International J. of Math. Sci. & Engg. Appls. (IJMSEA)
ISSN 0973-9424, Vol. 10 No. II (August, 2016), pp. 91-98

DOT PRODUCTS OF FOURTH ORDER STRONGLY MAGIC SQUARES

NEERADHA C.K.[1] AND MADHUKAR MALLAYYA[2]
[1] Assistant Professor, Dept. of Science & Humanities,
Mar Baselios College of Engineering & Technology
Thiruvananthapuram, India
[2] Professor & HOD, Dept. of Mathematics,
Mohandas College of Engineering & Technology,
Thiruvananthapuram, India

Abstract

Magic squares have turned up throughout history, some in a mathematical context and others in philosophical or religious contexts. A normal magic square is a square array of consecutive numbers from where the rows, columns, diagonal and co-diagonals add up to the same number. The constant sum is called magic constant or magic number. Along with the conditions of normal magic squares, strongly magic squares have a stronger property that the sum of the entries of the sub-squares taken without any gaps between the rows or columns is also the magic constant. There are many recreational aspects of strongly magic squares. But, apart from the usual recreational aspects, it is found that strongly magic squares possess advanced mathematical properties. In this paper a generic definition for strongly magic square is given and some advanced mathematical properties like dot products of 4×4 strongly magic squares are discussed. This can pave way for developing new horizons in the field of vector algebra.

Key Words : *Magic square, Magic constant, Strongly magic square, Dot products of strongly magic squares.*

2010 AMS Subject Classification : 15AXX.

1. Introduction

A magic square is a square array of numbers where the rows, columns and both diagonals add up to the same number. The constant sum is called magic constant or magic number [1]. There are many recreational aspects of magic squares. But, apart from the usual recreational aspects, it is found that these squares possess mathematical properties.

2. Notations and Mathematical Preliminaries

(A) Magic Square

A magic square of order n is an n^{th} order matrix $[a_{ij}]$ such that

$$\sum_{j=1}^{n} a_{ij} = \rho, \quad \text{for} \quad i = 1, 2, \cdots, n \tag{1}$$

$$\sum_{j=1}^{n} a_{ji} = \rho, \quad \text{for} \quad i = 1, 2, \cdots, n \tag{2}$$

$$\sum_{i=1}^{n} a_{ii} = \rho, \quad \sum_{i=1}^{n} a_{i,n-i+1} = \rho \tag{3}$$

Equation (1) represents the row sum, equation (2) represents the column sum, equation (3) represents the diagonal and co-diagonal sum and ρ represents the magic constant. [2].

(B) Magic Constant

The constant ρ in the above definition is known as the magic constant or magic number. The magic constant of the magic square A is denoted as $\rho(A)$.

For example Fig 1 is a well-known 10th-century 4×4 magic square on display in the Parshvanath Jain temple in Khajuraho, India, [4] and Fig 2 [3] refers to well-known Sri Rama Chakra; both having magic constant 34.

7	12	1	14
2	13	8	11
16	3	10	5
9	6	15	4

Fig. 1: The ChautisaYantra

9	16	5	4
7	2	11	14
12	13	8	1
6	3	10	15

Fig. 2: The Sri Rama Chakra

(C) Dot Products of Magic Squares

i. Column Dot Product of two Magic Squares

Let C and C' be any two columns of two magic squares A and A' of order n. If $a_1, a_2, \cdots, a_n$ and $a'_1, a'_2, \cdots, a'_n$ are the elements of the C and C' respectively, then the dot product of C and C' denoted by $C.C'$ is defined as

$$C.C' = \sum_{i=1}^{n} a_i, a'_i.$$

For example, two magic squares A and B are given in such a way that

$$A = \begin{bmatrix} 16 & 5 & 4 & 9 \\ 2 & 11 & 14 & 7 \\ 13 & 8 & 1 & 12 \\ 3 & 10 & 15 & 6 \end{bmatrix} \quad \text{and} \quad A' = \begin{bmatrix} 3 & 13 & 2 & 16 \\ 10 & 8 & 11 & 5 \\ 15 & 1 & 14 & 4 \\ 7 & 12 & 7 & 9 \end{bmatrix}$$

