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A PSEUDOCONTRACTIVE MAPPING AND K-SET CONTRACTION WITH FIXED POINTS IN REAL BANACH SPACE

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Abstract

The purpose of this paper is to study the existence and uniqueness of fixed point for a class of nonlinear mapping defined on real Banach space. Which among others, contains a class of separate of contractive mappings as well as an important class of k-set contraction and of pseudocontraction falls into this type of nonlinear mappings. we study fixed point theorems for k-set contractions, $k < 1$ and for pseudo-contractive mappings which are k-set-contractions for some $k > 0$. We also obtain fixed point theorem for certain k-set contractive mappings, $k < 1$, defined on closure of bounded open set that contains the origin in its closure and that satisfy suitable boundary condition. We obtained a fixed point for pseudocontractive mappings which are k-set contractive for some $k > 0$. In the last section we use the result obtained in previous to prove the fixed point theorem for pseudocontractive mapping defined on the subset of real Banach space. Then we will obtained fixed point result of pseudocontractive mapping defined on all of real Banach space.

Key Words : *Pseudo contractive mapping, Nonexpansive mapping, k-set contraction, Banach space, Fixed point.*

1. Introduction

Pseudocontractive mapping and fixed point theory plays a important role to solve the nonlinear equations. This theory has two main branches: On the one hand we may consider the results that are obtained by the topological properties and on other hand those results which may be deducted from metric assumption.

Regarding the topology branch the two main theorems are Brouwer's Theorem and its infinite dimensional version, Schauder's fixed point theorem. In both the theorem compactness plays the essential role. In 1955 Darbo [1] extends Schauder's theorem to the setting of non compact operator, introducing the notation of k-set contraction with $0 \leq k < 1$.

Concerning the metric branch the most important metric fixed point result is the Banach contraction principal [3] where kirk gives an overview of the sharpening of this result.

Although historically the two branches of fixed point theory have had a separated development in 1958, Kransoselskii [4] established that the sum of two operator has a fixed point in a non empty convex subset of a Banach space. This result combine both Banach contraction principle and Schauder's theorem and thus it is blend of two branches. Nevertheless, it is not hard to see that Kransoseleskii theorem is a particular case of Darbo's theorem. Namely, it appears that sum of two nonlinear mapping is a is a k-contraction with respect to the kuratowski measure of noncompactness. In 1967 Sadovskii [5] gave more general fixed point result than Darbofs theorem using the concept of condensing map. In this framework it is well known that for the limit case i.e. mapping which are k-set contraction for some measure of noncompactness, it is not possible to obtain a similar fixed point result like the above cases. Indeed it was proved in [6] that given an infinite dimensional Banach space there exist a fixed point free k-set contraction self mapping of the unit ball. Therefore to develop a theory of fixed point of k-set contraction similar to the same for nonexpansive mapping does not work. Nevertheless, therefore a degree theory of semiclosed k-contraction mapping.

In this paper we study existence of fixed point for a k-contraction and pseudocontractive mapping, proved a fixed point theorem for such a mappings defined on subset of Banach space and finally we obtain a fixed point result for pseudocontractive mapping defined on all of Banach space.

2. Preliminaries

In the sequel we shall make use of the following notation, definitions lemmas and theorems.

Notation 2.1 : Throughout this paper we denote Banach space by B, D as open subset of B, T as pseudocontractive mapping and we denote by $\overline{D}$ and ∂D the closure and boundary of D respectively.

Definition 2.1 : Let B be a real Banach space and D be non empty subset of B then a mapping $T : D \to B$ is said to be pseudo contractive if for $r > 0$ and $x, y \in D$

$$\|x - y\| \leq \|(1 + r)(x - y) - r(T(x) - T(y))\|.$$

Definition. 2.2 : A normed space B is called a Banach space if it is complete, i.e., if every Cauchy sequence is convergent.

Defination. 2.3 : Let B be a Complete Metric Space. Then a map $T : B \to B$ is called a contraction mapping on B if there exists $k \in (01)$ such that $d(T(x), T(y)) \leq kd(xy)$ for all x and y in B.

