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COMMON FIXED POINT RESULTS FOR A MAPPING IN 2-METRIC SPACES

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Abstract

2-metric spaces is an attractive nonlinear generalization of metric spaces which was studied in details by Gahler. In this note, some common fixed point results in 2-metric spaces are obtained. Our results generalize the theorem of Lahiri et al. [12], in the context of 2-metric spaces. Some results are also proved in different way as proved earlier.

1. Introduction and Preliminaries

The concept of 2-metric spaces was initiated by Gahler in a series of papers ([3]-[5]). The 2-metric space have a unique nonlinear structure, which is different from metric spaces. Gahler and White [20] extended the concept to 2-Banach spaces, while White established Hahn-Banach theorem in 2-Banach spaces. Further many of the authors studied and generalize the theorems in this 2-metric spaces. Iseki ([7]-[9]) obtained basic results on fixed point of operators in 2-metric spaces and 2-Banach spaces. After

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the work of Iseki, many authors studied and generalized fixed point theorems in 2-metric spaces and 2- Banach spaces. For more information on fixed point theory in the above spaces, one can refer ([6], [8], [13] - [18]).

For convenience we recall some basic definitions and properties from the theory of 2-metric spaces used in the sequel.

Definition 1.1 [3] : Let X be a non-empty set and let σ be a mapping from $X \times X \times X \to R$ i.e. $\sigma : X^3 \to R$ satisfying the following conditions:

1. for every pair of distinct points a, b there exist a point $c \in X$ such that $\sigma(a, b, c) \neq 0$.
2. $\sigma(a, b, c) = 0$ only if at least two of the three points are same.
3. $\sigma(a, b, c) = \sigma(a, c, b) = \sigma(b, c, a)$ for all $a, b, c \in X$.
4. $\sigma(a, b, c) \leq \sigma(a, b, d) + \sigma(a, d, c) + \sigma(d, b, c)$ for all $a, b, c, d \in X$.

Then σ is called a 2- metric on X and (X, σ) is called a 2- metric space which will sometimes be denoted by X, when there is no confusion.

It can be easily seen that σ is a non-negative function. we shall assume through out that X is an infinite set.

Definition 1.2 [3] : Let (X, σ) be a 2- metric space. Let $a, b \in X$ and $r > 0$. The subset

$$B_r(a, b) = \{c \in X; \sigma(a, b, c) < r\}$$

of X will be called a 2- ball concerted at a and b with radius r.

From the definition of a 2- metric, it is clear that $B_r(a, b)$ is a same as $B_r(b, a)$ Gahler [**?**] observe that the topology can be generated in X by taking the collection of all 2-balls as a sub-bassic, which we call here the 2-metric topology, to be denoted by τ. Thus (X, τ) is a 2- metric topological space. Members of τ are called 2- open sets and their complements, 2-closed sets.

Definition 1.3 [3] : For $A \subset (X, \tau)$, the 2- closure of A, denoted by $\bar{A}$ is defined to be the intersection of all 2- closed sets containing A.

Definition 1.4 [3] : $x \in (X, \tau)$ is called 2- limit point of $A \subset X$ if for any 2-open set U containing $x, A \cap (U - \{x\}) \neq \phi$.

As in the topological space, A can also be defined by $\bar{A} = A \cup \partial A$ where ∂A is the derived set of A that consists of all 2- limit points of A. For any $A \subset X, \bar{A}$ is clearly a 2- closed set.

Proportion 1.1 [3] : $A \subset (X, \tau)$, is 2- closed if and only if $\bar{A} = A$.

Proportion 1.2 [3] : (X, τ) is T_1.

Definition 1.5 [7] : A sequence $\{x_n\}$ in (X, σ) is said to converge to $x \in X$ if for any $a \in X, \sigma(x_n, x, a) \to 0$ as $n \to \infty$.

i.e. $x_n \to x$ as $n \to \infty$ or $lim_{n \to \infty} x_n = x$.

