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SOME COMMON FIXED POINT THEOREMS FOR THREE MAPPINGS IN TVS-VALUED CONE METRIC SPACES

A. K. DUBEY[1] AND MADHUBALA KASAR[2]
[1] Department of Mathematics,
Bhilai Institute of Technology,
Bhilai House, Durg 491001,Chhattisgarh, India
Department of Mathematics,
Dr. C.V. Raman University, Kota, Bilaspur,
Chhattisgarh, India

Abstract

We obtain sufficient conditions for the existence of coincidence points and common fixed points for three mappings satisfying generalized contractive conditions in TVS-valued cone metric spaces without the assumption of normality. Our results generalize several well-known recent results in the literature.

1. Introduction and Preliminaries

Huang and Zhang [9] generalized the concept of a metric space by replacing the set of real numbers by ordered Banach space and obtained some fixed point theorems for contractive type mappings in a normal cone metric space. Subsequently, several other

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authors [3,4,12,15] studied the existence of common fixed point of mappings satisfying a contractive type condition in normal cone metric spaces. Afterwards, Rezapour and Hamlbarani in [10] studied fixed point theorems of contractive type mappings by omitting the assumption of normality in cone metric spaces. Recently, Beg et al.[1] studied common fixed points of a pair of maps on TVS-valued cone metric space which is a larger class than that of introduced by Huang and Zhang [9]. Since then several papers deal with fixed point theorems for contractive type mappings in TVS-valued cone metric spaces [2,5,6,7,8,11]. In this paper we obtain common fixed points and points of coincidence of three mappings in TVS-valued cone metric spaces without the assumption of normality. Our results improve and generalize various comparable results in the literature e.g. [2,6,7,11]. The following definitions and results will be needed in the sequel.

Let (E, τ) be a topological vector space (TVS) and P a subset of E. Then, P is called a cone whenever

i) P is closed, non-empty, and $P \neq \{\theta\}$,

ii) $ax + by \in P$ for all $x, y \in P$ and non-negative real numbers a, b.

iii) $P \cap (-P) = \{\theta\}$.

For a given a cone $P \subseteq E$, we can define a partial ordering $\preceq$with respect to P by $x \preceq y$ if and only if $y - x \in P$. $x \prec y$ will stand for $x \preceq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in intP$, where $intP$ denotes the interior of P. A cone P is called solid if $int\,P$ is nonempty.

Definition 1.[1,2] : Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies

(d1) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$,

(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$,

(d3) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a TVS-valued cone metric on X and (X, d) is called a TVS-valued cone metric space.

If E is a real Banach space then $(X.d)$ is called cone metric space [9].

Definition 2 [1] : Let (X, d) be a TVS-valued cone metric space, $x \in X$ and $\{x_n\}_{n\geqslant 1}$ a sequence in X. Then

(i) $\{x_n\}_{n\geqslant 1}$converges to x whenever for every $c \in E$ *with* $\theta \ll c$, there is a natural number N such that $d(x_n, x) \ll c$ for all $n \geqslant N$. We denote this by $\lim_{n\to\infty} x_n = x$ *or* $x_n \to x$.

(ii)$\{x_n\}_{n\geqslant 1}$ is a Cauchy sequence whenever for every $c \in E$ with $\theta \ll c$, there is a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m \geqslant N$.

(iii) (X, d) is a complete TVS valued cone metric space if every Cauchy sequence is convergent.

A pair (F, T) of self-mappings on X is said to be weakly compatible if $FTx = TFx$ whenever $Fx = Tx$. A point $y \in X$ is called point of coincidence of a family $T_j, j \in J$, of self-mappings on X if there exists a point $x \in X$ such that $y = T_jx$ for all $j \in J$.

Lemma 3 [4] : Let X be a nonempty set and the mappings $S, T, F : X \to X$ have a unique point of coincidence $\vartheta \in X$. If (S, F) and (T, F) are weakly compatible, then S, T and F have a unique common fixed point.

2. Common Fixed Point Theorems

In this section we obtain common fixed points and points of coincidence of three mappings in TVS-valued cone metric spaces.

