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ON NEAR (ω)COMPACTNESS

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Abstract

A generalization of near compactness is defined for (ω)topological spaces and some characterizations of the notion are obtained.

1. Introduction

The notion of (ω)topological spaces is introduced and studied in Bose and Tiwari [1, 2]. The notion of connectedness in an (ω)topological space is studied in Bose and Tiwari [3], while in Tiwari and Bose [7] we study the notions of (ω)compactness and (ω)paracompactness in a product (ω)topological space. It is defined to be a set X equipped with a countable number of topologies $\mathcal{J}_n$ satisfying $\mathcal{J}_n \subset \mathcal{J}_{n+1}, \forall n \in N$ and is denoted by $(X, \{\mathcal{J}_n\})$. Singal and Mathur [5, 6] introduced and studied the notion of

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nearly compact spaces as a covering axiom weaker than compactness but stronger than almost compactness (Singal and Arya [4]. In this paper, we introduce the notion of near compactness on an (ω)topological spaces, we call it near (ω)compactness. Further, we obtain characterizations of near (ω)compactness, analogous to the characterizations of near compactness obtained in Singal and Mathur [5].

2. Preliminaries

The set of natural numbers and the set of real numbers is denoted by N and R respectively. The elements of N are denoted by i, j, k, l, m, n etc. The closure(resp. interior) of a set $A \subset X$ with respect to a topology $\mathcal{J}$ on X is denoted by $(\mathcal{J})clA$(resp. $(\mathcal{J})intA$). Throughout the paper, unless mentioned otherwise, X denotes the (ω)topological space $(X, \{\mathcal{J}_n\})$. For ready reference we give here the following definitions from Bose and Tiwari [1].

Definition 2.1 : A set $G(\subset X) \in \mathcal{J}_n$ for some n is called an (ω)open set. A set F is said to be (ω)closed if $X - F$ is (ω)open.

Definition 2.2 : X is said to be (ω)Hausdorff if for any two distinct points x, y of X, there exists an n such that for some $U, V \in \mathcal{J}_n$, we have $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Definition 2.3 : X is said to be (ω)compact if every (ω)open cover of X has a finite subcover.

We recall some of the notions introduced in Tiwari and Bose [8].

Definition 2.4 : If an (ω)open set G is regularly $(\mathcal{J}_n)$open for some n, then it is said to be a regularly (ω)open set. An (ω)closed set F is said to be regularly (ω)closed if it is regularly $(\mathcal{J}_n)$closed for some n.

Obviously a set A is regularly (ω)closed iff it is the complement of a regularly (ω)open set.

Definition 2.5 : X is said to be almost (ω)regular if for any regularly (ω)closed set F and any point x with $x \notin F$, there exists an n such that for some $U, V \in \mathcal{J}_n$, we have $x \in U$, $F \subset V$ and $U \cap V = \emptyset$.

As shown in Tiwari and Bose [8] we may alternatively define almost (ω)regularity as follows:

An (ω)topological space X is said to be almost (ω)regular if for any point $x \in X$ and any regularly (ω)open set A with $x \in A$, there exists an n such that for some $(\mathcal{J}_n)$open set G, we have $x \in G \subset (\mathcal{J}_n)clG \subset A$.

Definition 2.6 : X is said to be semi$-(\omega)$regular if for any $x \in X$ and any (ω)open set G with $x \in G$, there exists an n such that for some regularly $(\mathcal{J}_n)$open set H, we have $x \in H \subset G$.

Definition 2.7 : X is said to be almost (ω)compact if for each (ω)open cover $\mathcal{U} = \{U_\alpha | \alpha \in A\}$ of X has a finite subcollection $\mathcal{U}_0 = \{U_{\alpha_1}, U_{\alpha_2}, ..., U_{\alpha_m}\}$ with $\cup_{k=1}^{m}((\mathcal{J}_{n_0})clU_{\alpha_k}) = X$, where n_0 is any natural number such that $\mathcal{U}_0 \subset \mathcal{J}_{n_0}$.

3. Nearly (ω)compact Spaces

We now introduce the following notion.