Then one of the column dot products of A and A' are given by

$$C_1 \cdot C'_1 = 16 \times 3 + 2 \times 10 + 13 \times 15 + 3 \times 6 = 281$$

Also one of the row dot products of A and A' are given by

$$R_3 \cdot R'_4 = 13 \times 6 + 8 \times 12 + 1 \times 7 + 12 \times 9 = 289.$$

(D) Strongly Magic Square (SMS) : Generic Definition

Let $A = [a_{ij}]$ be a matrix of order $n^2 \times n^2$, such that

$$\sum_{j=1}^{n^2} a_{ij} = \rho \quad \text{for} \quad i = 1, 2, \cdots, n^2 \tag{4}$$

$$\sum_{j=1}^{n^2} a_{ji} = \rho \quad \text{for} \quad i = 1, 2, \cdots, n^2 \tag{5}$$

$$\sum_{i=1}^{n^2} a_{ii} = \rho \quad \sum_{i=1}^{n^2} a_{i,n^2-i+1} = \rho \tag{6}$$

$$\sum_{i=0}^{n-1} \sum_{k=0}^{n-1} a_{i+k,j+l} = \rho \quad \text{for} \quad i, j = 1, 2, \cdots, n^2 \tag{7}$$

where the subscripts are congruent modulo n^2.

Equation (4) represents the row sum, equation (5) represents the column sum, equation (6) represents the diagonal and co-diagonal sum, equation (7) represents the $n \times n$ sub-square sum with no gaps in between the elements of rows or columns which is denoted as $M_{OC}^{(n)}$ or $M_{OR}^{(n)}$ and ρ is the magic constant.

3. Properties of Dot Products of Strongly Magic Squares

Proposition 3.1 : If $A = [a_{ij}]$ be a SMS of order 4 and R_1, R_2, R_3, R_4 and C_1, C_2, C_3, C_4 be the rows and columns of SMS respectively, then

(i) $R_1R_2 = R_3R_4$ (ii) $R_1R_4 = R_2R_3$

(iii) $C_1C_2 = C_3C_4$ (iv) $C_1C_4 = C_2C_3$

or in general $R_iR_{i+1} = R_{i+2}R_{i+3}$ and $C_iC_{i+1} = C_{i+2}C_{i+3}$ where $i = 1, 2, 3, 4$ and the subscripts should be taken modulo 4.

Proof : The general form of a 4×4 Strongly Magic Square is given by

$$\begin{bmatrix} a & b & c & d \\ \rho & c+d-\rho & a-c+\rho & b+c-\rho \\ \frac{\rho}{2}-c & \frac{\rho}{2}-d & \frac{\rho}{2}-a & \frac{\rho}{2}-b \\ c-a-\frac{\rho}{2} & \frac{3\rho}{2}-b-c & -\frac{\rho}{2} & \frac{3\rho}{2}-c-d \end{bmatrix} \tag{8}$$

where ρ is the magic constant and $a, b, c, d \in R$.

$$\begin{aligned} \text{(i)} \quad R_3R_4 &= \rho^2 + 2bd + ac - c^2 - 2\rho b - 2\rho d + cd + bc \\ R_1R_2 &= (a+c)\rho + 2bd + ac - c^2 + cd + bc - b\rho - \rho d \\ &= [\rho - (b+d)]\rho + 2bd + ac - c^2 + cd + bc - b\rho - \rho d \ \ \text{since} \ \ a+b+c+d = \rho \\ &= \rho^2 + 2bd + ac - c^2 - 2\rho b - 2\rho d + cd + bc. \end{aligned}$$

Hence $R_1R_2 = R_3R_4$.

(ii) $R_1R_4 = ac - a^2 - \frac{a\rho}{2} + \frac{3b\rho}{2} - b^2 - bc - \frac{\rho c}{2} + \frac{3\rho d}{2} - cd - d^2 = R_2R_3$.