Theorem 4 : Let (X, d)be a complete TVS-valued cone metric space, P be a solid cone, and mappings, $S, T, F : X \to X$ satisfy:

$$d(Sx, Ty) \preceq a_1 d(Fx, Fy) + a_2 d(Sx, Fx) + a_3 d(Ty, Fy) + a_4[d(Sx, Fy) + d(Ty, Fx)] \quad (2.1)$$

for all $x, y \in X$, where a_1, a_2, a_3, a_4 are non-negative real numbers with $a_1 + a_2 + a_3 + 2a_4 < 1$. If $S(X) \cup T(X) \subseteq F(X)$ and $F(X)$ or $S(X) \cup T(X)$ is a complete subspace of X, then S, T and F have a unique point of coincidence. Moreover, if $(S, F)$and (T, F) are weakly compatible then S, T and F have a unique common fixed point.

Proof : We shall first show that, if S, T and F have a point of coincidence, then it is unique. For this, assume that there exist two distinct points of coincidence $\vartheta, \vartheta^\star$of mappings S, T and F in X. It follows that there exists $u, u^\star \in X$ such thaat

$$\vartheta = Fu = Su = Tu,$$

and

$$\vartheta^\star = Fu^* = Su^* = Tu^*.$$

From (2.1), we obtain

$$\begin{aligned}
d(\vartheta, \vartheta^*) &= d(Su, Tu^*) \\
&\preceq a_1 d(Fu, Fu^*) + a_2 d(Su, Fu) + a_3 d(Tu^*, Fu^*) + a_4[d(Su, Fu^*) + d(Tu^*, Su)] \\
&= a_1 d(\vartheta, \vartheta^*) + a_2 d(\vartheta, \vartheta) + a_3 d(\vartheta^*, \vartheta^*) \\
&\quad + a_4[d(\vartheta, \vartheta^*) + d(\vartheta^*, \vartheta)] \\
&\preceq (a_1 + 2a_4) d(\vartheta, \vartheta^*) (Since\, a_1 + a_2 + a_3 + 2a_4 < 1).
\end{aligned}$$

It implies that $\vartheta = \vartheta^*$, a contradiction.
Now, we prove the existence of a point of coincidence of the mappings S, T and F. Let x_0 be any arbitrary point in X. Choose a point x_1 in X such that $Fx_1 = Tx_0$. This can be done since $S(X) \cup T(X) \subseteq F(X)$. Similarly choose a point x_2 in X such that $Fx_2 = Sx_1$. Continuing this process having chosen x_n in X, we obtain x_{n+1}in X such that

$$\begin{aligned}
Fx_{2n+1} &= Tx_{2n} \\
Fx_{2n+2} &= Sx_{2n+1}, n \geq 0.
\end{aligned}$$

Suppose there exists n such that $Fx_{2n} = Fx_{2n+1}$. Then $Fx_{2n} = Tx_{2n}$ and from (2.1)

$$\begin{aligned}
d(Fx_{2n}, Sx_{2n}) &= d(Fx_{2n+1}, Sx_{2n}) \\
&= d(Tx_{2n}, Sx_{2n}) \\
&\preceq a_1 d(Fx_{2n}, Fx_{2n}) + a_2 d(Sx_{2n}, Fx_{2n}) + a_3 d(Tx_{2n}, Fx_{2n}) \\
&\quad + a_4[d(Sx_{2n}, Fx_{2n}) + d(Tx_{2n}, Fx_{2n})] \\
&\preceq a_2 d(Fx_{2n}, Sx_{2n}) + a_4 d(Fx_{2n}, Sx_{2n}) \\
&= (a_2 + a_4) d(Fx_{2n}, Sx_{2n})
\end{aligned}$$

which yields $Fx_{2n} = Sx_{2n}$ and so, $Fx_{2n} = Sx_{2n} = Tx_{2n} = y$ is the required unique point of coincidence of F, S and T. Similarly, if $Fx_{2n+1} = Fx_{2n+2}$ for some n. Then

$Fx_{2n+1} = Sx_{2n+1} = Tx_{2n+1} = y$ is the required point. Thus in this sequel of proof we can suppose that $Fx_n \neq Fx_{n+1}$ for some n, From (2.1)