Definition 3.1 : X is said to be nearly (ω)compact if for each (ω)open cover $\mathcal{U} = \{U_\alpha | \alpha \in A\}$ of X has a finite subcollection $\mathcal{U}_0 = \{U_{\alpha_1}, U_{\alpha_2}, ..., U_{\alpha_m}\}$ with $\cup_{k=1}^{m}(\mathcal{J}_{n_{\alpha_k}})int((\mathcal{J}_{n_{\alpha_k}})clU_{\alpha_k}) = X$, where n_{α_k} is any natural number such that $U_{\alpha_k} \in \mathcal{J}_{n_{\alpha_k}}$.

It is easy to see that any nearly (ω)compact space is almost (ω)compact while every (ω)compact space is nearly (ω)compact. However, the converse relations are not true as shown by the following examples.

Example 3.2 : Consider the (ω)topological space $(N, \{\mathcal{J}_n\})$ where the topologies $\{\mathcal{J}_n\}$ are defined by

$$\mathcal{J}_n = \{N\} \cup P\{1, 2, ..., n\}.$$

We see that $(N, \{\mathcal{J}_n\})$ is almost (ω)compact but not nearly (ω)compact.

Example 3.3 : Consider the (ω)topological space $(N, \{\mathcal{J}_n\})$ in which the topologies $\{\mathcal{J}_n\}$ are defined as follows:

$$\mathcal{J}_n = \{\emptyset, N\} \cup \{E \subset \{1, 2, ..., n\} | 1 \in E\}.$$

We see that $(N, \{\mathcal{J}_n\})$ is nearly (ω)compact but not (ω)compact.

The following theorem gives a set of characterizations for near (ω)compactness analogous to those obtained for near compactness by Singal and Mathur [5].

Theorem 3.4 : In an (ω)topological space X, the following statements are equivalent.

(a) X is nearly (ω)compact.

(*b*) Every basic (ω)open cover $\mathcal{U} = \{U_\alpha | \alpha \in A\}$ of X possesses a finite subcollection $\mathcal{U}_0 = \{U_{\alpha_1}, U_{\alpha_2}, ..., U_{\alpha_m}\}$ with $\cup_{k=1}^m (\mathcal{J}_{n_{\alpha_k}}) int((\mathcal{J}_{n_{\alpha_k}}) cl U_{\alpha_k}) = X$, where n_{α_k} is any natural number such that $U_{\alpha_k} \in \mathcal{J}_{n_{\alpha_k}}$.

(*c*) Every regular (ω)open cover of X has a finite subcover.

(*d*) Every regular (ω)closed collection with finite intersection property has a nonempty intersection.

(*e*) Every (ω)closed collection $\mathcal{F} = \{F_\alpha | \alpha \in A\}$ with the property that for any finite subfamily $\{F_{\alpha_1}, F_{\alpha_2}, ..., F_{\alpha_m}\}$ with $\cap_{k=1}^m (\mathcal{J}_{n_{\alpha_k}}) cl((\mathcal{J}_{n_{\alpha_k}}) int F_{\alpha_k}) \neq \emptyset$, has a nonempty intersection.

Proof : $(a) \Rightarrow (b)$: Obvious.
$(b) \Rightarrow (c)$: Let $\mathcal{G} = \{G_\alpha | \alpha \in A\}$ be a regularly (ω)open cover of X with $G_\alpha = (\mathcal{J}_{n_\alpha}) int((\mathcal{J}_{n_\alpha}) cl G_\alpha)$. For each $\alpha \in A$, let G_α be replaced by basic $(\mathcal{J}_{n_\alpha})$open sets which make the $(\mathcal{J}_{n_\alpha})$open set G_α. Then we get a basic (ω)open cover $\mathcal{U}$ of X. By (*b*), we obtain a finite subcollection $\mathcal{U}_0 = \{U_{\alpha_1}, U_{\alpha_2}, ..., U_{\alpha_m}\}$ with $U_{\alpha_k} \in \mathcal{J}_{n_{\alpha_k}}$ if $U_{\alpha_k} \subset G_{\alpha_k}$, having the following property:

$$\cup_{k=1}^m (\mathcal{J}_{n_{\alpha_k}}) int((\mathcal{J}_{n_{\alpha_k}}) cl U_{\alpha_k}) = X.$$