Definition 1.6 [7] : A sequence $\{x_n\}$ in (X, σ) is said to be a Cauchy sequence if for any $a \in X, \sigma(x_m, x_n, a) \to 0$ as $m, n \to \infty$.

Definition 1.7 ([7]-[8] : (X, σ) is said to be a complete if every Cauchy sequence in X converge to a point of X.

Definition 1.8 : (X, σ) is said to be a compact if every sequence in X has a convergent sub-sequence.

Definition 1.9 : $A \subset X$ is said to be dense in X if $\bar{A} = X$.

Definition 1.10 : $A \subset X$ is said to be no-where dense if $int(\bar{A}) = \phi$ where interior of a set B is defined to be the union of all 2- open sets contained in B.

Definition 1.11 : A mapping $T : (X, \sigma) \to (Y, \sigma_1)$ where (Y, σ_1) is another 2- metric space, is called continuous at $x \in X$ if for any 2- open sets V containing $f(x)$ in $\in Y$, there is a 2- open set U containing $x \in X$ such that $T(U) \subset V$.

Let $T : X \to X$ be a mapping. For $t > 0,\ St$ is defined as

$$S_t = \{x \in X; \sigma(x, Tx, y) \leq \epsilon\ \forall y \in X\}$$

Lemma 1.1 : Let (X, σ) be a $2 - metric$ space and T be continuous mapping of X into itself. Then S_t is closed.

Proof : Let $\{y_n\}$ be a sequence of points of the set S_t converging to $z \in X$. In order to prove S_t is closed. We shall show that $z \in S_t$

Let $\epsilon > 0$ be arbitrary. Then there exists a positive integer N such that for all $n \geq N$

$$\begin{aligned} \sigma(z, Ty_n, a) &\leq \sigma(z, Ty_n, y_n) + \sigma(z, y_n, a) + \sigma(y_n, Ty_n, a) \\ &< \epsilon + \epsilon + t \\ &= t_n + 2\epsilon \\ &\leq t. \end{aligned}$$

which implies that $\sigma(z, Tz, a) \le t_n \Rightarrow z \in S_t$. Hence S_t is closed.

2. Main Results

Theorem 2.1 : Let (X, σ) be a complete $2-$*metric space* and $T : X \to X$ is continuous mapping satisfying

$$\sigma(Tx, Ty, a) \le \alpha\sigma(x, y, a) + \beta\{\sigma(x, Tx, a) + \sigma(y, Ty, a)\} + \gamma\sigma(x, Ty, a) + \delta\sigma(y, Tx, a) \tag{2.1}$$

$\forall x, y, a \in X$ and $\alpha, \beta, \gamma, \delta$ are non negative reals such that $\alpha + \beta + \gamma + \delta < 1$. Then T has unique fixed point.

Proof : Let $\{t_n\}$ be a decreasing sequence of positive numbers converging to zero. Clearly $S_{t_{n+1}} \subseteq S_{t_n}$.

In view of its property $S_{t_n}(n = 1, 2, 3...)$ is $2-$*closed*.

Now we shall show that $\partial_a(S_{t_n}) \to 0$ as $n \to \infty$. For any $x, y \in S_{t_n}$ and $a \in X$, we have

$$\begin{aligned}
\sigma(x, y, a) &\le \sigma(x, Tx, a) + \sigma(x, y, Tx) + \sigma(Tx, y, a) \\
&\le 2tn + \sigma(Tx, y, a) \\
&\le 2tn + \sigma(Tx, Ty, a) + \sigma(Tx, y, Ty) + \sigma(Ty, y, a) \\
&\le 4tn + \sigma(Tx, Ty, a) \\
&\le 4tn + \alpha\sigma(x, y, a) + \beta\{\sigma(x, Tx, a) + \sigma(y, Ty, a)\} \\
&\quad +\gamma\sigma(x, Ty, a) + \delta\sigma(y, Tx, a) \\
(1-\alpha)\sigma(x, y, a) &\le 4tn + 2\beta tn + \gamma\sigma(x, Ty, a) + \delta\sigma(y, Tx, a) \\
&= 2(1+2\beta)tn + \gamma\sigma(x, Ty, a) + \delta\sigma(y, Tx, a). \qquad (2.2)
\end{aligned}$$