$$\begin{aligned}
d(Fx_{2n}, Fx_{2n+1}) &= d(Sx_{2n-1}, Tx_{2n}) \\
&\preceq a_1 d(Fx_{2n-1}, Fx_{2n}) + a_2 d(Sx_{2n-1}, Fx_{2n-1}) \\
&\quad + a_3 d(Tx_{2n}, Fx_{2n}) + a_4[d(Sx_{2n-1}, Fx_{2n}) + d(Tx_{2n}, Fx_{2n-1})] \\
&\preceq a_1 d(Fx_{2n-1}, Fx_{2n}) + a_2 d(Fx_{2n}, Fx_{2n-1}) + a_3 d(Fx_{2n}, Fx_{2n+1}) \\
&\quad + a_4[d(Fx_{2n}, Fx_{2n}) + d(Fx_{2n+1}, Fx_{2n-1})] \\
&\preceq \frac{a_1 + a_2 + a_4}{1 - a_3 - a_4} d(Fx_{2n-1}, Fx_{2n}) \\
&\preceq max\left\{\frac{a_1 + a_3 + a_4}{1 - a_2 - a_4}, \frac{a_1 + a_2 + a_4}{1 - a_3 - a_4}\right\} d(Fx_{2n-1}, Fx_{2n})
\end{aligned}$$

and

$$\begin{aligned}
d(Fx_{2n-1}, Fx_{2n}) &= d(Tx_{2n-2}, Sx_{2n-1}) \\
&\preceq a_1 d(Fx_{2n-2}, Fx_{2n-1}) + a_2 d(Sx_{2n-1}, Fx_{2n-1}) \\
&\quad + a_3 d(Tx_{2n-2}, Fx_{2n-2}) \\
&\quad + a_4[d(Sx_{2n-1}, Fx_{2n-2}) + d(Tx_{2n-2}, Fx_{2n-1})] \\
&\preceq a_1 d(Fx_{2n-2}, Fx_{2n-1}) + a_2 d(Fx_{2n}, Fx_{2n-1}) \\
&\quad + a_3 d(Fx_{2n-1}, Fx_{2n-2}) + a_4 d(Fx_{2n}, Fx_{2n-2}) \\
&\preceq \frac{a_1 + a_3 + a_4}{1 - a_2 - a_4} d(Fx_{2n-2}, Fx_{2n-1}) \\
&\preceq max\left\{\frac{a_1 + a_3 + a_4}{1 - a_2 - a_4}, \frac{a_1 + a_2 + a_4}{1 - a_3 - a_4}\right\} d(Fx_{2n-2}, Fx_{2n-1}).
\end{aligned}$$

It implies that $d(Fx_{2n}, Fx_{2n+1}) \preceq \lambda d(Fx_{2n-1}, Fx_{2n})$, where $\lambda = max\left\{\frac{a_1+a_3+a_4}{1-a_2-a_4}, \frac{a_1+a_2+a_4}{1-a_3-a_4}\right\}$ As $Fx_n \neq Fx_{n+1}$ and $A + B < 1$, therefore, $0 < \lambda < 1$ and for all n, $d(Fx_n, Fx_{n+1}) \preceq \lambda d(Fx_{n-1}, Fx_n) \preceq \lambda^2 d(Fx_{n-2}, Fx_{n-1}) \preceq - - - \preceq \lambda^n d(Fx_0, Fx_1)$. Now for any $m > n$,

$$\begin{aligned}
d(Fx_m, Fx_n) &\preceq d(Fx_n, Fx_{n+1}) + d(Fx_{n+1}, Fx_{n+2}) + - - - + d(Fx_{m-1}, Fx_m) \\
&\preceq [\lambda^n + \lambda^{n+1} + - - - + \lambda^{m-1}] d(Fx_0, Fx_1) \\
&\preceq [\frac{\lambda^n}{1 - \lambda}] d(Fx_0, Fx_1).
\end{aligned}$$

Let $\theta \ll c$ be given, choose a symmetric neighborhood $V\, of\, \theta$ such that $c + V \subseteq\ int\, P$.

Also, choose a natural number N_1 such that

$$[\frac{\lambda^n}{1-\lambda}]d(Fx_0, Fx_1) \in V \quad for\ all \quad n \geq N_1.$$