Since $U_{\alpha_k} \subset G_{\alpha_k}$, we get

$$\begin{aligned} \cup_{k=1}^m (\mathcal{J}_{n_{\alpha_k}}) int((\mathcal{J}_{n_{\alpha_k}}) cl G_{\alpha_k}) &= X \\ \Rightarrow \cup_{k=1}^m G_{\alpha_k} &= X. \end{aligned}$$

$(c) \Rightarrow (d)$: Let $\mathcal{F} = \{F_\alpha | \alpha \in A\}$ be a regular (ω)closed collection of subsets of X with

(*i*) $(\mathcal{J}_{n_\alpha}) int((\mathcal{J}_{n_\alpha}) cl(X - F_\alpha)) = X - F_\alpha$,

(*ii*) for any $m \in N, \cap_{k=1}^m F_{\alpha_k} \neq \emptyset$.

If possible, suppose $\cap\{F_\alpha | \alpha \in A\} = \emptyset$. Then $\{X - F_\alpha | \alpha \in A\}$ is a regularly (ω)open cover of X. Therefore by (*c*), $\{X - F_\alpha | \alpha \in A\}$ has a finite subcover $\{X - F_{\alpha_1}, X - F_{\alpha_2}, ..., X - F_{\alpha_m}\}$. So $\cap_{k=1}^m F_{\alpha_k} = \emptyset$. This contradicts our assumption. Thus $\cap\{F_\alpha | \alpha \in A\} \neq \emptyset$.
$(d) \Rightarrow (e)$: Let $\mathcal{F} = \{F_\alpha | \alpha \in A\}$ be an (ω)closed collection of subsets of X having the property mentioned in (*e*). Then $\{(\mathcal{J}_{n_\alpha}) cl((\mathcal{J}_{n_\alpha}) int F_\alpha) | \alpha \in A\}$ is a collection of

(ω)closed sets having finite intersection property. So by (d), $\cap(\mathcal{J}_{n_\alpha})cl((\mathcal{J}_{n_\alpha})intF_\alpha) \neq \emptyset$ which implies $\cap F_\alpha \neq \emptyset$.

$(e) \Rightarrow (a)$: Suppose $\mathcal{U} = \{U_\alpha | \alpha \in A\}$ is an (ω)open cover of X. If possible, X is not nearly (ω)compact. Then for every finite subcollection $\mathcal{U}_0 = \{U_{\alpha_1}, U_{\alpha_2}, ..., U_{\alpha_m}\}$ of $\mathcal{U}$, $\cup_{k=1}^m (\mathcal{J}_{n_{\alpha_k}})int((\mathcal{J}_{n_{\alpha_k}})clU_{\alpha_k}) \neq X$. Thus $\cap_{k=1}^m \{X - (\mathcal{J}_{n_{\alpha_k}})int((\mathcal{J}_{n_{\alpha_k}})clU_{\alpha_k})\} \neq \emptyset$. But $X - (\mathcal{J}_{n_{\alpha_k}})int((\mathcal{J}_{n_{\alpha_k}})clU_{\alpha_k}) \subset (\mathcal{J}_{n_{\alpha_k}})cl((\mathcal{J}_{n_{\alpha_k}})int(X - U_{\alpha_k}))$. Therefore, $\cap_{k=1}^m \{(\mathcal{J}_{n_{\alpha_k}})cl((\mathcal{J}_{n_{\alpha_k}})int(X - U_{\alpha_k}))\} \neq \emptyset$. Thus $\{X - U_\alpha | \alpha \in A\}$ is an (ω)closed collection of subsets of X satisfying the property of (e). Hence $\cap_\alpha \{X - U_\alpha\} \neq \emptyset$ which implies $\cup_\alpha U_\alpha \neq X$ which contradicts the fact that $\{U_\alpha | \alpha \in A\}$ is an (ω)open cover of X. Hence X is nearly (ω)compact. □

Theorem 3.5 : A semi$-(\omega)$regular, nearly (ω)compact space is (ω)compact.

