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BINOMIAL CONVOLUTIIONS IDENTITIES OF HYBRID FIBONACCI AND LUCAS POLYNOMIALS

R. RANGARAJAN[1], C. K. HONNEGOWDA [2] AND RANGASWAMY[3]
[1,2] Department of Studies in Mathematics,
University of Mysore, Manasagangotri, Mysuru - 570 006, India
[3] Department of Mathematics, B.M.S. College of Engineering,
Bull Temple Road, Basavanagudi,
Bengaluru - 560 019, India

Abstract

Two variable hybrid Fibonacci and Lucas polynomials display many interesting combinatorial properties which are generalizations of those of Fibonacci and Lucas numbers [6, 7, 8]. In the present paper, both the polynomials are shown to satisfy binomial convolution identities which will be a good addition to the current literature.

1. Introduction

Binomial coefficients, Fibonacci and Lucas numbers are basic combinatorial entities [2, 3, 9, 10, 11]. Several researchers are looking for their binomial convolution identities [4, 5]. Two variable generalization of Fibonacci and Lucas variables given by two Pascal

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like tables displaying the terms $(x+y)^n$ and $(x+2y)(x+y)^n$ are quite remarkable. They naturally contain F_n and L_n, two kinds of Fibonacci and Lucas polynomials as special cases . Recently these two variable generalization of F_n and L_n are studied as two variable hybrid Fibonacci and Lucas polynomials [6, 7].

The hybrid Fibonacci and Lucas polynomials in two variables x and y of degree n, are given by the following binet forms [2, 6, 7]:

$$f_n^{(H)}(x,y) = \frac{1}{\sqrt{x^2+4y}}\left[\Big(\frac{x+\sqrt{x^2+4y}}{2}\Big)^n - \Big(\frac{x-\sqrt{x^2+4y}}{2}\Big)^n\right] \tag{1.1}$$

$$l_n^{(H)}(x,y) = \left[\Big(\frac{x+\sqrt{x^2+4y}}{2}\Big)^n + \Big(\frac{x-\sqrt{x^2+4y}}{2}\Big)^n\right] \tag{1.2}$$

They satisfy the following 3 term recurrence relations :

$$f_{n+1}^{(H)}(x,y) = xf_n^{(H)}(x,y) + yf_{n-1}^{(H)}(x,y), \tag{1.3}$$

$$l_{n+1}^{(H)}(x,y) = x\ l_n^{(H)}(x,y) + y\ l_{n-1}^{(H)}(x,y) \tag{1.4}$$

$$\text{and} \quad l_n^{(H)}(x,y) = x\ f_n^{(H)}(x,y) + 2y\ f_{n-1}^{(H)}(x,y). \tag{1.5}$$

Put $\alpha = \left(\frac{x+\sqrt{x^2+4y}}{2}\right)$ and $\beta = \left(\frac{x-\sqrt{x^2+4y}}{2}\right)$. Then

$$f_n^{(H)}(x,y) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad , \qquad l_n^{(H)}(x,y) = \alpha^n + \beta^n. \tag{1.6}$$

Also,

$$\alpha + \beta \ = \ x, \quad \alpha - \beta \ = \ \sqrt{x^2+4y} \quad \text{and} \quad \alpha\beta \ = \ -\,y. \tag{1.7}$$

Hybrid Fibonacci and Lucas polynomials in two variables display many interesting combinatorial properties useful for research workers in combinatorics [2, 3, 9, 10, 11]. In the next section, a Bernoulli type identity for

$$B_n(m,x) = \sum_{k=0}^{n} \binom{n}{k}\ k^m\ x^k$$

is derived which will be used in the section 3 and 4. In the ensuing section, Binomial Convolution Identities of hybrid Fibonacci and Lucas polynomials in two variables with a fixed power of expanding variable, $k^m, m\ =\ 0\ ,\ 1$ are stated and proved . In the last section, Binomial Convolution Identities of hybrid Fibonacci and Lucas polynomials in two variables with a fixed power of expanding variable, $k^m, m = 2\ ,\ 3$ are stated and proved.