Then $\frac{\lambda^n}{1-\lambda}d(Fx_1, Fx_0) \ll c$, for all $n \geq N_1$. Thus,

$$d(Fx_m, Fx_n) \preceq [\frac{\lambda^n}{1-\lambda}]d(Fx_0, Fx_1) \ll c,$$

for all $m > n$. Therefore, $\{Fx_n\}_{n\geq 1}$ is a Cauchy sequence. Since FX is complete, there exist $u \in X$, $\vartheta \in FX$ such that $Fx_n \to \vartheta = Fu$ (this holds also if $S(X) \cup T(X)$ is complete with $\vartheta \in S(X) \cup T(X)$). Choose a natural number N_2 such that for all $n \geq N_2$,

$$d(Fx_{n+1}, Fx_n) \ll \frac{c(1-a_3-a_4)}{2(a_1+a_2+a_4)}$$

and

$$d(Fx_{n+1}, Fu) \ll \frac{c(1-a_3-a_4)}{2}.$$

xThen for all $n \geq N_2$

$$\begin{aligned}
d(Fu, Tu) &\preceq d(Fu, Fx_{2n+2}) + d(Fx_{2n+2}, Tu) \\
&\preceq d(Fu, Fx_{2n+2}) + d(Sx_{2n+1}, Tu) \\
&\preceq d(Fu, Fx_{2n+2}) + a_1 d(Fx_{2n+1}, Fu) + a_2 d(Sx_{2n+1}, Fx_{2n+1}) \\
&\quad + a_3 d(Tu, Fu) + a_4[d(Sx_{2n+1}, Fu) + d(Tu, Fx_{2n+1})] \\
&\preceq d(Fu, Fx_{2n+2}) + a_1 d(Fx_{2n+1}, Fu) + a_2 d(Fx_{2n+2}, Fx_{2n+1}) \\
&\quad + a_3 d(Tu, Fu) + a_4[d(Fx_{2n+2}, Fu) + d(Tu, Fx_{2n+1})] \\
&\preceq d(Fu, Fx_{2n+2}) + a_1 d(Fx_{2n+1}, Fu) + a_2 d(Fx_{2n+1}, Fx_{2n+2}) \\
&\quad + a_3 d(Tu, Fu) + a_4[d(Fx_{2n+2}, Fu) + d(Tu, Fu)] \\
&\preceq \frac{1}{1-a_3-a_4} d(Fu, Fx_{2n+2}) + \frac{a_1+a_2+a_4}{1-a_3-a_4} d(Fx_{2n+1}, Fx_{2n+2}) \\
&\ll \frac{c}{2} + \frac{c}{2} = c.
\end{aligned}$$

Thus

$$d(Fu, Tu) \ll \frac{c}{m} \quad for \quad all \quad m \geq 1.$$

So, $\frac{c}{m} - d(Fu, Tu) \in P$, for all $m \geq 1$. Since $\frac{c}{m} \to \theta$ (as $m \to \infty$) and P is closed, $-d(Fu, Su) \in P$. But $d(Fu, Tu) \in P$, therefore, $d(Fu, Tu) = \theta$. Hence

$$\vartheta = Fu = Tu,$$

and

$$\begin{aligned} d(Fu, Su) &= d(Tu, Su) \preceq a_1 d(Fu, Fu) + a_2 d(Fu, Su) \\ &\quad + a_3 d(Tu, Fu) + a_4[d(Su, Fu) + d(Tu, Fu)] \\ &\preceq (a_2 + a_4) d(Fu, Su) \end{aligned}$$

implies that ϑ is a unique point of coincidence of F, S and T. If (S, F) and (T, F) are weakly compatible, then by Lemma 3, ϑ is a unique common fixed point of S, T and F.

Theorem 5 : Let (X, d) be a complete TVS-valued cone metric space, P be a solid cone, and mappings, $S, T, F : X \to X$ satisfy:

$$d(Sx, Ty) \preceq a_1 d(Fx, Fy) + a_2 d(Sx, Fy) + a_3 d(Ty, Fx) + a_4[d(Sx, Fx) + d(Ty, Fy)] \quad (2.2)$$

for all $x, y \in X$, where a_1, a_2, a_3, a_4 are non-negative real numbers with $a_1 + a_2 + a_3 + 2a_4 < 1$. If $S(X) \cup T(X) \subseteq F(X)$ and $F(X)$ or $S(X) \cup T(X)$ is a complete subspace of X, then S, T and F have a unique point of coincidence. Moreover, if $(S, F)$and (T, F) are weakly compatible then S, T and F have a unique common fixed point.

