contains basis row vector e_k, whenever $b_{ik} = 1$. Then B is either a row partial permutation matrix or a matrix having rows with more than one non-zero elements. If B is a row partial permutation matrix then by Theorem 2.14, $< B >_L$ is a k-ideal. On the other hand if B is a matrix having rows with more than one non-zero element, then by Theorem 2.15, $< B >_L$ is a k-ideal. □

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