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ON THE LAPLACE TRANSFORM OF THE FUNCTIONS INVOLVING THE PRODUCT OF THE GENERALIZED STRUVE'S FUNCTION AND THE I-FUNCTION OF r-VARIABLES

P. C. SREENIVAS[1], T. M. VASUDEVAN NAMBISAN[2] AND M. SUNITHA[3]
[1] Department of Mathematics,
Payyanur College, Payyanur, Kannur Dist, Kerala- 670327, India
[2] Retired Professor & Head,
Department of Mathematics, NAS College, Kanhangad, India
[3] Department of Mathematics,
Govt. Brennen College,Dharmadam, Thalassery,
Kannur Dist, Kerala -670106, India

Abstract

The object of this paper is to obtain the Laplace transform of the functions involving the product of the generalized Struve's function and the I-function of several complex variables. On specializing the parameters similar results can be derived in the case of I-function of two variables and H functions of r and two-variables, which include the result proved by K. P. Shahul Hameed [4, p.69].

Key Words : *Laplace transform, I-functions of two and several complex variables, Multivariable H-functions, Generalized Struve's function.*

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1. Introduction

Notations used :

${}_1(a_j;\alpha_j,A_j)_p$ stands for $(a_1;\alpha_1,A_1),(a_2;\alpha_2,A_2),\cdots,(a_p;\alpha_p,A_p)$.

The generalized Fox's H-function, namely I-function of r-variables introduced by Prathima, et al. [3, p.38] is defined and represented in the following manner:

$$I[z_1,\because,z_r]=I^{0,N:m_1,n_1;\cdots;m_r,n_r}_{P,Q;p_1,q_1;\cdots,p_r,q_r}$$
$$\left[\begin{array}{c|c} z_1 & {}_1(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_p :{}_1(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{p_1};\cdots;{}_1(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{p_r} \\ \vdots & \\ z_r & {}_1(b_j,\beta_j^{(1)},\cdots,\beta_j^{(r)})_Q :{}_1(d_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{q_1};\cdots;{}_1(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{q_r}\end{array}\right]$$
$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\theta_1(s_1)\cdots\theta_r(s_r)\ \phi(s_1,\cdots,s_r)\ z_1^{s_1}\cdots z_r^{s_r}ds_1\cdots ds_r, \qquad (1.1)$$

where $\phi(s_1,\cdots,s_r)$ and $\theta_i(s_i), i=1,2,\cdots,r$ are given by

$$\phi(s_1,\cdots,s_r)=\frac{\prod\limits_{j=1}^{N}\Gamma^{A_j}\left(1-a_j+\sum\limits_{i=1}^{r}\alpha_j^{(i)}s_i\right)}{\prod\limits_{j=1}^{Q}\Gamma^{B_j}\left(1-b_j+\sum\limits_{i=1}^{r}\beta_j^{(i)}s_i\right)\prod\limits_{j=N+1}^{P}\Gamma^{A_j}\left(a_j-\sum\limits_{i=1}^{r}\alpha_j^{(i)}s_i\right)}, \qquad (1.2)$$

$$\theta_i(s_i)=\frac{\prod\limits_{j=1}^{m_i}\Gamma^{D_j^{(i)}}\left(d_j^{(i)}-\delta_j^{(i)}s_i\right)\prod\limits_{j=1}^{n_i}\Gamma^{C_j^{(i)}}(1-c_j^{(i)}+\gamma_j^{(i)}s_i)}{\prod\limits_{j=m_i+1}^{q_i}\Gamma^{D_j^{(i)}}(1-d_j^{(i)}+\delta_j^{(i)}s_i)\prod\limits_{j=n_i+1}^{p_i}\Gamma^{C_j^{(i)}}(c_j^{(i)}-\gamma_j^{(i)}s_i)} \qquad (1.3)$$