Proof : It can be easily seen that if S, T and F have a point of coincidence, then it is unique. Let x_0 be an arbitrary point in X. Choose a point x_1 in X such that $Fx_1 = Tx_0$. This can be done since $S(X) \cup T(X)1 \subseteq F(X)$. Similarly choose a point x_2 in X such that $Fx_2 = Sx_1$. Continuing this process having chosen x_n in X, we obtain x_{n+1} in X such that

$$Fx_{2n+1} = Tx_{2n} \text{ and } Fx_{2n+2} = Sx_{2n+1}, n \geq 0.$$

Suppose there exist n such that $Fx_{2n} = Fx_{2n+1}$. Then using (2.2), we obtain $Fx_{2n} = Sx_{2n} = Tx_{2n} = y(say)$ is the required unique point of coincidence of F, S and T. Similarly, if $Fx_{2n+1} = Fx_{2n+2}$ for some n. Then $Fx_{2n+1} = Sx_{2n+1} = Tx_{2n+1} = y$ is the required point. Thus in this sequel of proof we can suppose that $Fx_n \neq Fx_{n+1}$. From

(2.2), we obtain

$$\begin{aligned}
d(Fx_{2n}, Fx_{2n+1}) &= d(Sx_{2n-1}, Tx_{2n}) \\
&\preceq a_1 d(Fx_{2n-1}, Fx_{2n}) + a_2 d(Sx_{2n-1}, Fx_{2n}) \\
&\quad + a_3 d(Tx_{2n}, Fx_{2n-1}) + a_4[d(Sx_{2n-1}, Fx_{2n-1}) + d(Tx_{2n}, Fx_{2n})] \\
&= a_1 d(Fx_{2n-1}, Fx_{2n}) + a_2 d(Fx_{2n}, Fx_{2n}) + a_3 d(Fx_{2n+1}, Fx_{2n-1}) \\
&\quad + a_4[d(Fx_{2n}, Fx_{2n-1}) + d(Fx_{2n+1}, Fx_{2n})] \\
&\preceq a_1 d(Fx_{2n-1}, Fx_{2n}) + a_3[d(Fx_{2n-1}, Fx_{2n}) + d(Fx_{2n}, Fx_{2n+1})] \\
&\quad + a_4[d(Fx_{2n}, Fx_{2n-1}) + d(Fx_{2n+1}, Fx_{2n})] \\
&\quad d(Fx_{2n}, Fx_{2n+1}) \preceq \frac{a_1 + a_3 + a_4}{1 - a_3 - a_4} d(Fx_{2n-1}, Fx_{2n})
\end{aligned}$$

and

$$\begin{aligned}
d(Fx_{2n-1}, Fx_{2n}) &= d(Tx_{2n-2}, Sx_{2n-1}) \\
&= d(Sx_{2n-1}, Tx_{2n-2}) \\
&\preceq a_1 d(Fx_{2n-1}, Fx_{2n-2}) + a_2 d(Sx_{2n-1}, Fx_{2n-2}) \\
&\quad + a_3 d(Tx_{2n-2}, Fx_{2n-1}) + a_4[d(Sx_{2n-1}, Fx_{2n-1}) + d(Tx_{2n-2}, Fx_{2n-2})] \\
&\preceq a_1 d(Fx_{2n-2}, Fx_{2n-1}) + a_2 d(Fx_{2n}, Fx_{2n-2}) \\
&\quad + a_3 d(Fx_{2n-1}, Fx_{2n-1}) + a_4[d(Fx_{2n}, Fx_{2n-1}) + d(Fx_{2n-1}, Fx_{2n-2})] \\
&\quad d(Fx_{2n-1}, Fx_{2n}) \preceq \frac{a_1 + a_2 + a_4}{1 - a_2 - a_4} d(Fx_{2n-2}, Fx_{2n-1}).
\end{aligned}$$

It follows that

$$\begin{aligned}
d(Fx_{2n}, Fx_{2n+1}) &\preceq \frac{a_1 + a_3 + a_4}{1 - a_3 - a_4} d(Fx_{2n-1}, Fx_{2n}) \\
&\preceq \left[\frac{a_1 + a_3 + a_4}{1 - a_3 - a_4} . \frac{a_1 + a_2 + a_4}{1 - a_2 - a_4}\right] d(Fx_{2n-2}, Fx_{2n-1}) \\
&\preceq \left[\frac{a_1 + a_3 + a_4}{1 - a_3 - a_4} . \frac{a_1 + a_2 + a_4}{1 - a_2 - a_4}\right]^n d(Fx_0, Fx_1),
\end{aligned}$$

and

$$d(Fx_{2n+1}, Fx_{2n+2}) \preceq \frac{a_1 + a_2 + a_4}{1 - a_2 - a_4} d(Fx_{2n}, Fx_{2n+1})$$

$$\preceq \frac{a_1 + a_2 + a_4}{1 - a_2 - a_4} \left[\frac{a_1 + a_3 + a_4}{1 - a_3 - a_4} . \frac{a_1 + a_2 + a_4}{1 - a_2 - a_4}\right]^n d(Fx_0, Fx_1).$$

