International J. of Math. Sci. & Engg. Appls. (IJMSEA)
ISSN 0973-9424, Vol. 9 No. III (September, 2015), pp. 179-185

ON CERTAIN RESULTS INVOLVING CONTINUED FRACTIONS

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Abstract

In this paper, we have established continued fraction representations for certain special type of series.

1. Introduction Notations and Definitions

We shall use the following usual notations and definitions. Let,

$$(a,q)_n = \begin{cases} 1 & \text{if } n = 0; \\ (1-a)(1-aq)(1-aq^2)\ldots,(1-aq^{n-1}) & \text{if } n \geq 1. \end{cases}$$

We define the basic hypergeometric series as,

Key Words and Phrases : *Basic hypergeometric series, Continued fraction, Identities and false theta functions.*

AMS Subject Classification : Primary 33D15, 11P83.

$$
{}_r\Phi_s\left[\begin{array}{l} a_1, a_2, a_3, ..., a_r; q; z \\ b_1, b_2, b_3, ..., b_s \end{array}\right]
= \sum_{n=0}^{\infty} \frac{[a_1;q]_n[a_2;q]_n...[a_r;q]_n z^n}{[b_1;q]_n[b_2;q]_n...[b_s;q]_n(q;q)_n},
$$

where $|q| < 1$ and $|z| < 1$.

An expression of the form

$$
\frac{a_1}{b_1+}\frac{a_2}{b_2+}\frac{a_3}{b_3+}....\frac{a_n}{b_n}
$$

is said to be a terminating continued fraction and as $n \to \infty$, it is said to be an infinite continued fraction.

Following known results are needed in our analysis.

$$
\frac{{}_2\Phi_1\left[\begin{array}{l} \alpha, \beta; q; z \\ \gamma \end{array}\right]}{{}_2\Phi_1\left[\begin{array}{l} \alpha, \beta q; q; z \\ \gamma \end{array}\right]}
= 1 - \frac{z\beta(1-\alpha)}{(1-\gamma/\beta q)+} \frac{\frac{\gamma}{\beta q}(1-\beta q)\left(1-\frac{\alpha\beta z q}{\gamma}\right)}{1-}
$$

$$
\frac{z\beta q(1-\alpha q)}{(1-\gamma/\beta q)+} \frac{\frac{\gamma}{\beta q}(1-\beta q^2)\left(1-\frac{\alpha\beta z q^2}{\gamma}\right)}{1-} \frac{z\beta q^2(1-\alpha q^2)}{(1-\gamma/\beta q)+}... \tag{1.1}
$$

[S. N. Singh 3]

Rogers-Fine identity is,

$$
\sum_{n=0}^{\infty} \frac{(a;q)_n z^n}{(b;q)_n} = \sum_{n=0}^{\infty} \frac{(a;q)_n(azq/b;q)_n(bz)^n(1-azq^{2n})q^{n^2-n}}{(b;q)_n(z;q)_{n+1}}. \tag{1.2}
$$

[Andrews and Berndt 1; (9.1.1)]

On page 13 of Ramanujan's Lost Notebook [2] following five identities are given,

$$
\sum_{n=0}^{\infty} \frac{(-)^n q^{n(n+1)}(q;q^2)_n}{(-q;q)_{2n+1}} = \sum_{n=0}^{\infty} (-)^n q^{n(n+1)/2}, \tag{1.3}
$$

$$
\sum_{n=0}^{\infty} \frac{(q;q^2)_n^2 q^n}{(-q;q)_{2n+1}} = \sum_{n=0}^{\infty} (-)^n q^{n(n+1)}, \tag{1.4}
$$

$$\sum_{n=0}^{\infty} \frac{(q;q^2)_n q^n}{(-q;q)_{2n+1}} = \sum_{n=0}^{\infty} (-)^n q^{3n(n+1)/2}, \tag{1.5}$$

$$\sum_{n=0}^{\infty} \frac{(q;-q)_{2n}}{(-q;q)_{2n+1}} = \sum_{n=0}^{\infty} (-)^n q^{2n(n+1)}, \tag{1.6}$$

$$\sum_{n=0}^{\infty} \frac{(q;-q)_n(-q^2;q^2)_n q^n}{(-q;q)_{2n+1}} = \sum_{n=0}^{\infty} (-)^n q^{3n(n+1)}. \tag{1.7}$$

On the right hand side of the identities (1.3) - (1.7). There are false theta functions

$$\sum_{n=0}^{\infty} (-)^n q^{n(n+1)/2}$$

with q replaced by q, q^2, q^3, q^4 and q^6 respectively.