2. A Bernoulli Type Identity

By Bernoulli identity [1] we mean

$$(n+1)^m - 1 = \binom{m}{1} S_n(m-1) + \binom{m}{2} S_n(m-2) + \cdots + \binom{m}{m} S_n(0).$$
$$\text{where } S_n(m) = 1^m + 2^m + \cdots + n^m, S_n(0) = n, m = 2, 3, 4, \ldots.$$

The derivation is quite simple. Consider

$$\begin{aligned}(n+1)^m - 1 + S_n(m) &= \sum_{k=1}^{n} (k+1)^m \\ &= \sum_{k=1}^{n} \left[\binom{m}{0} k^m + \binom{m}{1} k^{m-1} + \cdots + \binom{m}{m} k^0\right] \\ &= S_n(m) + \binom{m}{1} S_n(m-1) + \cdots + \binom{m}{m} S_n(0).\end{aligned}$$

Hence by cancelling $S_n(m)$ on both sides one can get Bernoulli identity [1]. Following the same idea of derivation one can also derive a Bernoulli type identity :

$$\text{If } \ B_n(0) = \sum_{k=0}^{n} \binom{n}{k} \ = \ 2^n \ , \ B_n(m) = \sum_{k=0}^{n} \binom{n}{k} \ k^m, m = 1, 2, 3, \ldots, \ \text{ then}$$

$$\begin{aligned}\frac{1}{n+1} \ B_{n+1}(m+1) &= \frac{1}{n+1} \sum_{k=1}^{n} \binom{n+1}{k} \ k^{m+1} \\ &= \frac{1}{n+1} \sum_{k=0}^{n} \binom{n+1}{k+1} \ (k+1)^{m+1} \\ &= \sum_{k=0}^{n} \binom{n}{k} \ (k+1)^m \\ &= \sum_{k=0}^{n} \left[\binom{m}{0} k^m + \binom{m}{1} k^{m-1} + \cdots + \binom{m}{m} 1\right]\end{aligned}$$

Hence

$$\frac{1}{n+1} \ B_{n+1}(m+1) = \binom{m}{0} B_n(m) + \binom{m}{1} B_n(m-1) + \cdots + \binom{m}{m} B_n(0).$$
$$\textit{where } B_n(0) = 2^n, m = 0, 1, 2, 3, 4, \ldots.$$

Table 1 : First four sums

$B_n(0)$	2^n
$B_n(1)$	$n2^{n-1}$
$B_n(2)$	$n(3n-1)2^{n-2}$
$B_n(3)$	$n^2(n+3)2^{n-3}$

Let us define

$$\begin{aligned}
B_n(0,x) &= \sum_{k=0}^{n}\binom{n}{k}x^k = (x+1)^n \\
B_n(m,x) &= \sum_{k=0}^{n}\binom{n}{k}k^m x^k \\
\frac{1}{n+1}\,B_{n+1}(m+1,x) &= \frac{1}{n+1}\sum_{k=1}^{n+1}\binom{n+1}{k}k^{m+1}x^k \\
&= \frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k+1}(k+1)^{m+1}x^{k+1} \\
&= x\sum_{k=0}^{n}\binom{n}{k}(k+1)^m x^k \\
&= x\left[\binom{m}{0}B_n(m,x)+\binom{m}{1}B_n(m-1,x)+\cdots+\binom{m}{m}B_n(0,x)\right]
\end{aligned}$$

So, a Bernoulli type identity for

$$\begin{aligned}
B_n(0,x) &= \sum_{k=0}^{n}\binom{n}{k}x^k = (x+1)^n\ ,\ B_n(m,x) = \sum_{k=0}^{n}\binom{n}{k}k^m x^k \ \text{ is given by} \\
\frac{1}{n+1}\,B_{n+1}(m+1,x) &= x\left[\binom{m}{0}B_n(m,x)+\binom{m}{1}B_n(m-1,x)+\cdots+\binom{m}{m}B_n(0,x)\right] \\
\text{m=0,1,2,}\ldots &
\end{aligned}$$