Let

$$A = \frac{a_1 + a_2 + a_4}{1 - a_2 - a_4}, B = \frac{a_1 + a_3 + a_4}{1 - a_3 - a_4}$$

then as $Fx_n \neq Fx_{n+1}$ and $a_1 + a_2 + a_3 + 2a_4 < 1$,

$$\begin{aligned} 0 < AB &= \frac{a_1 + a_2 + a_4}{1 - a_2 - a_4} . \frac{a_1 + a_3 + a_4}{1 - a_3 - a_4} \\ &= \frac{a_1 + a_2 + a_4}{1 - a_3 - a_4} . \frac{a_1 + a_3 + a_4}{1 - a_2 - a_4} < 1.1 = 1. \end{aligned}$$

Now for any $n < m$, we have

$$\begin{aligned} d(Fx_{2n+1}, Fx_{2m+1}) &\preceq d(Fx_{2n+1}, Fx_{2n+2}) + d(Fx_{2n+2}, Fx_{2n+3}) \\ &\quad + - - - + d(Fx_{2m}, Fx_{2m+1}) \\ &\preceq A[AB]^n d(Fx_0, Fx_1) + [AB]^{n+1} d(Fx_0, Fx_1) \\ &\quad + - - - - - + [AB]^m d(Fx_0, Fx_1) \\ &\preceq \left[A \overset{[}{i} = n]m - 1 \sum (AB)^i + \overset{[}{i} = n+1]m \sum (AB)^i \right] d(Fx_0, Fx_1) \\ &\preceq \left[\frac{A(AB)^n}{1 - AB} + \frac{(AB)^{n+1}}{1 - AB} \right] d(Fx_0, Fx_1) \\ &\preceq (1 + B) \left[\frac{A(AB)^n}{1 - AB} \right] d(Fx_0, Fx_1). \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} d(Fx_{2n}, Fx_{2m+1}) &\preceq (1 + A) \left[\frac{(AB)^n}{1 - AB} \right] d(Fx_0, Fx_1), \\ d(Fx_{2n}, Fx_{2m}) &\preceq (1 + A) \left[\frac{(AB)^n}{1 - AB} \right] d(Fx_0, Fx_1), \end{aligned}$$

and

$$d(Fx_{2n+1}, Fx_{2m}) \preceq (1 + B) \left[\frac{A(AB)^n}{1 - AB} \right] d(Fx_0, Fx_1).$$

Hence, for $0 < n < m$

$$d(Fx_n, Fx_m) \preceq \left[\frac{2(AB)^p}{1 - AB} \right] d(Fx_0, Fx_1),$$

where p is the integer part of $\frac{n}{2}$. Let $\theta \ll c$ be given. Choose a symmetric neighborhood V of θ such that $c + V \subseteq int\, P$. Since

$$lt_{p \to \infty} \left[\frac{2(AB)^p}{1 - AB} \right] d(Fx_0, Fx_1) = \theta,$$

there exists a natural number N_1such that

$$\left[\frac{2(AB)^p}{1-AB}\right] d(Fx_0, Fx_1) \in V,$$

for all $p \geq N_1$ and so

$$\left[\frac{2(AB)^p}{1-AB}\right] d(Fx_0, Fx_1) \ll c, \quad for\ all \quad p \geq N_1.$$

Consequently, for all $n, m \in \mathbb{N}$ with $2N_1 < n < m$, we have $d(Fx_n, Fx_m) \ll c$. Therefore, $\{Fx_n\}_{n\geq 1}$ is a Cauchy sequence. Since $F(X)$ is complete, there exist $u \in X$, $\vartheta \in F(X)$ such that $Fx_n \to \vartheta = Fu$ (this holds also if $S(X) \cup T(X)$ is complete with $\vartheta \in S(X) \cup T(X)$). Choose a natural number N_2 such that for all $n \geq N_2, d(Fx_{n+1}, Fu) \ll \frac{c}{2M}$, where $M = max\left\{\frac{1+a_2+a_4}{1-a_3-a_4}, \frac{a_1+a_3+a_4}{1-a_3-a_4}\right\}$. Then for all $n \geq N_2$,