Also $z_i\neq 0$ $(i=1,\cdots,r)$, $\omega=\sqrt{-1}$, m_j,n_j,p_j,q_j $(j=1,\cdots,r)$, $N,P.Q$ are non-negative integers such that $0\le N\le P$, $Q\ge 0$, $0\le m_j\le q_j$, $0\le n_j\le p_j$ $(j=1,2,\cdots,r)$ (not all zero simultaneously). $\alpha_j^{(i)}$ $(j=1,2,\cdots,P,i=1,2,\cdots,r)$, $\beta_j^{(i)}$ $(j=1,2,\cdots,Q,i=1,2,\cdots,r)$, $\gamma_j^{(i)}$ $(j=1,2,\cdots,p_i,i=1,2,\cdots,r)$ and $\delta_j^{(i)}$ $(j=1,2,\cdots,q_i,i=1,2,\cdots,r)$ are positive numbers a_j $(j=1,2,\cdots,P)$, b_j $(j=1,2,\cdots,Q)$, $c_j^{(i)}$ $(j=1,2,\cdots,p_i,i=1,2,\cdots,r)$ and $d_j^{(i)}$ $(j=1,2,\cdots,q_i,i=1,2,\cdots,r)$ are complex numbers. The exponents A_j $(j=1,2,\cdots,P)$, B_j $(j=1,2,\cdots,Q)$, $C_j^{(i)}$ $(j=1,2,\cdots,p_i,i=1,2,\cdots,r)$ and $D_j^{(i)}$ $(j=1,2,\cdots,q_i,i=1,2,\cdots,r)$ of various gamma functiions may take non integer values. The I-functiion of r-variables is analytic if

$$\Psi_i=\sum_{j=1}^{P}A_j\alpha_j^{(i)}-\sum_{j=1}^{Q}B_j\beta_j^{(i)}+\sum_{j=1}^{p_1}C_j^{(i)}\gamma_j^{(i)}-\sum_{j=1}^{q_i}D_j^{(i)}\delta_j^{(i)}\le 0,\ i=1,2,\cdots,r.$$

The integral (1.1) converges absolutely if $|arg(z_i)| < \frac{1}{2}\Delta_i\pi, \quad i = 1, 2, \cdots, r$ where

$$\Delta_i = -\sum_{j=n+1}^{P} A_j\alpha_j^{(i)} - \sum_{j=1}^{Q} B_j\beta_j^{(i)} + \sum_{j=1}^{m_i} D_j^{(i)}\delta_j^{(i)} - \sum_{j=m_i+1}^{q_i} D_j^{(i)}\delta_j^{(i)}$$

$$+\sum_{j=1}^{n_i} C_j^{(r)}\gamma_j^{(i)} - \sum_{j=n_i+1}^{p_i} C_j^{(i)}\gamma_j^{(i)} > 0. \tag{1.4}$$

The $\overline{I}$-functionof r-variables introduced by Prathima, [3, p. 42] is defined and represented in the following manner.

$$\overline{I}[z_1, \cdots, z_r] = I^{0,N:m_1,n_1;\cdots;m_r,n_r}_{P,Q;p_1,q_1;\cdots,p_r,q_r}$$

$$\left[\begin{array}{c} z_1 \\ \vdots \\ z_r \end{array}\middle| \begin{array}{c} {}_1(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_p :_1 (c_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{p_1}; \\ {}_1(b'_j\beta_j^{(1)},\cdots,\beta_j^{(r)})_Q :_1 (d_j^{(1)},\delta_j^{(1)};1)_{m_1,m_1+1}(d_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{q_1} \end{array}\right.$$

$$\left.\begin{array}{c} \cdots;\ {}_1(c_j^{(r)},\gamma_j^{(r)};c_j^{(r)})_{p_r} \\ \cdots\ {}_1(d_j^{(r)},\delta_j^{(r)};1)_{m_r,m_r+1}(d_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{q_r} \end{array}\right]$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\overline{\theta}_1(s_1)\cdots\overline{\theta}_r(s_r)\phi(s_1,\cdots,s_r)\ z_1^{s_1}\cdots z_r^{s_r}ds_1\cdots ds_r, \tag{1.5}$$