Table 2 : First four sums $B_n(m,x)$

$B_n(0,x)$	$(x+1)^n$
$B_n(1,x)$	$(nx)(x+1)^{n-1}$
$B_n(2,x)$	$(nx)(nx+1)(x+1)^{n-2}$
$B_n(3,x)$	$(nx)^2(nx+3)(x+1)^{n-3}-(nx)(x-1)(x+1)^{n-3}$

For $x = 1$, we get back the Table (1) for $B_n(m)$. The above table is very useful to derive binomial convolution identities satisfied by $f_{(n)}^{(H)}(x,y)$ and $l_{(n)}^{(H)}(x,y)$ (c.f. (1.1) and (1.2)) which will be stated and proved in the next two sections.

3. Binomial Convolution Identities at the Levels $m = 0$ and $m = 1$

The Binomial Convolution Identities at $m = 0$ and $m = 1$ are stated and proved in Theorems(1) and (2) respectively.

Theorem 1 : The convolution identities at the level $m = 0$ are

$$
\begin{aligned}
(1.1)\ \sum_{k=0}^{n}\binom{n}{k} f_k^{(H)}(x,y) f_{n-k}^{(H)}(x,y) &= \frac{2^n\ l_n^{(H)}(x,y) - 2x^n}{(x^2+4y)} \\
(1.2)\ \sum_{k=0}^{n}\binom{n}{k} l_k^{(H)}(x,y) l_{n-k}^{(H)}(x,y) &= 2^n l_n^{(H)}(x,y) + 2x^n \\
(1.3)\ \sum_{k=0}^{n}\binom{n}{k} l_k^{(H)}(x,y) f_{n-k}^{(H)}(x,y) &= 2^n f_n^{(H)}(x,y) \\
(1.4)\ \sum_{k=0}^{n}\binom{n}{k} f_k^{(H)}(x,y) l_{n-k}^{(H)}(x,y) &= 2^n f_n^{(H)}(x,y)
\end{aligned}
$$

Proof : The result in Table 1 for $m = 0$, Table 2 for $B_n(0, \frac{\alpha}{\beta}), B_n(0, \frac{\beta}{\alpha})$ and the equations (1.6) and (1.7) will take us through the derivation step by step for all four identities.

$$
\begin{aligned}
(1.1): \quad & \sum_{k=0}^{n}\binom{n}{k} f_k^{(H)}(x,y) f_{n-k}^{(H)}(x,y) = \sum_{k=0}^{n}\binom{n}{k}\left(\frac{\alpha^k-\beta^k}{\alpha-\beta}\right)\left(\frac{\alpha^{n-k}-\beta^{n-k}}{\alpha-\beta}\right) \\
&= \frac{1}{(\alpha-\beta)^2}\Big[\sum_{k=0}^{n}\binom{n}{k}(\alpha^n+\beta^n) - \beta^n\sum_{k=0}^{n}\binom{n}{k}\Big(\frac{\alpha}{\beta}\Big)^k - \alpha^n\sum_{k=0}^{n}\binom{n}{k}\Big(\frac{\beta}{\alpha}\Big)^k\Big] \\
&= \frac{1}{(x^2+4y)}\Big[2^n l_n^{(H)}(x,y) - \beta^n\Big(1+\frac{\alpha}{\beta}\Big)^n - \alpha^n\Big(1+\frac{\beta}{\alpha}\Big)^n\Big] \\
&= \frac{1}{(x^2+4y)}\Big[2^n l_n^{(H)}(x,y)) - 2(\alpha+\beta)^n\Big] \\
&= \frac{2^n l_n^{(H)}(x,y) - 2x^n}{2(x^2+4y)}.
\end{aligned}
$$