Using (2.1) and (2.2), we have

$$\begin{aligned}
&(1-\alpha)\sigma(x, y, a) \le 2(1+2\beta)t_n + \gamma\{\sigma(x, Ty, Tx) + \sigma(x, Tx, a) + \sigma(Tx, Ty, a)\} \\
&+\delta\{\sigma(y, Tx, Ty) + \sigma(y, Ty, a) + \sigma(Ty, Tx, a)\} \\
&\le 2(1+2\beta)t_n + \gamma\{2t_n + \sigma(Tx, Ty, a)\} + \delta\{2t_n + \sigma(Tx, Ty, a)\} \\
&= 2(1+2\beta)t_n + 2(\gamma+\delta)t_n + (\gamma+\delta)\sigma(Tx, Ty, a)
\end{aligned}$$

$$\leq 2(1+2\beta)t_n + 2(\gamma+\delta)t_n + (\gamma+\delta)[\alpha\sigma(x,y,a) + \beta\{\sigma(x,Tx,a) + \sigma(y,Ty,a)\}$$
$$+\gamma\sigma(x,Ty,a) + \delta\sigma(y,Tx,a)]$$
$$\leq 2(1+2\beta)t_n + 2(\gamma+\delta)t_n + (\gamma+\delta)[\alpha\{\sigma(x,y,Tx) + \sigma(x,Tx,y) + \sigma(Tx,y,a)\}$$
$$+2\beta t_n + \gamma\sigma(x,Ty,a) + \delta\sigma(y,Tx,a)]$$
$$\leq 2(1+2\beta)t_n + 2(\gamma+\delta)t_n + (\gamma+\delta)[\alpha\{2t_n + \sigma(Tx,y,a)\} + 2\beta t_n$$
$$+\gamma\sigma(x,Ty,a) + \delta\sigma(y,Tx,a)]$$
$$= 2(1+2\beta)t_n + 2(\gamma+\delta)t_n + 2(\alpha+\beta)(\gamma+\delta)t_n + (\gamma+\delta)\gamma\sigma(x,Ty,a)$$
$$+(\alpha+\delta)(\gamma+\delta)\sigma(y,Tx,a)$$
$$= 2(1+2\beta)t_n + 2(\gamma+\delta)(1+\alpha+\beta)t_n + (\gamma+\delta)\{\gamma\sigma(x,Ty,a) + (\alpha+\delta)\sigma(y,Tx,a)\}. \quad (2.3)$$