(iii) $C_1C_2 = 2ab + cd + ac - bc - c^2 + \frac{\rho b}{2} + \frac{5\rho c}{2} - \frac{3\rho^2}{2} - \frac{3a\rho}{2} + \frac{\rho d}{2} = C_3C_4$. Similarly other result can also be verified.

Proposition 3.2 : If $A = [a_{ij}]$ be an SMS of order 4 and if R_1, R_2, R_3, R_4 and C_1, C_2, C_3, C_4 be the rows and columns then

(i) $R_1C_1 = R_3C_3$ (ii) $R_1C_2 = R_3C_4$ (iii) $R_1C_1 = R_3C_1$
(iv) $R_1C_4 = R_3C_2$ (v) $R_2C_1 = R_4C_3$ (iv) $R_2C_2 = R_4C_4$
(vii) $R_2C_3 = R_4C_1$ (viii) $R_2C_4 = R_4C_2$ or in general $R_iC_j = R_{i+2}C_{j+2}$
where $i, j = 1, 2, 3, 4$ and the subscripts should be taken modulo 4.

Proof : From the general form of a 4×4 SMS as in Proposition 3.1

(i) $R_1C_1 = a^2 - c^2 + b\rho + \frac{\rho c}{2} + cd - ad - \frac{\rho d}{2}$

$$\begin{aligned} R_3C_3 &= a^2 -^2 + \frac{\rho c}{2} + cd - ad - \frac{\rho(a+c)}{2} + \frac{\rho b}{2} - \rho d + \frac{\rho^2}{2} \quad \text{(From eq. (8))} \\ &= a^2 - c^2 + \frac{\rho c}{2} + cd - ad - \frac{\rho(\rho - (b+d))}{2} + \frac{\rho b}{2} - \rho d + \frac{\rho^2}{2} \\ &= a^2 - c^2 + b\rho + \frac{\rho c}{2} + cd - ad - \frac{\rho d}{2} = R_1C_1 \end{aligned}$$

(ii) $R_1C_2 = ab + bc - b\rho + \frac{c\rho}{2} - 2cd + \frac{3\rho d}{2}$. Now

$$\begin{aligned} R_3C_4 &= ab + bc - 2cd + \frac{\rho^2}{2} + \rho d - \frac{\rho a}{2} - \frac{3\rho b}{2} \quad \text{(From Eq. (8))} \\ &= ab + bc - 2cd + \frac{\rho^2}{2} + \frac{3\rho d}{2} - \frac{\rho d}{2} - \frac{\rho a}{2} - \frac{\rho b}{2} - \rho b \\ &= ab + bc - 2cd + \frac{\rho^2}{2} + \frac{3\rho d}{2} - \frac{\rho}{2}(a+b+d) - \rho b \\ &= ab + bc - 2cd + \frac{\rho^2}{2} + \frac{3\rho d}{2} - \frac{\rho}{2}(\rho - c) - \rho b \\ &= R_1C_2. \end{aligned}$$

(iii) $R_1C_3 = ab - bc + b\rho - \frac{\rho d}{2} + \frac{c\rho}{2}$ (From Eq. (8)). Now

$$\begin{aligned} R_3C_1 &= ab - bc + \frac{b\rho}{2} - \frac{a\rho}{2} + \frac{\rho^2}{2} - \rho d \quad \text{(From Eq. (8))} \\ &= ab - bc + b\rho - \frac{\rho}{2}(a+b) + \frac{\rho^2}{2} - \rho d \\ &= ab - bc + b\rho - \frac{\rho}{2}(\rho - (c+d)) + \frac{\rho^2}{2} - \rho d \\ &= ab - bc + b\rho - \frac{\rho d}{2} + \frac{c\rho}{2} = R_1C_3. \end{aligned}$$