where

$$\overline{\theta}_i(s_i) = \frac{\prod_{j=1}^{m_i}\Gamma(d_j^{(i)} - \delta_j^{(i)}s_i)\prod_{j=1}^{n_i}\Gamma^{C_j^{(i)}}(1 - c_j^{(j)} + \gamma_j^{(i)}s_i)}{\prod_{j=m_i+1}^{q_i}\Gamma^{D_j^{(i)}}(1 - d_j^{(i)} + \delta_j^{(i)}s_i)\prod_{j=n_i+1}^{p_i}\Gamma^{C_j^{(i)}}(c_j^{(i)} - \gamma_j^{(i)}s_i)}.$$

The integral converges absolutely if $|arg(z_i)| < \frac{1}{2}\Delta'_i\pi, i = 1, 2, \cdots, r$ where

$$\Delta'_i = -\sum_{j=n+1}^{P} A_j\alpha_j^{(i)} - \sum_{j=1}^{Q} B_j\beta_j^{(i)} + \sum_{j=1}^{m_i}\delta_j^{(i)} - \sum_{j=m_i+1}^{q_i} D_j^{(i)}\delta_j^{(i)}$$

$$+\sum_{j=1}^{n_i} C_j^{(r)}\gamma_j^{(i)} - \sum_{j=n_i+1}^{p_i} C_j^{(i)}\gamma_j^{(i)} > 0.$$

The Generalized Struve's function defined by Kanth [1, p. 18]

$$H^{\lambda,k}_{v,y,\mu}(z) = \sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{z}{2}\right)^{v+2m+1}}{\Gamma(km+y)\Gamma(v+\lambda m+\mu)} \tag{1.6}$$

where $Re(k) > 0,\ Re(\lambda) > 0, Re(y) > 0, Re(v + \mu) > 0.$

2. Laplace Transform of the Functions Involving the Product of the Generalized Struve's Function and the I-function of Several Complex Variables

$$\int_0^\infty e^{-\eta x}\overline{I}[z_1x^{\sigma_1}, z_2x^{\sigma_2}, \cdots, z_rx^{\sigma_r}]H^{\lambda,k}_{v,y,\mu}(zx^\rho)dx$$
$$= \frac{1}{\eta}H^{\lambda,k}_{v,y,\mu}\left(\frac{z}{\eta^\rho}\right) \times I^{-0,N+1:m_1,n_1;\cdots;m_r,n_r}_{P+1,Q:p_1,q_1;\cdots;p_r,q_r}$$
$$\left[\begin{array}{c} \frac{z_1}{\eta^{\sigma_1}} \\ \vdots \\ \frac{z^r}{\eta^{\sigma_r}} \end{array}\middle| \begin{array}{l} (-\rho(v+2m+1);\sigma_1,\cdots,\sigma_r;1),\ {}_1(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_P : \\ {}_1(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{p_1};\cdots;\ {}_1(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{p_r} \\ {}_1(b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_Q :_1 (d_j^{(1)},\delta_j^{(1)};1)_{m_1,m_r+1}(d_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{q_1} \\ ;\cdots;\ {}_1(d_j^{(r)},\delta_j^{(r)};1)_{m_r,m_r+1}(d_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{q_r} \end{array}\right]$$

(2.1)

provided, $Re(\eta) > 0, Re(k) > 0,\ Re(\lambda) > 0, Re(y) > 0,\ Re(v+\mu) > 0,\ \sigma_i > 0$, $i = 1, 2, \cdots, r$ and

$$Re\left(1 + \sum_{i=1}^{r}\frac{\sigma_i d_j^{(i)}}{\delta_j^{(i)}}\right) > 0, \quad j = 1, 2, \cdots, m_r.$$