Definition. 2.4 : Let B be a Banach space, D a subset of B. Then a mapping $T : D \to B$ is said to be non expansive if for all $x, y \in D$

$$\|T(x) - T(x)\| \leq \|x - y\|.$$

Definition. 2.5 : If E is a bounded subset of B then measure of noncompactness of E is defined by,

$$\gamma(E) = \{d > 0 : E \text{ can be covered by finite number of sets each of diameter} \leq d\}.$$

Definition. 2.6 : If D be a subset of B, $V : D \to B$ is continuous mapping and $k > 0$ then we say that V is k-set contraction if for each bounded sub set A of we have $\gamma(V(A)) = k\gamma(A)$ and when $\gamma(A) \neq 0$, $\gamma(V(A)) < \gamma(A)$ then V is said to be condensing.

Theorem 2.1 : Let D be an open bounded subset of B and suppose that $0 \in D$. Let $V : \overline{D} \to B$ be a k-set contraction, $k < 1$ such that:

(1) $V(x) = \lambda x,\ x \in \partial D,\ x \neq 0 \Rightarrow \lambda \leq 1$

(2) $\exists \beta \in (0, 1]$ such that βV has a fixed point in $\overline{D}$.

(3) $I - tV$ is one to one for all $t \in [\beta, 1]$.

Then V has a fixed point in $\overline{D}$.

Proof : Suppose that $V(0) \neq 0$ and V has no fixed point in ∂D. Therefore by (2) βV has a fixed point in D. Also we suppose that $\beta \in (0,1)$. Since V is k-contractive and $k < 1$ therefore if $t \in (0,1)$ then tV is also k- contractive. Now by (3) $I - tV$ is one to one for all $t \in [\beta, 1]$ therefore by applying Nussabaum's Invariant Domain theorem [9] $Dt = (I - tV)D$ is open for all $t \in [\beta, 1]$. Also tV is k-contractive and $k < 1$ therefore if $t \in [\beta, 1]$ then

$$\overline{D}_t = (I - tV)\overline{D} \ \text{ and } \ \partial\overline{D} = (I - tV)D.$$

Let

$$U = \{t \in [\beta, 1] : 0 \in Dt\}.$$

Then we observe that $\beta \in U$ and if $t \in [\beta, 1]$ then either $0 \in D_t$ or $0 \notin \overline{D}_t$.
Let $\alpha = Sup\ U$ then $\exists$ a sequence $\{\alpha_n\}$ of element of U such that $\alpha_n \to \alpha$. For each $\exists$ a sequence $x_n \in D$ such that $x_n - \alpha_n V(x_n) = 0$i.e. $x_n = \alpha_n V(x_n)$. We consider $\{x)_n\}_{n=1}^{\infty}$. Let

$$d = \gamma(\{x_n\}_{n=1}^{\infty}), \quad d' = \gamma\{\alpha_n V(x_n)\}_{n=1}^{\infty}.$$

Then

$$d = \gamma(\{\alpha_n V(x_n)\}_{n=1}^{\infty}) \ \text{ and } \ d' \leq kd.$$

Since $\overline{D}$ is bounded therefore $V(\overline{D})$ is also bounded and $\exists M > 0$ such that

$$\|V(x)\| \leq M \ \text{ for all } \ x \in D.$$

Since sequence $\{\alpha_n\}$ is convergent consequently it is a cauchy sequence therefore for given $\epsilon > 0$ $\exists$ a positive integer M such that $\|\alpha_n - \alpha_m\| \frac{e}{2M}$ $\forall$ $m, n \geq N$.
We see that

$$\gamma(\{\alpha_n V(x_n)\}_{n=1}^{\infty}) = \gamma(\{\alpha_n V(x_n)\}_{n=N}^{\infty} = d$$

$$\gamma(\{V(x_n)\}_{n=1}^{\infty}) = \gamma(\{V(x_n)\}_{n=N}^{\infty} = d'.$$

Also by definition of measure of noncompactness $\exists$ a finite number of sets say $E_1, E_2, E_3, \cdots E_r$ of diameter less than or equal to $d' + \frac{\epsilon}{2}$ that covers $\{V(x_n)\}_{n=N}^{\infty}$ then if $i \in \{1, 2, 3, \cdots, r\}$