Taking $\beta = 1$ and replacing α by a and γ by b in (1.1) and also using (1.2) we get,

$$\sum_{n=0}^{\infty} \frac{(a;q)_n z^n}{(b;q)_n} = \sum_{n=0}^{\infty} \frac{(a;q)_n (azq/b;q)_n (bz)^n (1-azq^{2n}) q^{n^2-n}}{(b;q)_n (z;q)_{n+1}}$$

$$= \frac{1}{1-} \frac{z(1-a)}{(1-b/q)+} \frac{\frac{b}{q}(1-q)\left(1-\frac{azq}{b}\right)}{1-}$$

$$\frac{zq(1-aq)}{(1-b/q)+} \frac{\frac{b}{q}(1-q^2)\left(1-\frac{azq^2}{b}\right)}{1-} \frac{zq^2(1-aq^2)}{(1-b/q)+} \cdots \tag{1.8}$$

2. Main Results

Putting z/a for z, taking $a \to \infty$ and b=0 in (1.8) we get,

$$\sum_{n=0}^{\infty} (-)^n q^{n(n-1)/2} z^n = \sum_{n=0}^{\infty} (1-zq^{2n}) z^{2n} q^{2n^2-n}$$

$$= \frac{1}{1+} \frac{z}{1-} \frac{(1-q)z}{1+} \frac{zq^2}{1-} \frac{(1-q^2)zq}{1+} \frac{zq^4}{1-} \cdots \quad . \tag{2.1}$$

Putting z=q in (2.1) we have

$$\sum_{n=0}^{\infty} (-)^n q^{n(n+1)/2} = \sum_{n=0}^{\infty} (1-q^{2n+1}) q^{2n^2+n}$$

$$= \frac{1}{1+} \frac{q}{1-} \frac{(1-q)q}{1+} \frac{q^3}{1-} \frac{(1-q^2)q^2}{1+} \frac{q^5}{1-} \cdots \quad . \tag{2.2}$$

Making use of (1.3) we have

$$\sum_{n=0}^{\infty}(-)^n q^{n(n+1)/2} = \sum_{n=0}^{\infty}\frac{(-)^n q^{n(n+1)}(q;q^2)_n}{(-q;q)_{2n+1}}$$

$$= \frac{1}{1+}\frac{q}{1-}\frac{(1-q)q}{1+}\frac{q^3}{1-}\frac{(1-q^2)q^2}{1+}\frac{q^5}{1-}\ldots \quad . \tag{2.3}$$

From (2.3) and (1.4) it is easy to have,

$$\sum_{n=0}^{\infty}(-)^n q^{n(n+1)} = \sum_{n=0}^{\infty}\frac{(q;q^2)_n^2 q^n}{(-q;q)_{2n+1}}$$

$$= \frac{1}{1+}\frac{q^2}{1-}\frac{(1-q^2)q^2}{1+}\frac{q^6}{1-}\frac{(1-q^4)q^4}{1+}\frac{q^{10}}{1-}\ldots \quad . \tag{2.4}$$

Similar results can be established for (1.5), (1.6) and (1.7).

For z=-q in (2.1) we find,

$$\sum_{n=0}^{\infty} q^{n(n+1)/2} = \Psi(q) = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \sum_{n=0}^{\infty}(1+q^{2n+1})q^{2n^2+n}$$

$$= \frac{1}{1-}\frac{q}{1+}\frac{(1-q)q}{1-}\frac{q^3}{1+}\frac{(1-q^2)q^2}{1-}\frac{q^5}{1+}\ldots \quad , \tag{2.5}$$

where $\Psi(q)$ is Ramanujan's Theta function defined in [Andrews and Berndt 1; (1.1.7) p. 11].