Proof :

$$\int_0^\infty e^{-\eta x}\overline{I}[z_1x^{\sigma_1}, z_2x^{\sigma_2}, \cdots, z_rx^{\sigma_r}]H^{\lambda,k}_{v,y,\mu}(z\ x^\rho)dx$$
$$= \int_0^\infty e^{-\eta x}\left(\frac{1}{2\pi\omega}\right)^r \int_{L_1}\cdots\int_{L_r}\phi(s_1,\cdots,s_r)\overline{\theta}_1(s_1)\cdots\overline{\theta}_r(s_r)$$
$$(z_1x^{\sigma_1})^{s_1}\cdots(z_rx^{\sigma_r})^{s_r}ds_1\cdots ds_r\sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{zx^\rho}{2}\right)^{v+2m+1}}{\Gamma(km+y)\Gamma(v+\lambda m+\mu)}dx$$
$$= \sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{z}{2}\right)^{v+2m+1}}{\Gamma(km+y)\Gamma(v+\lambda m+\mu)}\left(\frac{1}{2\pi\omega}\right)^r$$
$$\int_{L_1}\cdots\int_{L_r}\phi(s_1,\cdots,s_r)\overline{\theta}_1(s_1)\cdots\overline{\theta}_r(s_r)z_1^{s_1}\cdots z_r^{s_r}\times$$
$$\left[\int_0^\infty e^{-\eta x}x^{\sigma_1s_1+\cdots+\sigma_rs_r+\rho(v+2m+1)}dx\right]ds_1\cdots ds_r$$

(by changing the order of integration)

$$= \frac{1}{\eta}\sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{z}{2}\right)^{v+2m+1}}{\Gamma(km+y)\Gamma(v+\lambda m+\mu)}\times\frac{1}{\eta^{\rho(v+2m+1)}}\left(\frac{1}{2\pi\omega}\right)^r$$
$$\int_{L_1}\cdots\int_{L_r}\phi(s_1,\cdots,s_r)\overline{\theta}_1(s_1)\cdots\overline{\theta}_r(s_r)\left(\frac{z_1}{\eta^{\sigma_1}}\right)^{s_1}\cdots\left(\frac{z_r}{\eta^{\sigma_r}}\right)^{s_r}\times$$
$$\Gamma(\sigma_1 s_1+\cdots+\sigma_r s_r+\rho(v+2m+1)+1)ds_1\cdots ds_r$$

from which the result is obtained by using (1.5) and (1.6).

Special Cases :

1. Taking $r=2$, (2.1) reduces to the result involving I function of 2 variables as

$$\int_0^\infty e^{-\eta x}\overline{I}[z_1x^{\sigma_1},z_2x^{\sigma_2},]H^{\lambda,k}_{v,y,\mu}(zx^\rho)dx$$
$$=\frac{1}{\eta}H^{\lambda,k}_{v,y,\mu}\left(\frac{z}{\eta^\rho}\right)\times\overline{I}^{0,N+1:m_1,n_1;m_2,n_2}_{P+1,Q:p_1,q_1;p_2,q_2}$$
$$\left[\begin{array}{c|l}\frac{z_1}{\eta^{\sigma_1}} & (-\rho(v+2m+1);\sigma_1,\sigma_2;1),\ {}_1(a_j;\alpha_j^{(1)},\alpha_j^{(2)};A_j)_P: \\ & {}_1(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{p_1};\ {}_1(c_j^{(2)},\gamma_j^{(2)};C_j^{(2)})_{p_2} \\ \frac{z_2}{\eta^{\sigma_2}} & {}_1(b_j;\beta_j^{(1)},\beta_j^{(2)};B_j)_Q:{}_1(d_j^{(1)},\delta_j^{(1)};1)_{m_1,m_i+1}(d_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{q_1} \\ & {}_1(d_j^{(2)},\delta_j^{(2)};1)_{m_2,m_2+1}(d_j^{(2)},\delta_j^{(2)};D_j^{(2)})_{q_2}\end{array}\right] \tag{2.2}$$

provided, $Re(\eta)>0, Re(k)>0,\ Re(\lambda)>0, Re(y)>0,\ Re(v+\mu)>0,\ \sigma_i>0,\ i=1,2$ and

$$Re\left(1+\sum_{i=1}^{2}\frac{\sigma_i d_j^{(i)}}{\delta_j^{(i)}}\right)>0,\quad j=1,2,\cdots,m_r.$$