Using (.1) and (2.3), we have

$$(1-\alpha)\sigma(x,y,a) \leq 2(1+2\beta)t_n + 2(\gamma+\delta)(1+\alpha+\beta)t_n$$
$$+(\gamma+\delta)[\gamma\{\sigma(x,Ty,Tx) + \sigma(x,Tx,a) + \sigma(Tx,Ty,a)\}$$
$$+(\alpha+\delta)\{\sigma(y,Tx,Ty) + \sigma(y,Ty,a) + \sigma(Ty,Tx,a)\}]$$
$$\leq 2(1+2\beta)t_n + 2(\gamma+\delta)(1+\alpha+\beta)t_n + (\gamma+\delta)[\gamma\{2t_n + \sigma(Tx,Ty,a)\}$$
$$+(\alpha+\delta)\{2t_n + \sigma(Tx,Ty,a)\}]$$
$$= 2(1+2\beta)t_n + 2(\gamma+\delta)(1+\alpha+\beta)t_n + (\gamma+\delta)[2(\alpha+\gamma+\delta)t_n$$
$$+(\alpha+\gamma+\delta)\sigma(Tx,Ty,a)]$$
$$= 2(1+2\beta)t_n + 2(\gamma+\delta)(1+\alpha+\beta)t_n + 2(\gamma+\delta)(\alpha+\gamma+\delta)t_n$$
$$+(\gamma+\delta)(\alpha+\gamma+\delta)\sigma(Tx,Ty,a)$$
$$\leq 2(1+2\beta)t_n + 2(\gamma+\delta)(1+\alpha+\beta)t_n + 2(\gamma+\delta)(\alpha+\gamma+\delta)t_n$$
$$+(\gamma+\delta)(\alpha+\gamma+\delta)[\alpha\sigma(x,y,a) + \beta\{\sigma(x,Tx,a) + \sigma(y,Ty,a)\}$$
$$+\gamma\sigma(x,Ty,a) + \delta\sigma(y,Tx,a)]$$
$$\leq 2(1+2\beta)t_n + 2(\gamma+\delta)(1+\alpha+\beta)t_n + 2(\gamma+\delta)(\alpha+\gamma+\delta)t_n$$
$$+(\gamma+\delta)(\alpha+\gamma+\delta)[\alpha\{\sigma(x,y,Tx) + \sigma(x,Tx,y) + \sigma(Tx,y,a)\}$$
$$+2\beta t_n + \gamma\sigma(x,Ty,a) + \delta\sigma(y,Tx,a)]$$

$$\leq 2(1+2\beta)t_n + 2(\gamma+\delta)(1+\alpha+\beta)t_n + 2(\gamma+\delta)(\alpha+\gamma+\delta)t_n$$
$$+(\gamma+\delta)(\alpha+\gamma+\delta)[2\alpha t_n + \alpha\sigma(Tx,y,a) + 2\beta t_n + \gamma\sigma(x,Ty,a) + \delta\sigma(y,Tx,a)]$$
$$\leq 2(1+2\beta)t_n + 2(\gamma+\delta)(1+\alpha+\beta)t_n + 2(\gamma+\delta)(\alpha+\gamma+\delta)t_n$$
$$+2(\gamma+\delta)(\alpha+\gamma+\delta)(\alpha+\beta)t_n$$
$$+(\gamma+\delta)(\alpha+\gamma+\delta)\{\gamma\sigma(x,Ty,a) + (\alpha+\delta)\sigma(y,Tx,a)\}$$
$$= 2(1+2\beta)t_n + 2(\gamma+\delta)(1+\alpha+\beta)t_n + 2(\gamma+\delta)(1+\alpha+\beta)t_n$$
$$+(\gamma+\delta)(\alpha+\gamma+\delta)\{\gamma\sigma(x,Ty,a) + (\alpha+\delta)\sigma(y,Tx,a)\}. \tag{2.4}$$