$$\begin{aligned}
d(Fu, Tu) &\preceq d(Fu, Fx_{2n+2}) + d(Fx_{2n+2}, Tu) \\
&\preceq d(Fu, Fx_{2n+2}) + d(Sx_{2n+1}, Tu) \\
&\preceq d(Fu, Fx_{2n+2}) + a_1 d(Fx_{2n+1}, Fu) + a_2 d(Sx_{2n+1}, Fu) \\
&\quad + a_3 d(Tu, Fx_{2n+1}) + a_4[d(Sx_{2n+1}, Fx_{2n+1}) + d(Tu, Fu)] \\
&\preceq d(Fu, Fx_{2n+2}) + a_1 d(Fx_{2n+1}, Fu) + a_2 d(Fx_{2n+2}, Fu) \\
&\quad + a_3 d(Tu, Fx_{2n+1}) + a_4[d(Fx_{2n+2}, Fx_{2n+1}) + d(Tu, Fu)] \\
&\preceq d(Fu, Fx_{2n+2}) + a_1 d(Fu, Fx_{2n+1}) + a_2 d(Fu, Fx_{2n+2}) \\
&\quad + a_3[d(Fu, Tu) + d(Fu, Fx_{2n+1})] \\
&\quad + a_4[d(Fu, Fx_{2n+2}) + d(Fu, Fx_{2n+1}) + d(Fu, Tu)] \\
&\preceq \frac{1+a_2+a_4}{1-a_3-a_4} d(Fu, Fx_{2n+2}) + \frac{a_1+a_3+a_4}{1-a_3-a_4} d(Fx_{2n+1}, Fu) \\
&\quad \ll \frac{c}{2} + \frac{c}{2} = c.
\end{aligned}$$

By a similar argument $Fu = Tu = Su$, which implies that ϑ is a unique point of coincidence of F, S, and T. If $(S.F)$ and (T, F) are weakly compatible, then by Lemma 3, ϑ is a unique common fixed point of $S, T\, and\, F$.

Corollary 6 : Let (X, d) be a complete TVS-valued cone metric space, P be a solid cone, and mappings $S, T : X \to X$ satisfy:

$$d(Sx, Sy) \preceq a_1 d(Tx, Ty) + a_2 d(Sx, Tx) + a_3 d(Sy, Ty) + a_4[d(Sx, Ty) + d(Sy, Tx)]$$

for all $x, y \in X$, where $a_1, a_2, a_3, a_4 \in [0, 1)$ satisfying $a_1 + a_2 + a_3 + 2a_4 < 1$. If $S(X) \subset T(X)$ and one of $S(X)$ or $T(X)$ is a complete subspace of X, then S and T have a unique point of coincidence in X. Moreover, if S and T are weakly compatible, then S and T have a unique common fixed point.

Corollary 7 : Let (X, d) be a complete TVS-valued cone metric space with a cone P having the non-empty interior. If a mappings $S : X \to X$ satisfies:

$$d(Sx, Sy) \preceq a_1 d(x, y) + a_2 d(Sx, x) + a_3 d(Sy, y) + a_4[d(Sx, y) + d(Sy, x)]$$

for all $x, y \in X$, where $a_1, a_2, a_3, a_4 \in [0, 1)$ satisfying $a_1 + a_2 + a_3 + 2a_4 < 1$. If $S(X)$ is a complete subspace of X, then S has a unique fixed point in X.

Now, we give one example to validate Theorem 4:

Example 8 : Let $E = (C_{[0,1]}, R)$

$$P = (\varphi \in E : \varphi \geq 0) \subset E,$$

$X = [0, 1]$ and $d : X \times X \to E$ defined by $d(x, y)(t) = (|x - y|)e^t$, where $e^t \in E$. Then (X, d) is a cone metric space. Consider three mappings $S, T, F : X \to X$ defined by

$$Sx = \frac{x}{8}, Tx = \frac{x}{12}, Fx = \frac{x}{2}.$$

Clearly $S(X) \cup T(X) \subseteq F(X)$ for all $x, y \in X$.