2. When all the exponents are equal to unity, (2.1) reduces to the formula involving the H-function of r variables as

$$\int_0^\infty e^{-\eta x}H[z_1x^{\sigma_1},z_2x^{\sigma_2}]H^{\lambda,k}_{v,y,\mu}(zx^\rho)dx$$
$$=\frac{1}{\eta}H^{\lambda,k}_{v,y,\mu}\left(\frac{z}{\eta^\rho}\right)\times H^{0,N+1:m_1,n_1;\cdots;m_r,n_r}_{P+1,Q:p_1,q_1;\cdots;p_r,q_r}$$
$$\left[\begin{array}{c|l}\frac{z_1}{\eta^{\sigma_1}} & (-\rho(v+2m+1);\sigma_1,\cdots;\sigma_r),\ {}_1(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_P: \\ \vdots & {}_1(c_j^{(1)};\gamma_j^{(1)})_{p_1},\cdots,(c_j^{(r)},\gamma_j^{(r)})_{p_r}:{}_1(b_j^{(1)},\beta_j^{(1)},\cdots,\beta_j^{(r)})_Q; \\ \frac{z_r}{\eta^{\sigma_r}} & {}_1(d_j^{(1)},\delta_j^{(1)})_{q_1};\cdots;\ {}_1(d_j^{(r)},\delta_j^{(r)})_{q_r}\end{array}\right] \tag{2.3}$$

provided, $Re(\eta) > 0, Re(k) > 0$, $Re(\lambda) > 0, Re(y) > 0$, $Re(v + \mu) > 0$, $\sigma_i > 0$, $i = 1, 2, \cdots, r$ and

$$Re\left(1 + \sum_{i=1}^{r} \frac{\sigma_i d_j^{(i)}}{\delta_j^{(i)}}\right) > 0, \quad j = 1, 2, \cdots, m_r.$$

3. When $r = 2$, (2.3) reduces to corresponding result for H function of 2 variables given by K. P. Shahul Hameed [4, p.69] as

$$\int_0^\infty e^{-\eta x} H[z_1 x^{\sigma_1}, z_2 x^{\sigma_2}] H_{v,y,\mu}^{\lambda,k}(z x^{\rho}) dx$$
$$= \frac{1}{\eta} H_{v,y,\mu}^{\lambda,k}\left(\frac{z}{\eta^{\rho}}\right) \times H_{P+1,Q:p_1,q_1;p_2,q_2}^{0,N+1:m_1,n_1;m_2,n_2}$$
$$\left[\begin{array}{c|l} \frac{z_1}{\eta^{\sigma_1}} & (-\rho(v+2m+1); \sigma_1, \sigma_2),\ {}_1(a_j; \alpha_j^{(1)}, \alpha_j^{(2)})_P :\ {}_1(c_j^{(1)}; \gamma_j^{(1)})_{p_1} :_1 (c_j^{(2)}, \gamma_j^{(2)})_{p_2}; \\ \frac{z^2}{\eta^{\sigma_2}} & {}_1(b_j, \beta_j^{(1)}, \beta_j^{(2)})_Q;\ {}_1(d_j^{(1)}, \delta_j^{(1)})_{q_1};\ {}_1(d_j^{(2)}, \delta_j^{(2)})_{q_2} \end{array}\right] \tag{2.4}$$

provided, $Re(\eta) > 0, Re(k) > 0$, $Re(\lambda) > 0, Re(y) > 0$, $Re(v + \mu) > 0$, $\sigma_i > 0$, $i = 1, 2, \cdots, r$ and

$$Re\left(1 + \sum_{i=1}^{2} \frac{\sigma_i d_j^{(i)}}{\delta_j^{(i)}}\right) > 0, \quad j = 1, 2, \cdots, m_2.$$

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