By (2.1) and (2.4), we have

$$(1-\alpha)\sigma(x,y,a) \leq 2(1+2\beta)t_n + 2(\gamma+\delta)(1+\alpha+\beta)t_n$$
$$+2(1+\alpha+\beta)(\gamma+\delta)(\alpha+\gamma+\delta)t_n$$
$$+(\gamma+\delta)(\alpha+\gamma+\delta)[\gamma\{\sigma(x,Ty,Tx) + \sigma(x,Tx,a) + \sigma(Tx,Ty,a)\}$$
$$+(\alpha+\delta)\{\sigma(y,Tx,Ty) + \sigma(y,Ty,a) + \sigma(Ty,Tx,a)\}]$$
$$\leq 2(1+2\beta)t_n + 2(\gamma+\delta)(1+\alpha+\beta)t_n + 2(1+\alpha+\beta)(\gamma+\delta)(\alpha+\gamma+\delta)t_n$$
$$+(\gamma+\delta)(\alpha+\gamma+\delta)[\gamma\{2t_n + \sigma(Tx,Ty,a)\} + (\alpha+\delta)\{2t_n + \sigma(Tx,Ty,a)\}]$$
$$= 2(1+2\beta)t_n + 2(\gamma+\delta)(1+\alpha+\beta)t_n + 2(1+\alpha+\beta)(\gamma+\delta)(\alpha+\gamma+\delta)t_n$$
$$+2(\gamma+\delta)(\alpha+\gamma+\delta)^2 t_n + (\gamma+\delta)(\alpha+\gamma+\delta)^2\sigma(Tx,Ty,a)$$
$$\leq 2(1+2\beta)t_n + 2(\gamma+\delta)(1+\alpha+\beta)t_n + 2(1+\alpha+\beta)(\gamma+\delta)(\alpha+\gamma+\delta)t_n$$
$$+2(\gamma+\delta)(\alpha+\gamma+\delta)^2 t_n + (\gamma+\delta)(\alpha+\gamma+\delta)^2[\alpha\sigma(x,y,a)$$
$$+\beta\{\sigma(x,Tx,a) + \sigma(y,Ty,a)\} + \gamma\sigma(x,Ty,a) + \delta\sigma(y,Tx,a)]$$
$$\leq 2(1+2\beta)t_n + 2(\gamma+\delta)(1+\alpha+\beta)t_n + 2(1+\alpha+\beta)(\gamma+\delta)(\alpha+\gamma+\delta)t_n$$
$$+2(\gamma+\delta)(\alpha+\gamma+\delta)^2 t_n + (\gamma+\delta)(\alpha+\gamma+\delta)^2[\alpha\{\sigma(x,y,Tx)$$
$$+\sigma(x,Tx,y) + \sigma(Tx,y,a)\} + 2\beta t_n + \gamma\sigma(x,Ty,a) + \delta\sigma(y,Tx,a)]$$
$$\leq 2(1+2\beta)t_n + 2(\gamma+\delta)(1+\alpha+\beta)t_n + 2(1+\alpha+\beta)(\gamma+\delta)(\alpha+\gamma+\delta)t_n$$
$$+2(\gamma+\delta)(\alpha+\gamma+\delta)^2 t_n + (\gamma+\delta)(\alpha+\gamma+\delta)^2[2\alpha t_n + \alpha\sigma(Tx,y,a)$$
$$+2\beta t_n + \gamma\sigma(x,Ty,a) + \delta\sigma(y,Tx,a)]$$

$$= 2(1+2\beta)t_n + 2(\gamma+\delta)(1+\alpha+\beta)t_n + 2(1+\alpha+\beta)(\gamma+\delta)(\alpha+\gamma+\delta)t_n$$
$$+2(\gamma+\delta)(\alpha+\gamma+\delta)^2 t_n + 2(\gamma+\delta)(\alpha+\gamma+\delta)^2(\alpha+\beta)$$
$$+2(\gamma+\delta)(\alpha+\gamma+\delta)^2[\gamma\sigma(x,Ty,a) + (\alpha+\delta)\sigma(y,Tx,a)]$$
$$= 2(1+2\beta)t_n + 2(\gamma+\delta)(1+\alpha+\beta)t_n + 2(1+\alpha+\beta)(\gamma+\delta)(\alpha+\gamma+\delta)t_n$$
$$+2(1+\alpha+\beta)(\gamma+\delta)(\alpha+\gamma+\delta)^2 t_n$$
$$+(\gamma+\delta)(\alpha+\gamma+\delta)^2[\gamma\sigma(x,Ty,a) + (\alpha+\delta)\sigma(y,Tx,a)]. \quad (2.5)$$