$$
\begin{aligned}
(1.3): \quad & \sum_{k=0}^{n}\binom{n}{k} l_k^{(H)}(x,y) f_{n-k}^{(H)}(x,y) = \sum_{k=0}^{n}\binom{n}{k}(\alpha^k+\beta^k)\left(\frac{\alpha^{n-k}-\beta^{n-k}}{\alpha-\beta}\right) \\
= & \ \frac{1}{(\alpha-\beta)}\Big[\sum_{k=0}^{n}\binom{n}{k}(\alpha^n-\beta^n) - \beta^n\sum_{k=0}^{n}\binom{n}{k}\Big(\frac{\alpha}{\beta}\Big)^k + \alpha^n\sum_{k=0}^{n}\binom{n}{k}\Big(\frac{\beta}{\alpha}\Big)^k\Big] \\
= & \ \sum_{k=0}^{n}\binom{n}{k}\Big(\frac{\alpha^n-\beta^n}{\alpha-\beta}\Big) - \frac{\beta^n}{(\alpha-\beta)}\Big(1+\frac{\alpha}{\beta}\Big)^n + \frac{\alpha^n}{(\alpha-\beta)}\Big(1+\frac{\beta}{\alpha}\Big)^n \\
= & \ 2^n f_n^{(H)}(x,y) + \frac{1}{(\alpha-\beta)}[-(\alpha+\beta)^n + (\alpha+\beta)^n] \\
= & \ 2^n f_n^{(H)}(x,y).
\end{aligned}
$$

The proofs of (1.2) and (1.4) are similar to that of (1.1) and (1.3) respectively.

Theorem 2 : The convolution identities at the level $m = 1$ are

$$
\begin{aligned}
(2.1)\ \sum_{k=0}^{n}\binom{n}{k}k\ f_k^{(H)}(x,y) f_{n-k}^{(H)}(x,y) &= \frac{2^{n-1}n\ l_n^{(H)}(x,y) - nx^n}{(x^2+4y)} \\
(2.2)\ \sum_{k=0}^{n}\binom{n}{k}k\ l_k^{(H)}(x,y) l_{n-k}^{(H)}(x,y) &= 2^{n-1}n\ l_n^{(H)}(x,y) + nx^n \\
(2.3)\ \sum_{k=0}^{n}\binom{n}{k}k\ l_k^{(H)}(x,y) f_{n-k}^{(H)}(x,y) &= 2^{n-1}n\ f_n^{(H)}(x,y) - nx^{n-1} \\
(2.4)\ \sum_{k=0}^{n}\binom{n}{k}k\ f_k^{(H)}(x,y) l_{n-k}^{(H)}(x,y) &= 2^{n-1}n\ f_n^{(H)}(x,y) + nx^{n-1}
\end{aligned}
$$

Proof : The result in Table 1 for $m = 1$, Table 2 for $B_n(1, \frac{\alpha}{\beta}), B_n(1, \frac{\beta}{\alpha})$ and the equations (1.6) and (1.7) will take us through the derivation step by step for all four identities.

$$
\begin{aligned}
(2.1): \quad & \sum_{k=0}^{n}\binom{n}{k}k\ f_k^{(H)}(x,y) f_{n-k}^{(H)}(x,y) = \sum_{k=0}^{n}\binom{n}{k}k\left(\frac{\alpha^k-\beta^k}{\alpha-\beta}\right)\left(\frac{\alpha^{n-k}-\beta^{n-k}}{\alpha-\beta}\right) \\
= & \ \frac{1}{(\alpha-\beta)^2}\Big[\sum_{k=0}^{n}\binom{n}{k}k(\alpha^n+\beta^n) - \beta^n\sum_{k=0}^{n}\binom{n}{k}k\Big(\frac{\alpha}{\beta}\Big)^k - \alpha^n\sum_{k=0}^{n}\binom{n}{k}k\Big(\frac{\alpha}{\beta}\Big)^k\Big] \\
= & \ \frac{1}{(x^2+4y)}\Big[n2^{n-1}l_n^{(H)}(x,y) - \beta^n n\frac{\alpha}{\beta}\Big(1+\frac{\alpha}{\beta}\Big)^{n-1} - \alpha^n n\frac{\beta}{\alpha}\Big(1+\frac{\beta}{\alpha}\Big)^{n-1}\Big] \\
= & \ \frac{1}{(x^2+4y)}\Big[n2^{n-1}l_n^{(H)}(x,y)) - n(\alpha+\beta)^{n-1}(\alpha+\beta)\Big] \\
= & \ \frac{n2^{n-1}l_n^{(H)}(x,y) - nx^n}{(x^2+4y)}.
\end{aligned}
$$