and $V(x_n), V(x_m) \in E_i$ we have

$$\begin{aligned}\|x_n - x_m\| &= \|\alpha_n V(x_n) - \alpha_m V(x_m)\| \\ &= \|\alpha_n V(x_n) - \alpha_n V(x_m) + \alpha_n V(x_m) - \alpha_m V(x_m)\| \\ &= \|\alpha_n (V(x_n) - V(x_m)) + (\alpha_n - \alpha_m) V(x_m)\| \\ &\leq |\alpha_n| \|V(x_n) - V(x_m)\| + |\alpha_n - \alpha_m| \|V(x_m)\| \\ &\leq \alpha \left(d' + \frac{\epsilon}{2}\right) + \frac{\epsilon}{2M} M \\ &\leq \alpha d' + \epsilon.\end{aligned}$$

Therefore by Definition 2.6

$$\gamma(\{V(x_n)\}_{n=1}^{\infty}) \leq \alpha d' + \epsilon.$$

Since ϵ is arbitrary therefore we must have:

$$d \leq \alpha \ d' \leq d' \leq kd$$

which is possible only if $d = 0$. Therefore the closure of $\{x_n\}_{n=1}^{\infty}$ is compact, consequently this sequence have convergent subsequence, say $\{x_{n_i}\}_{n=1}^{\infty}$. Let x_0 be the limit of this subsequence where $x_0 \in \partial \overline{D}$. Since the sequence $\{\alpha_n V\}_{n=1}^{\infty}$ converges uniformly to αV, we must have

$$\begin{aligned}\lim_{i \to \infty} \alpha_{n_i} V(x_{n_i}) &= \alpha V(x) \\ &= x_0.\end{aligned}$$

Since $\beta \leq \alpha \leq 1$, we must have $x_0 \in D$ therefore $\alpha \in V$ and we observe that if $t \in [0,1]$, then

$$\begin{aligned}\|x_0 - tV(x_0)\| &= \|x_0 - tV(x_0) - x_0 + x_0\| \\ &= \|x_0 - tV(x_0) - x_0 + \alpha V(x_0)\| \\ &= \| - (t - \alpha) V(x_0)\| \\ &= |t - \alpha| \|V(x_0)\| \\ &\leq |t - \alpha| M.\end{aligned}$$

Now we have two cases either $\alpha < 1$ or $\alpha = 1$. Suppose $\alpha < 1$ then $0 \in \overline{D}_t$ for all $t \in (\alpha, 1]$. Choose $t_0 \in (\alpha, 1)$ and a sequence $\{t_n\}_{n=1}^{\infty}$ of elements of (t_0) such that $t_n \to \alpha$.

Since $0 \in \overline{D}_{t_n}$ for all n $\exists$ $y_n \in \partial D_n$ such that $\|y_n\| \leq |t_n - \alpha|M$. Now $\partial D_{t_n} = (I - t_nV)(\partial D)$ so there exist $x_n \in \partial D$ such that $x_n - t_nV(x_n) = y_n$;

$$\|x_n - t_nV(x_n)\| \leq |t_n - \alpha|M.$$

Let

$$d = (\{x_n\}_{n=1}^{\infty}), \quad d' = \gamma(\{V(x_n)\}_{n=1}^{\infty}).$$

It is easy to see that $d' \leq kd$ and as before $\exists$ a convergent subsequence $\{x_{n_i}\}_{n=1}^{\infty}$ of the sequence $\{x_n\}_{n=1}^{\infty}$ and let x be limit of subsequence then $x \in \partial D$ and as before we can say that

$$x - \alpha V(x) = 0.$$

This is contradiction therefore $\alpha < 1$ is not possible.

Hence we must have $\alpha = 1$ and this completes the proof of the theorem.

Corollary 2.2 : Let D be a open subset of B with $0 \in \overline{D}$ and let $V : \overline{D} \to B$ be the contraction mapping such that :

(1) $V(x) = \lambda x,\ x \in \partial D, x \neq 0 \Rightarrow \lambda \leq 1.$

(2) $\exists\ \beta \in (0, 1]$ such that βV has a fixed point in $\overline{D}$.

Then V has a fixed point in $\overline{D}$.

Proof : Consider the set $E = \{x \in \overline{D} : V(x) = \lambda x$ for some $\lambda > 1\}$. Then as in [5] set E is bounded, so there is no loss of generality in assuming that D is bounded and by this the hypothesis (3) of theorem 2.1 is satisfied. Hence V has a fixed point in $\overline{D}$.