Utilizing (2.1) and (2.5), we have

$$(1-\alpha)\sigma(x,y,a) \leq 2(1+2\beta)t_n + 2(\gamma+\delta)(1+\alpha+\beta)t_n$$
$$+2(1+\alpha+\beta)(\gamma+\delta)(\alpha+\gamma+\delta)t_n$$
$$+2(1+\alpha+\beta)(\gamma+\delta)(\alpha+\gamma+\delta)^2 t_n + (\gamma+\delta)(\alpha+\gamma+\delta)^2[\gamma\{\sigma(x,Ty,Tx) +$$
$$\sigma(x,Tx,a) + \sigma(Tx,Ty,a)\} + (\alpha+\delta)\{\sigma(y,Tx,Ty) + \sigma(y,Ty,a) + \sigma(Ty,Tx,a)\}]$$
$$\leq 2(1+2\beta)t_n + 2(\gamma+\delta)(1+\alpha+\beta)t_n + 2(1+\alpha+\beta)(\gamma+\delta)(\alpha+\gamma+\delta)t_n$$
$$+2(1+\alpha+\beta)(\gamma+\delta)(\alpha+\gamma+\delta)^2 t_n$$
$$+(\gamma+\delta)(\alpha+\gamma+\delta)^2[\gamma\{2t_n + \sigma(Tx,Ty,a)\} + (\alpha+\delta)\{2t_n + \sigma(Tx,Ty,a)\}]$$
$$= 2(1+2\beta)t_n + 2(\gamma+\delta)(1+\alpha+\beta)t_n + 2(1+\alpha+\beta)(\gamma+\delta)(\alpha+\gamma+\delta)t_n$$
$$+2(1+\alpha+\beta)(\gamma+\delta)(\alpha+\gamma+\delta)^2 t_n$$
$$+2(\gamma+\delta)(\alpha+\gamma+\delta)^3 t_n + (\gamma+\delta)(\alpha+\gamma+\delta)^3\sigma(Tx,Ty,a)$$
$$\leq 2(1+2\beta)t_n + 2(\gamma+\delta)(1+\alpha+\beta)t_n + 2(1+\alpha+\beta)(\gamma+\delta)(\alpha+\gamma+\delta)t_n$$
$$+2(1+\alpha+\beta)(\gamma+\delta)(\alpha+\gamma+\delta)^2 t_n$$
$$+2(\gamma+\delta)(\alpha+\gamma+\delta)^3 t_n + (\gamma+\delta)(\alpha+\gamma+\delta)^3[\alpha\sigma(x,y,a) +$$
$$+\beta\{\sigma(x,Tx,a) + \sigma(y,Ty,a)\} + \gamma\sigma(x,Ty,a) + \delta\sigma(y,Tx,a)]$$
$$\leq 2(1+2\beta)t_n + 2(\gamma+\delta)(1+\alpha+\beta)t_n + 2(1+\alpha+\beta)(\gamma+\delta)(\alpha+\gamma+\delta)t_n$$
$$+2(1+\alpha+\beta)(\gamma+\delta)(\alpha+\gamma+\delta)^2 t_n$$
$$+2(\gamma+\delta)(\alpha+\gamma+\delta)^3 t_n + (\gamma+\delta)(\alpha+\gamma+\delta)^3[\alpha\{\sigma(x,y,Tx)$$
$$+\sigma(x,Tx,y) + \sigma(Tx,y,a)\} + 2\beta t_n + \gamma\sigma(x,Ty,a) + \delta\sigma(y,Tx,a)]$$

$$\leq 2(1+2\beta)t_n + 2(\gamma+\delta)(1+\alpha+\beta)t_n + 2(1+\alpha+\beta)(\gamma+\delta)(\alpha+\gamma+\delta)t_n$$
$$+2(1+\alpha+\beta)(\gamma+\delta)(\alpha+\gamma+\delta)^2 t_n$$
$$+2(\gamma+\delta)(\alpha+\gamma+\delta)^3 t_n + (\gamma+\delta)(\alpha+\gamma+\delta)^3[2\alpha t_n + \alpha\sigma(Tx,y,a)$$
$$+2\beta t_n + \gamma\sigma(x,Ty,a) + \delta\sigma(y,Tx,a)]$$
$$= 2(1+2\beta)t_n + 2(\gamma+\delta)(1+\alpha+\beta)t_n + 2(1+\alpha+\beta)(\gamma+\delta)(\alpha+\gamma+\delta)t_n$$
$$+2(1+\alpha+\beta)(\gamma+\delta)(\alpha+\gamma+\delta)^2 t_n$$
$$+2(\gamma+\delta)(\alpha+\gamma+\delta)^3 t_n + 2(\alpha+\beta)(\gamma+\delta)(\alpha+\gamma+\delta)^3 t_n$$
$$+(\gamma+\delta)(\alpha+\gamma+\delta)^3[\gamma\sigma(x,Ty,a) + (\gamma+\delta)\sigma(y,Tx,a)].$$