$$(2.3)\qquad \sum_{k=0}^{n}\binom{n}{k}k\ l_k^{(H)}(x,y)f_{n-k}^{(H)}(x,y)=\sum_{k=0}^{n}\binom{n}{k}k\left(\alpha^k+\beta^k\right)\left(\frac{\alpha^{n-k}-\beta^{n-k}}{\alpha-\beta}\right)$$

$$= \frac{1}{(\alpha-\beta)}\Big[\sum_{k=0}^{n}\binom{n}{k}k(\alpha^n-\beta^n)-\beta^n\sum_{k=0}^{n}\binom{n}{k}k\Big(\frac{\alpha}{\beta}\Big)^k+\alpha^n\sum_{k=0}^{n}\binom{n}{k}k\Big(\frac{\beta}{\alpha}\Big)^k\Big]$$

$$= \sum_{k=0}^{n}\binom{n}{k}k\Big(\frac{\alpha^n-\beta^n}{\alpha-\beta}\Big)-\frac{\beta^n}{(\alpha-\beta)}n\frac{\alpha}{\beta}\Big(1+\frac{\alpha}{\beta}\Big)^{n-1}+\frac{\alpha^n}{(\alpha-\beta)}n\frac{\beta}{\alpha}\Big(1+\frac{\beta}{\alpha}\Big)^{n-1}$$

$$= n2^{n-1}f_n^{(H)}(x,y)+\frac{1}{\alpha-\beta}(n)(\alpha+\beta)^{n-1}(\beta-\alpha)$$

$$= n2^{n-1}f_n^{(H)}(x,y)-nx^{n-1}.$$

The proofs of (2.2) and (2.4) are similar to that of (2.1) and (2.3) respectively.

4. Binomial Convolution Identities at the Levels $m = 2$ and $m = 3$

In this section, we continue the computation of the Convolution identities at next two higher levels.

Theorem 3 : The convolution identities at the level $m = 2$ are

$$(3.1)\qquad \sum_{k=0}^{n}\binom{n}{k}k^2\ f_k^{(H)}(x,y)f_{n-k}^{(H)}(x,y)$$

$$= \frac{1}{(x^2+4y)}\Big[2^{n-2}n(3n-1)l_n^{(H)}(x,y)-[n(n-1)(x^2+2y)x^{n-2}+nx^n]\Big]$$

$$(3.2)\qquad \sum_{k=0}^{n}\binom{n}{k}k^2\ l_k^{(H)}(x,y)l_{n-k}^{(H)}(x,y)$$

$$= 2^{n-2}n(3n-1)\ l_n^{(H)}(x,y)+n(n-1)x^{n-2}(x^2+2y)+nx^n$$

$$(3.3)\qquad \sum_{k=0}^{n}\binom{n}{k}k^2\ l_k^{(H)}(x,y)f_{n-k}^{(H)}(x,y)=2^{n-2}n(3n-1)f_n^{(H)}(x,y)-n^2x^{n-1}$$

$$(3.4)\qquad \sum_{k=0}^{n}\binom{n}{k}k^2\ f_k^{(H)}(x,y)l_{n-k}^{(H)}(x,y)=2^{n-2}n(3n-1)f_n^{(H)}(x,y)+n^2x^{n-1}$$

Proof : The result in Table 1 for $m = 2$, Table 2 for $B_n(2,\frac{\alpha}{\beta}), B_n(2,\frac{\beta}{\alpha})$ and the equations (1.6) and (1.7) will take us through the derivation step by step for all four identities.

