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FIXED POINT THEOREMS IN COMPLETE G-METRIC SPACE

N. SURENDER[1] AND B. KRISHNA REDDY[2]
[1,2] Department of Mathematics,
University College of Science,
Osmania University, Hyderabad, India
E-mail: [1] surender_sri2002@yahoo.co.in, [2] bkrbkr_07@yahoo.com

Abstract
In this paper, we prove some fixed point theorems in compLete G-Metric Space for self mapping satisfying various contractive conditions. We also discuss that these mapping are G continuous on such a fixed point.

1. Introduction

Some generalizations of the notion of a metric space have been proposed by some authors. Gahler [1, 2] coined the term of 2-metric spaces. This is extended to D-metric space by Dhage (1992) [3, 4]. Dhage proved many fixed point theorems in D-metric space. Recently, Mustafa and Sims [7] showed that most of the results concerning Dhage's G-metric spaces are invalid. Therefore, in 2006 they introduced a new notion of generalized metric space called G-metric space [5]. In fact, Mustafa et al. studied many fixed point results for a self mapping in G-metric spaces under certain conditions; see [5, 6, 7, 8 and 9].

Now, we give preliminaries and basic definitions which are used through-out the paper.

Key Words : *G-Metric Spaces, Fixed Point, G convergent.*

2. Definitions and Preliminaries

Definition 2.1 [5] : Let X be a non empty set, and Let $G : X \times X \times X \to [0, \infty)$ be a function satisfying the following axioms

$(G1)$ $G(x, y, z) = 0$ if $x = y = z$,

$(G2)$ $G(x, x, y) > 0$ for all $x, y \in X$, with $x \neq y$.

$(G3)$ $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$, with $y \neq z$.

$(G4)$ $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$, (symmetry in all three variables)

$(G5)$ $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$ (rectangular inequality)

Then the function G is called a generalized metric, or more specially a G-metric on X, and the pair (X, G) is called a G-metric space.

Example : Let (X, d) be a usual metric space. Then (X, G_s) and (X, G_m) are G-metric spaces, where

$$G_s(x, y, z) = d(x, y) + d(y, z) + d(x, z) \quad \text{for all} \quad x, y, z \in X$$

and

$$G_m(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\} \quad \text{for all} \quad x, y, z \in X.$$

Definition 2.2 [5] : Let (X, G) and (X', G') be G-metric spaces and let $f : (X, G) \to (X', G')$ be a function, then f is said to be G-continuous at a point $a \in X$ if given $\epsilon > 0$ there exist $\delta > 0$ such that $x, y \in X, G(a, x, y) < \delta$ implies that $G'(fa, fx, fy) < \epsilon$. A function f is G-continuous on X if and only if it is G-continuous at all $a \in X$.

Definition 2.3 [5] : Let (X, G) be a G–metric space, and let $\{x_n\}$ be a sequence of points of X, therefore; we say that $\{x_n\}$ is G-convergent to x if $\lim_{n,m\to\infty} G(x, x_n, x_m) = 0$; that is, for any $\epsilon > 0$, there exist $N \in N$ such that $G(x, x_n, x_m) < \epsilon$ for all $n, m \geq N$. We call x is the limit of the sequence $\{x_n\}$ and we write $x_n \to N$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = x$.

Proposition 2.4 [5] : Let (X, G) and (X', G') be G-metric spaces, then a function $f : X \to X$ is said to be G-continuous at a point $x \in X$ if and only if it is G-sequentially continuous, that is, whenever $\{x_n\}$ is G–convergent to x, $\{fx_n\}$ is G-convergent to $f(x)$.

Proposition 2.5 [5] : Let (X, G) be a G-metric space. Then the following statements are equivalent.

(a) $\{x_n\}$ is G-convergent to x.

(b) $G(x_n, x_n, x) \to 0$ as $n \to \infty$.

(c) $G(x_n, x, x) \to 0$ as $n \to \infty$.

(d) $G(x_n, x_m, x) \to 0$ as $n \to \infty$.

Proposition 2.6 [5] : Let (X, G) be a G-metric space. A sequence $\{x_n\}$ is called G-cauchy sequence if given $\epsilon > 0$, there is $N \in N$ such that $G(x_n, x_m, x_l) < \epsilon)$ for all $n, m, l \geq N$; that is if $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.

Proposition 2.7 [5] : In a G-metric space (X, G), the following two statements are equivalent.

(1) The sequence $\{x_n\}$ is G-cauchy.

(2) For every $\epsilon > 0$, there exist $N \in N$ such that $G(x_n, x_m, x_m) < \epsilon$ for all $n, m \geq N$.