Proof : Let X be a semi$-(\omega)$regular nearly (ω)compact space, and let $\mathcal{U} = \{U_\alpha | \alpha \in A\}$ be an (ω)open cover of X. For each $x \in X$, there exists an $\alpha_x \in A$ such that $x \in U_{\alpha_x}$. Since X is semi$-(\omega)$regular, therefore there exists for some $n \in N$ a regularly $(\mathcal{J}_n)$open set G_x such that $x \in G_x \subset U_{\alpha_x}$. $\{G_x | x \in X\}$ is a regularly (ω)open cover of X and has therefore a finite subcover $\{G_{x_i} | i = 1, 2, ..., m\}$. Then $\{U_{\alpha_{x_i}} | i = 1, 2, ..., m\}$ is a finite subcover of $\mathcal{U}$. Hence X is (ω)compact. □

Theorem 3.6 : An almost (ω)regular, almost (ω)compact space is nearly (ω)compact.

Proof : Let X be an almost (ω)regular, almost (ω)compact space and $\mathcal{U} = \{U_\alpha | \alpha \in A\}$ be a regularly (ω)open cover of an X. For each $x \in X$, there exists an $\alpha_x \in A$ such that $x \in U_{\alpha_x}$. Since X is almost (ω)regular, there exists for some $n_x \in N$ a $(\mathcal{J}_{n_x})$open set V_x such that $x \in V_x \subset (\mathcal{J}_{n_x})clV_x \subset U_{\alpha_x}$. Then $\{V_x | x \in X\}$ forms an (ω)open cover of X. Now since X is an almost (ω)compact space, there exists an finite subfamily $\{V_{x_i} | i = 1, 2, ..., m\}$ of $\mathcal{V}$ such that $\cup_{i=1}^m (\mathcal{J}_{n_0})clV_{x_i} = X$, for $n_0 = \max\{n_{x_1}, n_{x_2}, ..., n_{x_m}\}$. Then clearly $\{U_{\alpha_{x_i}} | i = 1, 2, ...m\}$ is a finite subcover of $\mathcal{U}$. Therefore X is nearly (ω)compact. □

Lemma 3.7 : Each (ω)open cover $\mathcal{U} = \{U_\alpha | \alpha \in A\}$ of a regularly (ω)closed subset Y of a nearly (ω)compact space X has a finite subcollection $\mathcal{U}' = \{U_{\alpha_1}, U_{\alpha_2}, ..., U_{\alpha_m}\}$ such that $\cup_{i=1}^m (\mathcal{J}_{n_{\alpha_i}})int((\mathcal{J}_{n_{\alpha_i}})clU_{\alpha_i}) \supset Y$, where n_{α_i} is any natural number such that $U_{\alpha_i} \in \mathcal{J}_{n_{\alpha_i}}$.

Proof : Straightforward. □

Theorem 3.8 : An (ω)Hausdorff nearly (ω)compact space is almost (ω)regular.

Proof : Let X be an (ω)Hausdorff nearly (ω)compact space. Let y be any point of X and let A be a regularly (ω)open subset of X containing y. Then for $x \in X - A$, there exists an $n_x \in N$ such that for some $U_x, V_x \in \mathcal{J}_{n_x}$, we have $x \in U_x, y \in V_x$ and $U_x \cap V_x = \emptyset$. Then $\mathcal{U} = \{U_x | x \in X - A\}$ is an (ω)open cover of the regularly (ω)closed set $X - A$. Therefore by the Lemma 3.7, $\mathcal{U}$ has a finite subcover $\mathcal{U}' = \{U_{x_i} | i = 1, 2, ..., m\}$ such that $X - A \subset \cup_{i=1}^{m} (\mathcal{J}_{n_{x_i}}) int((\mathcal{J}_{n_{x_i}}) cl U_{x_i})$, where n_{x_i} is any natural number such that $U_{x_i} \in \mathcal{J}_{n_{x_i}}$. Let $G = \cup_{i=1}^{m} (\mathcal{J}_{n_{x_i}}) int((\mathcal{J}_{n_{x_i}}) cl U_{x_i})$ and $H = \cap_{i=1}^{m} V_{x_i}$. Then obviously G, H are $(\mathcal{J}_{n_0})$open, where $n_0 = \max\{n_{x_1}, n_{x_2}, ..., n_{x_m}\}$. Also $y \in H$ and $X - A \subset G \subset X - (\mathcal{J}_{n_0}) cl H$. So it follows that $(\mathcal{J}_{n_0}) cl H \subset A$. Thus $y \in H \subset (\mathcal{J}_{n_0}) cl H \subset A$. Thus X is almost (ω)regular. □

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