$$
\begin{aligned}
(3.1)\qquad & \sum_{k=0}^{n}\binom{n}{k}k^2\ f_k^{(H)}(x,y)f_{n-k}^{(H)}(x,y) == \sum_{k=0}^{n}\binom{n}{k}k^2\left(\frac{\alpha^k-\beta^k}{\alpha-\beta}\right)\left(\frac{\alpha^{n-k}-\beta^{n-k}}{\alpha-\beta}\right)\\
= \ & \frac{1}{(\alpha-\beta)^2}\Big[\sum_{k=0}^{n}\binom{n}{k}k^2(\alpha^n+\beta^n)-\beta^n\sum_{k=0}^{n}\binom{n}{k}k^2\Big(\frac{\alpha}{\beta}\Big)^k-\alpha^n\sum_{k=0}^{n}\binom{n}{k}k^2\Big(\frac{\alpha}{\beta}\Big)^k\Big]\\
= \ & \frac{1}{(x^2+4y)}\Big[n(3n-1)2^{n-2}l_n^{(H)}(x,y)-\beta^n\Big(n(n-1)\frac{\alpha^2}{\beta^2}\Big(1+\frac{\alpha}{\beta}\Big)^{n-2}+n\frac{\alpha}{\beta}\Big(1+\frac{\alpha}{\beta}\Big)^{n-1}\Big)\\
- \ & \alpha^n\Big(n(n-1)\frac{\beta^2}{\alpha^2}\Big(1+\frac{\beta}{\alpha}\Big)^{n-2}+n\frac{\beta}{\alpha}\Big(1+\frac{\beta}{\alpha}\Big)^{n-1}\Big)\Big]\\
= \ & \frac{1}{(x^2+4y)}\Big[n(3n-2)2^{n-2}l_n^{(H)}(x,y)\\
- \ & \Big(n(n-1)(\alpha+\beta)^{n-2}(\alpha^2+\beta^2)+n(\alpha+\beta)^{n-1}(\alpha+\beta)\Big]\\
= \ & \frac{1}{(x^2+4y)}\Big[2^{n-2}n(3n-1)l_n^{(H)}(x,y)-[n(n-1)(x^2+2y)x^{n-2}+nx^n]\Big].
\end{aligned}
$$

(by repeated deductions using (1.3) , (1.4) and (1.5)) .

$$
\begin{aligned}
(3.3)\qquad & \sum_{k=0}^{n}\binom{n}{k}k^2\ l_k^{(H)}(x,y)f_{n-k}^{(H)}(x,y) = \sum_{k=0}^{n}\binom{n}{k}k^2\left(\alpha^k+\beta^k\right)\left(\frac{\alpha^{n-k}-\beta^{n-k}}{\alpha-\beta}\right)\\
= \ & \frac{1}{(\alpha-\beta)}\Big[\sum_{k=0}^{n}\binom{n}{k}k^2(\alpha^n-\beta^n)-\beta^n\sum_{k=0}^{n}\binom{n}{k}k^2\Big(\frac{\alpha}{\beta}\Big)^k+\alpha^n\sum_{k=0}^{n}\binom{n}{k}k^2\Big(\frac{\beta}{\alpha}\Big)^k\Big]\\
= \ & \sum_{k=0}^{n}\binom{n}{k}k^2\Big(\frac{\alpha^n-\beta^n}{\alpha-\beta}\Big)-\frac{\beta^n}{(\alpha-\beta)}\Big[n(n-1)\frac{\alpha^2}{\beta^2}\Big(1+\frac{\alpha}{\beta}\Big)^{n-2}\\
+ \ & n\frac{\alpha}{\beta}(1+\frac{\alpha}{\beta})^{n-1}\Big]+\frac{\alpha^n}{(\alpha-\beta)}\Big[n(n-1)\frac{\beta^2}{\alpha^2}\Big(1+\frac{\beta}{\alpha}\Big)^{n-2}+n\frac{\beta}{\alpha}\Big(1+\frac{\beta}{\alpha}\Big)^{n-1}\Big]\\
= \ & n(3n-1)2^{n-1}f_n^{(H)}(x,y)+\frac{1}{\alpha-\beta}\Big[n(n-1)(\alpha+\beta)^{n-2}(\beta^2-\alpha^2)\\
+ \ & n(\alpha+\beta)^{n-1}(\beta-\alpha)\Big]\\
= \ & n(3n-1)2^{n-2}f_n^{(H)}(x,y)-n^2x^{n-1}.
\end{aligned}
$$

(by repeated deductions using (1.3) , (1.4) and (1.5)) .