Taking b=q in (1.8) we have

$$\sum_{n=0}^{\infty}\frac{(a;q)_n z^n}{(q;q)_n} = \frac{(az;q)_\infty}{(z;q)_\infty} = \sum_{n=0}^{\infty}\frac{(a;q)_n(az;q)_n z^n q^{n^2}(1-azq^{2n})}{(q;q)_n(z;q)_{n+1}}$$

$$= \frac{1}{1-}\frac{z(1-a)}{0+}\frac{(1-q)(1-az)}{1-}\frac{zq(1-aq)}{0+}\frac{(1-q^2)(1-azq)}{1-}\ldots \quad . \tag{2.6}$$

For a=0, (2.6) yields

$$\sum_{n=0}^{\infty}\frac{z^n}{(q;q)_n} = \frac{1}{(z;q)_\infty} = \sum_{n=0}^{\infty}\frac{z^n q^{n^2}}{(q;q)_n(z;q)_{n+1}}$$

$$= \frac{1}{1-}\frac{z}{0+}\frac{(1-q)}{1-}\frac{zq}{0+}\frac{(1-q^2)}{1-}\frac{zq^2}{0+}\ldots \quad . \tag{2.7}$$

Taking z=q in (2.7) we get

$$\sum_{n=0}^{\infty}\frac{q^n}{(q;q)_n} = \frac{1}{(q;q)_\infty} = \sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q)_n(q;q)_{n+1}}$$

$$= \sum_{n=0}^{\infty} p(n) q^n$$

$$= \frac{1}{1-}\frac{q}{0+}\frac{(1-q)}{1-}\frac{q^2}{0+}\frac{(1-q^2)}{1-}\frac{q^3}{0+}\ldots \quad , \tag{2.8}$$

where p(n) stands for the number of partitions of n.

Again, replacing q by q^2 and then taking z=q in (2.7) we have

$$\sum_{n=0}^{\infty} \frac{q^n}{(q^2;q^2)_n} = \frac{1}{(q;q^2)_\infty} = \sum_{n=0}^{\infty} \frac{q^{2n^2+n}}{(q^2;q^2)_n (q;q^2)_{n+1}}$$

$$= \sum_{n=0}^{\infty} p_0(n) q^n$$

$$= \frac{1}{1-}\frac{q}{0+}\frac{(1-q^2)}{1-}\frac{q^3}{0+}\frac{(1-q^4)}{1-}\frac{q^5}{0+}\ldots \quad , \tag{2.9}$$

where $p_0(n)$ stands for the number of partitions of n into odd parts only.

Replacing q by q^2 and then taking $z = q^2$ in (2.7) we get,

$$\sum_{n=0}^{\infty} \frac{q^{2n}}{(q^2;q^2)_n} = \frac{1}{(q^2;q^2)_\infty} = \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q^2;q^2)_n (q^2;q^2)_{n+1}}$$

$$= \frac{1}{1-}\frac{q^2}{0+}\frac{(1-q^2)}{1-}\frac{q^4}{0+}\frac{(1-q^4)}{1-}\frac{q^6}{0+}\ldots \quad , \tag{2.10}$$

where $\frac{1}{(q^2;q^2)_\infty}$ is the generating function of the partitions into even parts.

Putting z/a for z and then taking $a \to \infty$ in (2.6) we find,

$$\sum_{n=0}^{\infty} \frac{(-)^n q^{n(n-1)/2} z^n}{(q;q)_n} = (z;q)_\infty = \sum_{n=0}^{\infty} \frac{(-)^n q^{\frac{3}{2}n^2 - \frac{1}{2}n} (z;q)_n z^n (1 - zq^{2n})}{(q;q)_n}$$

$$= \frac{1}{1+}\frac{z}{0+}\frac{(1-q)(1-z)}{1+}\frac{zq^2}{0+}\frac{(1-q^2)(1-zq)}{1+}\ldots \quad . \tag{2.11}$$