(iv) $R_3C_2 = ad + b^2 - d^2 - cd + \frac{\rho^2}{2} + \rho d - \frac{a\rho}{2} - \frac{3\rho b}{2}$ (From Eq. (8)). Now,

$$\begin{aligned} R_1C_4 &= ad + b^2 - d^2 - cd - b\rho + \frac{c\rho}{2} + \frac{3\rho d}{2} \quad \text{(From Eq. (8))} \\ &= ad + b^2 - d^2 - cd - b\rho + \frac{\rho}{2}(\rho - (a+b+d)) + \frac{3\rho d}{2} \\ &= ad + b^2 - d^2 - cd + \frac{\rho^2}{2} + \rho d - \frac{a\rho}{2} - \frac{3\rho b}{2} = R_3C_2. \end{aligned}$$

(v) $R_4C_3 = 2c^2 - 2ac + bc + 2a\rho - 2c\rho - ab + \frac{\rho^2}{2} - \frac{\rho c}{2} - b\rho + \frac{\rho d}{2}$ (From Eq. (8)). Now,

$$\begin{aligned} R_2C_1 &= 2c^2 - 2ac + bc + 2a\rho - 2c\rho - ab + \rho d + \frac{a\rho}{2} - \frac{b\rho}{2} \quad \text{(From Eq. (8))} \\ &= 2c^2 - 2ac + bc + 2a\rho - 2c\rho - ab + \rho d + \frac{\rho}{2}(\rho - (b+c+d)) - \frac{b\rho}{2} \\ &= 2c^2 - 2ac + bc + 2a\rho - 2c\rho - ab + \frac{\rho^2}{2} - \frac{\rho c}{2} - b\rho + \frac{\rho d}{2} = R_4C_3. \end{aligned}$$

(vi) $R_2C_2 = 3cd + d^2 - ad - b^2 - 2bc + \frac{7\rho b}{2} - 3\rho d + \frac{a\rho}{2}$ (From Eq. (8)). Now,

$$\begin{aligned} R_4C_4 &= 3cd + d^2 - ad - b^2 - 2bc - \frac{7\rho d}{2} + 3\rho b + \frac{\rho^2}{2} - \frac{\rho c}{2} \quad \text{(From Eq. (8))} \\ &= 3cd + d^2 - ad - b^2 - 2bc + \frac{7\rho(b-d)}{2} - \frac{\rho(b+c)}{2} + \frac{\rho^2}{2} \\ &= 3cd + d^2 - ad - b^2 - 2bc + \frac{7\rho(b-d)}{2} - \frac{\rho(\rho-(a+d))}{2} + \frac{\rho^2}{2} \\ &= 3cd + d^2 - ad - b^2 - 2bc + \frac{7\rho b}{2} - 3\rho d + \frac{a\rho}{2} = R_2C_2. \end{aligned}$$

(vii) $R_4C_1 = 2\rho c + 2ac - c^2 - a^2 + ad - cd - 2a\rho + \frac{\rho^2}{2} - \rho b - \frac{\rho c}{2} + \frac{\rho d}{2}$ (From Eq. (8)). Now

$$\begin{aligned} R_2C_3 &= 2\rho c + 2ac - c^2 - a^2 + ad - cd - 2a\rho + \rho d + \frac{\rho a}{2} - \frac{\rho b}{2} \quad \text{(From Eq. (8))} \\ &= 2\rho c + 2ac - c^2 - a^2 + ad - cd - 2a\rho + \rho d + \frac{\rho(\rho - (b+c+d))}{2} - \frac{\rho b}{2} \\ &= 2\rho c + 2ac - c^2 - a^2 + ad - cd + 2a\rho + \frac{\rho^2}{2} - \rho b - \frac{\rho c}{2} + \frac{\rho d}{2} = R_4C_1. \end{aligned}$$

(viii) $R_2C_4 = \rho d - \frac{\rho b}{2} + \frac{\rho a}{2} + bc - ab - \frac{\rho^2}{2}$ (From Eq. (8)). Now,

$$\begin{aligned} R_4C_2 &= \frac{\rho^2}{2} - \rho b + \frac{\rho d}{2} - \frac{\rho c}{2} + bc - ab \quad \text{(From EQ. (8))} \\ &= \frac{\rho^2}{2} - \frac{\rho b}{2} - \frac{\rho(b+c)}{2} + \frac{\rho d}{2} + bc - ab \\ &= \frac{\rho^2}{2} - \frac{\rho b}{2} - \frac{\rho(\rho-(a+d))}{2} + \frac{\rho d}{2} + bc - ab \\ &= \rho d - \frac{\rho b}{2} + \frac{\rho a}{2} + bc - ab - \frac{\rho^2}{2} = R_2C_4. \end{aligned}$$