Therefore

$$(1-\alpha)\sigma(x,y,a) \leq 2(1+2\beta)t_n + 2(\gamma+\delta)(1+\alpha+\beta)t_n$$
$$+2(1+\alpha+\beta)(\gamma+\delta)(\alpha+\gamma+\delta)t_n$$
$$+2(1+\alpha+\beta)(\gamma+\delta)(\alpha+\gamma+\delta)^2 t_n + 2(1+\alpha+\beta)(\gamma+\delta)(\alpha+\gamma+\delta)^3 t_n$$
$$+(\gamma+\delta)(\alpha+\gamma+\delta)^3[\gamma\sigma(x,Ty,a) + (\alpha+\delta)\sigma(y,Tx,a)]. \tag{2.6}$$

Continuing in this way in n^{th} step, we get

$$(1-\alpha)\sigma(x,y,a) \leq 2(1+2\beta)t_n + 2(\gamma+\delta)(1+\alpha+\beta)t_n$$
$$+2(1+\alpha+\beta)(\gamma+\delta)(\alpha+\gamma+\delta)t_n$$
$$+2(1+\alpha+\beta)(\gamma+\delta)(\alpha+\gamma+\delta)^2 t_n + ... + 2(1+\alpha+\beta)(\gamma+\delta)(\alpha+\gamma+\delta)^{n-1} t_n$$
$$+2(\gamma+\delta)(\alpha+\gamma+\delta)^{n-1}[\gamma\sigma(x,Ty,a) + (\alpha+\delta)\sigma(y,Tx,a)]$$
$$= 2(1+2\beta)t_n + 2(\gamma+\delta)(1+\alpha+\beta)[1 + (\alpha+\gamma+\delta) + (\alpha+\gamma+\delta)^2 + ...$$
$$+(\alpha+\gamma+\delta)^{n-1}]t_n$$
$$+(\gamma+\delta)(\alpha+\gamma+\delta)^{n-1}[\gamma\sigma(x,Ty,a) + (\alpha+\delta)\sigma(y,Tx,a)]$$
$$\leq 2(1+2\beta)t_n + 2(\gamma+\delta)(1+\alpha+\beta)[1 + (\alpha+\gamma+\delta) + (\alpha+\gamma+\delta)^2 + ...]t_n$$
$$+(\gamma+\delta)(\alpha+\gamma+\delta)^{n-1}[\gamma\sigma(x,Ty,a) + (\alpha+\delta)\sigma(y,Tx,a)]$$
$$\sigma(x,y,a) \leq \frac{2(1+2\beta)t_n}{(1-\alpha)} + \frac{2(\gamma+\delta)(1+\alpha+\beta)}{\{1-(\alpha+\gamma+\delta)\}(1-\alpha)} t_n$$
$$+\frac{(\gamma+\delta)(\alpha+\gamma+\delta)^{n-1}}{1-\alpha}[\gamma\sigma(x,Ty,a) + (\alpha+\delta)\sigma(y,Tx,a)].$$
$$\to 0 \; as \; n \to \infty. \tag{2.7}$$

Hence $\delta_a(S_{t_n}) \to 0$ as $n \to \infty$. So $\{S_{t_n}\}$ is a sequence of sets such that

1. S_{t_n} is 2-closed set;
2. $S_{t_{n+1}} \subseteq St_n \ \forall \ n = 1, 2, 3...$;
3. $\delta_a(S_{t_n}) \to 0 \ n \to \infty$.