Corollary 2.3 : Let S be a closed convex subset of B and W be a close subspace generated by S. Let $D \subseteq S$ be open relative to S, bounded with $0 \in D$ and suppose that $V : \overline{D} \to S$ is k-set contraction, $k < 1$ such that:

(1) $V(x) = \lambda x, x \in \partial_w D, x \neq 0 \Rightarrow \lambda \leq 1$ where $\partial_w D$ is boundary of D relative to W.

(2) $I - tV$ is one to one for all $t \in (0, 1]$.

Then V has a fixed point in D.

Proof : Let $\overline{D}$ be bounded then $V(\overline{D})$ is also bounded therefore $\exists M > 0$ such that $\|V(x)\| \leq M \forall x \in \overline{D}$. Now $x \in D \subseteq \overline{D}$ which implies that x is interior point of $\overline{D}$ relative to H therefore $\exists r > 0$ and $r < M$ such that $\emptyset \neq B_r[0] \cap D = B_r[0] \cap H$ where

$B_r[0]$ is closed ball of radius r centred at origin. Now $r < M \to \frac{r}{M} < 1$ therefore $\left(\frac{r}{m}\right) V$ maps $B_r[0] \cap H$ on to itself and $\left(\frac{r}{m}\right) V$ is ρ-set contraction, $\rho < 1$ and $B - r[0] \cap H$ is closed, bounded and convex subset of the Banach space W so that it has a fixed point by Darbo [2].

Our main results are as follows:

Theorem 3.1 : Let D be open subset of B, $0 \in \overline{D}$ and $V : \overline{D} \to B$be a pseudocontractive mapping which is also k-set contractive for $k > 0$ such that:

(1) $V(x) = \lambda x, x \in \partial D, x \neq 0 \Rightarrow \lambda \leq 1$

(2) $\exists\ \beta \in \left(0, \frac{1}{k}\right)$ such that βV has a fixed point in D.

(3) $(I - V)(\overline{D})$ is closed.

Then V has a fixed point in $\overline{D}$.

Proof : Let $r > 0$ be so small that $rk < 1$ and $r \in [\beta, 1]$. Again let $S = (1 - r)I$ and $T = (I - rV)$. Then for $x, y \in \overline{D}$ we have

$$\begin{aligned} \|S(x) - S(y)\| &= \|(1 - r)Ix - (1 - r)Iy\| \\ &\leq \|(i - rV)x - (I - rV)y\| \\ &= \|Tx - Ty\| \end{aligned}$$

which shows that T is one to one and if $H = ST^{-1}$ then

$$\begin{aligned} \|H(x) - H(y)\| &= \|ST^{-1}(x) - ST^{-1}(y)\| \\ &\leq \|TT^{-1}(x) - TT^{-1}(x)\| \\ &= \|Ix - Iy\| \\ &= \|x - y\|. \end{aligned}$$

Thus

$$\|H(x) - H(y)\| \leq \|x - y\|.$$

Hence H is nonexpansive in $T(\overline{D})$. Since rV is ρ-set contraction, $\rho < 1$ and $I - rV$ is one to one therefore by Nussbaum's Invariance of Domain Theorem [9] $T[D]$ is open. Further $T(\overline{D})$ is closed hence we get $\partial T(D) = T(\partial D)$.

Let $W = T(D)$ then W is open and $\overline{W} = T(\overline{D})$ and $\partial W = T(\partial D)$. Also rV is ρ-set contraction, $\rho < 1$ and $I - t(rV)$ one to one therefore by therefore 2.1, $0 \in W$. It can be seen that H satisfies all the condition of corollary 2.3 which implies that H has a fixed point in $\overline{W}$, say y then $H(y) = y$. Let $x \in D$ be such that $Tx = y$. Now

$$\begin{aligned}
&H(y) = y \\
&\Rightarrow ST^{-1}T(x) = T(x) \\
&\Rightarrow S(x) = T(x) \\
&\Rightarrow (1-r)Ix = (I - rV)x \\
&\Rightarrow V(x) = x \\
&\Rightarrow V \quad \text{has a fixed in } D
\end{aligned}$$

Theorem 3.2 : Let $V : B \to B$ be a k-set contraction for $k > 0$ such that there exist $D \subseteq B$, bounded for $(I - V)\overline{D}$ is closed and suppose that

(1) If $\lambda > 1$ is an eigen value for V then there exist $x \in \overline{D}$. that $V(x) = \lambda x$.