Proposition 3.3 : If $A = [a_{ij}]$ be a SMS of order 4 and R_1, R_2, R_3, R_4 and C_1, C_2, C_3, C_4 be the rows and columns of SMS respectively, then

(i) $R_iR_i = R_{i+2}R_{i+2}$ (ii) $C_iC_i = C_{i+2}C_{i+2}$

i.e. (i.1) $R_1R_1 = R_3R_3$ (i.2) $R_2R_2 = R_4R_4$ (ii.1) $C_1C_1 = C_3C_3$ (ii.2) $C_2C_2 = C_4C_4$.

Proof : From the general form of a 4×4 SMS as in Proposition 3.1

(i.1) $R_1R_1 = a^2 + b^2 + c^2 + d^2$ (From Eq. (8)). Now,

$$\begin{aligned} R_3R_3 &= \left(\frac{\rho}{2} - c\right)^2 + \left(\frac{\rho}{2} - d\right)^2 + \left(\frac{\rho}{2} - a\right)^2 + \left(\frac{\rho}{2} - b\right)^2 \quad \text{(From Eq. (8))} \\ &= a^2 + b^2 + c^2 + d^2 = R_1R_1. \end{aligned}$$

(i.2) $R_2R_2 = (\rho)^2 + (c + d - \rho)^2 + (a - c + \rho)^2 + (b + c - \rho)^2$ (From Eq. (8))
$= a^2 + b^2 + d^2 + 3c^2 + 2bc - 2ac + 2cd + 4\rho^2 - 6\rho c + 2\rho a - 2\rho b - 2\rho d$. Now,

$$\begin{aligned} R_4R_4 &= a^2 + b^2 + d^2 + 3c^2 + 2bc - 2ac + 2cd + 5\rho^2 - 7\rho c + \rho a - 3\rho b - 3\rho d \\ &= R_2R_2 - 2\rho a + \rho^2 - \rho c - \rho b - \rho d + \rho a \\ &= R_2R_2 - 2\rho a + \rho^2 - \rho c - \rho(\rho - (a + c)) + \rho a = R_2R_2 \end{aligned}$$

(ii.1) $C_1C_1 = 2a^2 + \frac{3}{2}\rho^2 + 2c^2 - 2\rho c - 2ac + a\rho = C_3C_3$ (From Eq. (8)).
(ii.2) $C_2C_2 = 2b^2 + 2c^2 + 2d^2 + \frac{7}{2}\rho^2 + 2cd - 5c\rho - 3\rho d + 2bc - 3\rho b = C_4C_4$. (From Eq. (8)).

4. Conclusion

Magic squares may be treated somewhat more seriously in different mathematical courses. The study of strongly magic squares is an emerging innovative area in which mathematical analysis can be done. This paper aims to explore some of the advanced mathematical properties of strongly magic squares. Certainly more can be done in the context of vector algebra.

Acknowledgement

The authors express sincere gratitude for the valuable suggestions given by Dr. Ramaswamy Iyer, Former Professor in Chemistry, Mar Ivanios College, Trivandrum, in preparing this paper.

References

[1] Andrews W. S., Magic Squares and Cubes, 2nd Rev.Ed., New York, Dover, (1960).

[2] Charles Small, Magic squares over fields,h The American Mathematical Monthly, 95(7) (Aug.-Sep., 1988), 621-625.

[3] Padmakumar T.V., Strongly Magic Squareh, Applications of Fibonacci Number, Vol. 6. New Publication, Proceedings of The Sixth International Research Conference on Fibonacci Numbers and Their Applications,Washington State University, Pullman, Washington, USA, (July 18-22, April 1995).

[4] Anderson D. L., Magic Squares [online] (2001). Available from http: //illuminations.nctm.org/ LessonDetail.aspx?id=L263.