Hence by $Cantor's$ Theorem in 2− metric space $\bigcap_{n=1}^{\infty} S_{t_n}$ contains exactly one point. Let
$x_0 \in \bigcap_{n=1}^{\infty} S_{t_n}$ then
$\sigma(x_0, Tx_0, a) \leq t_n \ \forall \ n = 1, 2, 3... \ and \ \forall \ a \in X$
$\Rightarrow \sigma(x_0, Tx_0, a) = 0 \ \forall \ a \in \ X$
$\Rightarrow Tx_0 = x_0$
Hence x_0 is a fixed point in T.
To prove uniqueness, let u and v be two distinct fixed point of T, then for a point $a \in \ X, \ a \neq u \ or \ v$, we have

$$\begin{aligned}
\sigma(u, v, a) &= \sigma(Tu, Tv, a) \\
&\leq \alpha\sigma(u, v, a) + \beta\{\sigma(u, Tu, a) + \sigma(v, Tv, a)\} + \gamma\sigma(u, Tv, a) + \delta\sigma(v, Tu, a) \\
&= \alpha\sigma(u, v, a) + \beta\{\sigma(u, u, a) + \sigma(v, v, a)\} + \gamma\sigma(u, v, a) + \delta\sigma(v, u, a) \\
&= \alpha\sigma(u, v, a) + \gamma\sigma(u, v, a) + \delta\sigma(u, v, a)
\end{aligned}$$

or $\{1 - (\alpha + \gamma + \delta)\}\sigma(u, v, a) \leq 0$.
Which implies that $\sigma(u, v, a) = 0 \ \Rightarrow \ u = v$.
Hence T has a unique fixed point.
Remark 2.1 : If $\beta = \gamma = \delta = 0$. Then we get result of B. K. Lahiri [12].
If $\alpha = 0$, we get the following corollary.
Corollary 2.1 : Let (X, σ) be a complete $2-$*metric space* and $T : X \to X$ is continuous map satisfying

$$\sigma(Tx, Ty, a) \leq \beta\{\sigma(x, Tx, a) + \sigma(y, Ty, a)\} + \gamma\sigma(x, Ty, a) + \delta\sigma(y, Tx, a) \tag{2.8}$$

$\forall x, y, a \in X$ and β, γ, δ are non negative reals such that $\beta + \gamma + \delta < 1$. Then T has unique fixed point If $\alpha = \beta = 0$, we get the following corollary.

Corollary 2.2 : Let (X, σ) be a complete $2-$*metric space* and $T : X \rightarrow X$ is continuous mapping satisfying

$$\sigma(Tx, Ty, a) \leq \gamma\sigma(x, Ty, a) + \delta\sigma(y, Tx, a) \tag{2.9}$$

$\forall x, y, a \in X$ and γ, δ are non negative reals such that $\gamma + \delta < 1$. Then T has unique fixed point

If $\gamma = \delta = 0$, we get the following corollary

Corollary 2.3 : Let (X, σ) be a complete $2-$*metric space* and $T : X \rightarrow X$ is continuous mapping satisfying

$$\sigma(Tx, Ty, a) \leq \alpha\sigma(x, y, a) + \beta\{\sigma(x, Tx, a) + \sigma(y, Ty, a)\} \tag{2.10}$$

$\forall x, y, a \in X$ and α, β, are non negative reals such that $\alpha + \beta < 1$. Then T has unique fixed point.

If $\beta = 0$, we get the following corollary

Corollary 2.4 : Let (X, σ) be a complete $2-$*metric space* and $T : X \rightarrow X$ is continuous mapping satisfying

$$\sigma(Tx, Ty, a) \leq \alpha\sigma(x, y, a) + \gamma\sigma(x, Ty, a) + \delta\sigma(y, Tx, a) \tag{2.11}$$

$\forall x, y, a \in X$ and α, γ, δ are non negative reals such that $\alpha + \gamma + \delta < 1$. Then T has unique fixed point.

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