The proofs of (3.2) and (3.4) are similar to that of (3.1) and (3.3) respectively.

Theorem 4 : The convolution identities at the level $m = 3$ are

$$\begin{aligned}
(4.1) \qquad & \sum_{k=0}^{n} \binom{n}{k} k^3 \ f_k^{(H)}(x,y) f_{n-k}^{(H)}(x,y) \\
= \quad & \frac{1}{(x^2+4y)} \Big[2^{n-3} n^2 (n+3) l_n^{(H)}(x,y) - [(3n^3 - 3n^2) x^{n-2} y + n^3 x^n] \Big] \\
(4.2) \qquad & \sum_{k=0}^{n} \binom{n}{k} k^3 \ l_k^{(H)}(x,y) l_{n-k}^{(H)}(x,y) \\
= \quad & 2^{n-3} n^2 (n+3) \ l_n^{(H)}(x,y) + (3n^3 - 3n^2) x^{n-2} y + n^3 x^n \\
(4.3) \qquad & \sum_{k=0}^{n} \binom{n}{k} k^3 \ l_k^{(H)}(x,y) f_{n-k}^{(H)}(x,y) \\
= \quad & 2^{n-3} n^2 (n+3) f_n^{(H)}(x,y) - [(n^3 - 3n^2 + 2n) x^{n-3} y + n^3 x^{n-1}] \\
(4.4) \qquad & \sum_{k=0}^{n} \binom{n}{k} k^3 \ f_k^{(H)}(x,y) l_{n-k}^{(H)}(x,y) \\
= \quad & 2^{n-3} n^2 (n+3) f_n^{(H)}(x,y) + [(n^3 - 3n^2 + 2n) x^{n-3} y + n^3 x^{n-1}]
\end{aligned}$$

Proof : The result in Table 1 for $m = 3$, Table 2 for $B_n(3, \frac{\alpha}{\beta}), B_n(3, \frac{\beta}{\alpha})$ and the equations (1.6) and (1.7) will take us through the derivation step by step for all four identities.

$$\begin{aligned}
(4.1) \qquad & \sum_{k=0}^{n} \binom{n}{k} k^3 \ f_k^{(H)}(x,y) f_{n-k}^{(H)}(x,y) = \sum_{k=0}^{n} \binom{n}{k} k^3 \left(\frac{\alpha^k - \beta^k}{\alpha - \beta} \right) \left(\frac{\alpha^{n-k} - \beta^{n-k}}{\alpha - \beta} \right) \\
= \quad & \frac{1}{(\alpha-\beta)^2} \Big[\sum_{k=0}^{n} \binom{n}{k} k^3 (\alpha^n + \beta^n) - \beta^n \sum_{k=0}^{n} \binom{n}{k} k^3 \Big(\frac{\alpha}{\beta} \Big)^k - \alpha^n \sum_{k=0}^{n} \binom{n}{k} k^3 \Big(\frac{\alpha}{\beta} \Big)^k \Big] \\
= \quad & \frac{1}{(x^2+4y)} \Big[n^2 (n+3) 2^{n-3} l_n^{(H)}(x,y) - \beta^n \Big(n(n-1)(n-2) \frac{\alpha^3}{\beta^3} \Big(1 + \frac{\alpha}{\beta} \Big)^{n-3} \\
+ \quad & 3n(n-1) \frac{\alpha^2}{\beta^2} \Big(1 + \frac{\alpha}{\beta} \Big)^{n-2} + n \frac{\alpha}{\beta} \Big(1 + \frac{\alpha}{\beta} \Big)^{n-1} \Big) - \alpha^n \Big(n(n-1) \frac{\beta^3}{\alpha^3} \Big(1 + \frac{\beta}{\alpha} \Big)^{n-3} \\
+ \quad & 3n(n-1) \frac{\beta^2}{\alpha^2} \Big(1 + \frac{\beta}{\alpha} \Big)^{n-2} + n \frac{\beta}{\alpha} \Big(1 + \frac{\beta}{\alpha} \Big)^{n-1} \Big) \Big]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{(x^2+4y)}\Big[n^2(n+3)2^{n-3}l_n^{(H)}(x,y) - \Big(n(n-1)(n-2)(\alpha+\beta)^{n-3}(\alpha^3+\beta^3)\\
&+ 3n(n-1)(\alpha+\beta)^{n-2}(\alpha^2+\beta^2) + n(\alpha+\beta)^{n-1}(\alpha+\beta)\Big)\Big]\\
&= \frac{n^2(n+3)2^{n-3}l_n^{(H)}(x,y) - \Big[(3n^3-3n^2)x^{n-2}y + n^3x^n\Big]}{(x^2+4y)}.
\end{aligned}
$$