$$\begin{aligned}
d(Sx, Ty)(t) &= (|\frac{x}{8} - \frac{y}{12}|)e^t = \frac{1}{8}(|x - \frac{2y}{3}|)e^t \\
d(Fx, Fy)(t) &= (|\frac{x}{2} - \frac{y}{2}|)e^t \\
d(Sx, Fx)(t) &= (|\frac{x}{8} - \frac{x}{2}|)e^t = (|\frac{3x}{8}|)e^t \\
d(Ty, Fy)(t) &= (|\frac{y}{12} - \frac{y}{2}|)e^t = (|\frac{5y}{12}|)e^t \\
d(Sx, Fy) + d(Ty, Fx) &= (|\frac{x}{8} - \frac{y}{2}| + |\frac{y}{12} - \frac{x}{2}|)e^t
\end{aligned}$$

and so

$$\begin{aligned}
d(Sx, Ty)(t) &= \frac{1}{8}(|x - \frac{2y}{3}|)e^t \\
&\leq \frac{1}{6}(|\frac{x}{2} - \frac{y}{2}|)e^t + \frac{1}{6}(|\frac{3x}{8}|)e^t + \frac{1}{6}(|\frac{5y}{12}|)e^t \\
&\quad + \frac{1}{6}(|\frac{x}{8} - \frac{y}{2}| + |\frac{y}{12} - \frac{x}{2}|)e^t \\
&= a_1 d(Fx, Fy)(t) + a_2 d(Sx, Fx)(t) + a_3 d(Ty, Fy)(t) \\
&\quad + a_4[d(Sx, Fy)(t) + d(Ty, Fx)(t)].
\end{aligned}$$

Thus all the conditions of Theorem 4 are satisfies with $a_1 + a_2 + a_3 + 2a_4 = \frac{5}{6} < 1$. Note that 0 is the unique common fixed point of the mapping S, T and F.

References

[1] Beg I., Azam A. and Arshad M., Common fixed points for maps on topological vector space valued cone metric spaces, Int. J. Math. Math.Sci. 2009 (2009), Article ID 560264, 8 pages.

[2] Azam A., Beg I. and Arshad M., Fixed point in topological vector space valued cone metric spaces, Fixed Point Theory Appl. 2010 (2010), Article ID 604084, 9 pages.

[3] Abbas M. and Jungck G., Common fixed point results for non commuting mappings without continuity in cone metric spaces, J. Math. Anal. Appl. 341 (2008), 416-420.

[4] Arshad M., Azam A. and Vetro P., Some common fixed point results in cone metric spaces, Fixed Point Theory Appl., 2009(2009), Article ID 493965, 11 pages.

[5] Djordjevic M., Djoric D., Kadelburg Z., Radenovic S. and Spasic D., Fixed point results under c-distance in tvs-cone metric spaces, Fixed Point Theory Appl., 2011:29, doi:10.1186/1687- 1812-2011-29(2011).

[6] Abbas M., Cho Y. J. and Nazir T., Common fixed point theorems for four mappings in tvs-valued cone metric spaces, J. of Mathematical Inequalities, 5(2)(2011),287-299.

[7] Kadelburg Z., Radenovic S., Rakocevic V., Topological vector spaces valued cone metric spaces and fixed point theorems, Fixed Point Theory Appl., 2010, Article ID 170253, (2010), 18 pages.

[8] Kadelburg Z. and Radenovic S., Coupled fixed point results under TVS-cone metric and W-cone distance, Adv. Fixed Point Theory, 2(1)(2012), 29-46.

[9] Huang L. G. and Zhang X., Cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl. 332 (2007), 1468-1476.

[10] Rezapour S. and Hamlbarani R., Some notes on the paper cone metric spaces and fixed point theorem of contractive mappings, J. Math. Anal. Appl., 345 (2008), 719-724.

[11] Azam A. and Beg I., Kannan type mapping in TVS-valued cone metric spaces and their application to Urysohn integral equations, Sarajevo Journal of Mathematics, 9 (22)(2013), 243-255.

[12] Radenovic S., Common fixed points under contractive conditions in cone metric spaces, Comput. Math Appl., (2009) doi:10.1016/j.camwa.2009.07.035.

[13] Radenovic S. and Rhoades B. E., Fixed point theorem for two non-self mappings in cone metric spaces, comput. Math. Appl. 57 (2009), 1701-1707.

[14] Jankovic S., Kadelburg Z. and Radenovic S., On cone metric spaces, a survey, Non-linear Anal., 74 (2011), 2591-2601.

[15] Vetro P., Common fixed points in cone metric spaces, Rendiconti del Circolo Matematico di Palermo, 56(3) (2007), 464-468.