(2) There exist $\delta > 0$ such that $r > 0$, $T_r = (1 + r)I - rV$ has an inverse which is δ-set contraction on its domain. Then V has a fixed point in $\overline{D}$.

Proof : $H = \{t \in [0, \infty) : T_t$ maps B on to $B\}$ then it is clear that $0 \in H$ therefore $H \neq \phi$.

Choose $r_0 < \frac{1}{(k+1)\delta}$ and let $t \in H$ and suppose that $s \geq 0$, $|s - t| < r_0$ then T_t is one to one, consequently T_t is invertible and let R be its inverse which is k-set contraction and if $w \in R$ then:

$$\begin{aligned}
T_s(R_w) &= T_s(R_w) - T_t(R_w) + T_t(R_w) \\
&= (T_s - T_t)R_w + w \\
&= S_s(w) + w
\end{aligned}$$

where $S_s = (T_s - T_t)R$.

then $I + S_sE$ is one to one, since if $x + S_s(x) = y + S_s(y)$ then

$$\begin{aligned}
& x + (T_s - T_t)R(x) = y + (T_s - T_t)R(y) \\
& \Rightarrow x + T_s(Rx) - T_t(Rx) = y + T_s(Ry) - T_t(Ry) \\
& \Rightarrow x + T_s(Rx) - x = y + T_s(Ry) - y \;\because\; R \text{ is inverse of } T_t \\
& \Rightarrow T_s(Rx) = T_s(Ry) \\
& \Rightarrow Rx = Ry \quad \because \; T_s \text{ is one to one} \\
& \Rightarrow x = y \quad \because \; R \text{ is one to one.}
\end{aligned}$$

Also for any bounded sub set a of Banach space B we have

$$\begin{aligned}
\gamma(S_s(A)) &= \gamma[(T_s - T_t)(R(A)) \\
&= \gamma[(1+s)I - sV - (1+t)I + tV](R(A)) \\
&= \gamma[(s-t)(I-V)(R(A))] \\
&= |s-t|\gamma[(I-V)R(A)] \\
&\leq |s-t|(1+k)\delta_\gamma(A).
\end{aligned}$$

Now

$$\begin{aligned}
& |s-t| < r_0 \text{ and } r_0 < \frac{1}{(k+1)\delta} \\
& \Rightarrow |s-t| < \frac{1}{(k+1)\delta} \\
& \Rightarrow |s-t|(k+1)\delta < 1 \\
& \Rightarrow S_s \text{ is a } \rho\text{-set contraction}, \quad \rho < 1 \\
& \Rightarrow (I + S_s)X \text{ is closed} \\
& \Rightarrow (I + S_s)X = X.
\end{aligned}$$

Now $T_s(x) = y$ if and only if $\omega + S_s(\omega)$ where $\omega = Ru$. Therefore T_s maps B on to B. We have shown that if $t \in H$ and if $|t - s| < r_0, s \geq 0$ then $s \in H$. It follows that

$H = [0, \infty]$ so we can say that if $r > 0$ then there exist $x_r \in B$ such that

$$\begin{aligned}
&T_r x_r = 0\\
&\Rightarrow [(1+r)I - rV]x_r = 0\\
&\Rightarrow r(I - V)x_r = x_r\\
&\Rightarrow (I - V)x_r = \frac{x_r}{r} \to \infty \quad \text{as} \quad r \to \infty\\
&\Rightarrow V(x_r) = x_r\\
&V \quad \text{has fixed point in} \quad \overline{D}.
\end{aligned}$$

4. Conclusion

Finding fixed points of nonlinear mappings especially, nonexpansive mappings has received vast investigations due to its extensive applications in a variety of applied areas of inverse problem, partial differential equations, image recovery and signal processing. It is well known that k-set contraction and pseudocontractive mappings have more powerful applications than nonexpansive mappings in solving different problems. In this paper, we devote to construct the methods for computing the fixed points of k-set contraction and pseudocontractive mappings.

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