(by repeated deductions using (1.3) , (1.4) and (1.5)) .

(4.3)
$$
\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}k^3\ l_k^{(H)}(x,y)f_{n-k}^{(H)}(x,y) = \sum_{k=0}^{n}\binom{n}{k}k^3\left(\alpha^k+\beta^k\right)\left(\frac{\alpha^{n-k}-\beta^{n-k}}{\alpha-\beta}\right)\\
&= \frac{1}{(\alpha-\beta)}\Big[\sum_{k=0}^{n}\binom{n}{k}k^3(\alpha^n-\beta^n) - \beta^n\sum_{k=0}^{n}\binom{n}{k}k^3\Big(\frac{\alpha}{\beta}\Big)^k + \alpha^n\sum_{k=0}^{n}\binom{n}{k}k^3\Big(\frac{\beta}{\alpha}\Big)^k\Big]\\
&= \sum_{k=0}^{n}\binom{n}{k}k^3\Big(\frac{\alpha^n-\beta^n}{\alpha-\beta}\Big) - \frac{\beta^n}{(\alpha-\beta)}\Big[n(n-1)(n-2)\frac{\alpha^3}{\beta^3}\Big(1+\frac{\alpha}{\beta}\Big)^{n-3}\\
&+ 3n(n-1)\frac{\alpha^2}{\beta^2}\Big(1+\frac{\alpha}{\beta}\Big)^{n-2} + n\frac{\alpha}{\beta}(1+\frac{\alpha}{\beta})^{n-1}\Big] + \frac{\alpha^n}{(\alpha-\beta)}\Big[n(n-1)(n-2)\frac{\beta^3}{\alpha^3}\Big(1+\frac{\beta}{\alpha}\Big)^{n-3}\\
&+ 3n(n-1)\frac{\beta^2}{\alpha^2}\Big(1+\frac{\beta}{\alpha}\Big)^{n-2} + n\frac{\beta}{\alpha}\Big(1+\frac{\beta}{\alpha}\Big)^{n-1}\Big]\\
&= n^2(n+3)2^{n-3}f_n^{(H)}(x,y) + \frac{1}{\alpha-\beta}\Big[n(n-1)(n-2)(\alpha+\beta)^{n-3}(\beta^3-\alpha^3)\\
&+ 3n(n-1)(\alpha+\beta)^{n-2}(\beta^2-\alpha^2) + n(\alpha+\beta)^{n-1}(\beta-\alpha)\Big]\\
&= n^2(n+3)2^{n-3}f_n^{(H)}(x,y) - [(n^3-3n^2+2n)x^{n-3}y + n^3x^{n-1}].
\end{aligned}
$$

(by repeated deductions using (1.3) , (1.4) and (1.5)) .

The proofs of (4.2) and (4.4) are similar to that of (4.1) and (4.3) respectively.
The same procedure of employing generalized Binomial summation can be applied to compute convolution identities at any level.

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