Putting z=-q in (2.11) we have

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2} z^n}{(q;q)_n} = (-q;q)_\infty = \sum_{n=0}^{\infty} \frac{q^{\frac{3}{2}n^2 + \frac{n}{2}} (-q;q)_n (1 + q^{2n+1})}{(q;q)_n}$$

$$= \frac{1}{1-}\frac{q}{0+}\frac{1-q^2}{1-}\frac{q^3}{0+}\frac{1-q^4}{1-}\frac{q^5}{0+}\frac{1-q^6}{1-}\ldots \quad , \tag{2.12}$$

where $(-q;q)_\infty$ is the generating function of the partitions into distinct parts.
For z=q, (2.11) yields

$$\sum_{n=0}^{\infty}\frac{(-)^n q^{n(n+1)/2}}{(q;q)_n}=(q;q)_\infty$$

$$=\frac{1}{1+}\frac{q}{0+}\frac{(1-q)^2}{1+}\frac{q^3}{0+}\frac{(1-q^2)^2}{1+}\frac{q^5}{0+}\ldots \quad . \tag{2.13}$$

Putting b=aq in (1.8) we get

$$(1-a)\sum_{n=0}^{\infty}\frac{z^n}{1-aq^n}=\frac{1}{1-}\frac{z(1-a)}{(1-a)+}\frac{a(1-q)(1-z)}{1-}\frac{zq(1-aq)}{(1-a)+}$$

$$\frac{a(1-q^2)(1-zq)}{1-}\frac{zq^2(1-aq^2)}{(1-a)+}\ldots \quad . \tag{2.14}$$

Replacing q by $q^5, a=z=q^j$ in (2.14) we have

$$(1-q^j)\sum_{n=0}^{\infty}\frac{q^{jn}}{1-q^{5n+j}}=\frac{1}{1-}\frac{q^j(1-q^j)}{(1-q^j)+}\frac{q^j(1-q^5)(1-q^j)}{1-}\frac{q^{5+j}(1-q^{5+j})}{(1-q^j)+}$$

$$\frac{q^j(1-q^{10})(1-q^{2j})}{1-}\frac{q^{10+j}(1-q^{10+j})}{(1-q^j)+}\ldots \quad . \tag{2.15}$$

Comparing (2.15) with [Andrews and Berndt 1; lemma (4.4.1) p. 117] we get,

$$(1-q^j)\sum_{n=0}^{\infty}q^{5n^2+2nj}\left(\frac{1+q^{5n+j}}{1-q^{5n+j}}\right)$$

$$=\frac{1}{1-}\frac{q^j(1-q^j)}{(1-q^j)+}\frac{q^j(1-q^5)(1-q^j)}{1-}\frac{q^{5+j}(1-q^{5+j})}{(1-q^j)+}$$

$$\frac{q^j(1-q^{10})(1-q^{2j})}{1-}\frac{q^{10+j}(1-q^{10+j})}{(1-q^j)+}\ldots \quad , \tag{2.16}$$

Replacing q by q^5 and then putting $a=q^j$ and $z=q^i$ in (2.14) we get

$$(1-q^j)\sum_{n=0}^{\infty}\frac{q^{in}}{1-q^{5n+j}}=\frac{1}{1-}\frac{q^i(1-q^j)}{(1-q^j)+}\frac{q^i(1-q^5)(1-q^i)}{1-}$$

$$\frac{q^{5+i}(1-q^{5+j})}{(1-q^j)+}\frac{q^i(1-q^{10})(1-q^{5+i})}{1-}\frac{q^{10+i}(1-q^{10+j})}{(1-q^j)+}\ldots \quad . \tag{2.17}$$

Similar other results can also be established.

Acknowledgement

The first author is thankful to The Department of Science and Technology, Government of India, New Delhi, for support under a major research project No. SR/ S4/ MS : 735 / 2011 dated 7^{th} May 2013, entitled "A study of transformation theory of q-series, modular equations, continued fractions and Ramanujan's mock-theta functions," under which this work has been done.

References

[1] Andrews G. E. and Berndt B. C., Ramanujan's Lost Notebook, Part I, Springer (2005).

[2] Ramanujan S., The 'Lost' Notebook and other unpublished papers, Narosa, New Delhi, (1988).

[3] Singh S. N., Basic Hypergeometric series and continued fractions, Math. Student, 56(1-4) (1988), 91-96.