Definition 2.8 [5] : A G-metric space (X, G) is said to be G-complete (or a compLete G-metric pace) if every G-cauchy sequence in (X, G) is G-convergent in (X, G).

Proposition 2.9 [5] : Let (X, G) be a G-metric space. Then the function $G(x, y, z)$ is jointly continuous in all three of its variables.

Definition 2.10 [5] : A G-metric space (X, G) is called a symmetric G-metric space if

$$G(x, y, y) = G(y, x, x) \quad \text{for all} \quad x, y \in X.$$

Proposition 2.11 [5] : Every G-metric space (X, G) defines a metric space (X, d_G) by

$$d_G(x, y) = G(x, y, y) + G(y, x, x) \quad \text{for all} \quad x, y \in X.$$

Note that, if $(X, G$ is a symmetric space G-metric space, the

$$d_G(x, y) = 2G(x, y, y) \quad \text{for all} \quad x, y \in X.$$

However, if (X, G) is not asymmetric space, then it holds by the G–metric properties that

$$\frac{3}{2}G(x, y, y) \leq d_G(x, y) \leq 3G(x, y, y) \quad \text{for all} \quad x, y \in X.$$

In general, these inequalities cannot be improved.

Proposition 2.12 [5] : A G-metric space (X, G) is G-complete if and only if (X, d_G) is a complete metric space.

Proposition 2.13 [5] : Let (X, G) be a G-metric space. Then for any $, y, z, a \in X$, it follows that

(1) If $G(x,y,z)=0$ then $x=y=z$

(2) $G(x,y,z) \leq G(x,x,y)+G(x,x,z)$.

(3) $G(x,y,y) \leq 2G(y,x,x)$.

(4) $G(x,y,z) \leq G(x,a,z)+G(a,y,z)$.

(5) $G(x,y,z) \leq \frac{2}{3}\{G(x,a,a)+G(y,a,a)+G(z,a,a)\}$.

3. Main Result

Theorem 3.1 : Let (X,G) be a complete G-metric space and $T:X \to X$ be a mapping which satisfies the following condition for all $x,y,z \in X$,

$$\begin{aligned} G(Tx,Ty,Tz) &\leq k\max\{G(x,Tx,Tx)+G(y,Ty,Ty)+G(z,Tz,Tz), \\ &\quad G(x,Tx,Tx)+G(y,Tx,Tx)+G(z,Tx,Tx), \\ &\quad G(x,Ty,Ty)+G(z,Ty,Ty), G(x,Tz,Tz)+G(y,Tz,Tz)\}. \end{aligned} \quad (3$$

where $0 \leq k < \frac{1}{3}$, then T has a unique fixed point (say u), and T is G-continuous at u.

Proof : Suppose that T satisfies the condition (3.1). Let $x_0 \in X$ be an ordinary point, and define the sequence $\{x_n\}$ by $x_n = T^n x_0$, that is

$$\begin{aligned} x_1 &= T^1x_0 = Tx_0, \\ x_2 &= T^2x_0 = T(Tx_0) = Tx_1, \\ x_3 &= T^3x_0 = T(T^2x_0) = Tx_2, \\ &\vdots \\ x_n &= Tx_{n-1}, x_{n+1} = Tx_n. \end{aligned}$$

$$G(x_n,x_{n+1},x_{n+1}) = G(Tx_{n-1},Tx_n,Tx_n),$$

then by (3.1), we have,

$$\begin{aligned} G(x_n,x_{n+1},x_{n+1}) &\leq k\max\{G(x_{n-1},Tx_{n-1},Tx_{n-1})+G(x_n,Tx_n,Tx_n) \\ &\quad +G(x_n,Tx_n,Tx_n), G(x_{n-1},Tx_{n-1},Tx_{n-1}) \\ &\quad +G(x_n,Tx_{n-1},Tx_{n-1})+G(x_n,Tx_{n-1},Tx_{n-1}), \\ &\quad G(x_{n-1},Tx_n,Tx_n)+G(x_n,Tx_n,Tx_n), G(x_{n-1},Tx_n,Tx_n)+G(x_n,Tx_n,Tx_n)\}. \end{aligned}$$

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) \leq\ & k \max\{G(x_{n-1}, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1}) + G(x_n, x_{n+1}, x_{n+1}), \\ & G(x_{n-1}, x_n, x_n) + G(x_n, x_n, x_n) + G(x_n, x_n, x_n), \\ & G(x_{n-1}, x_{n+1}, x_{n+1}) + G(x_n, x_{n+1}, x_{n+1}), \\ & G(x_{n-1}, x_{n+1}, x_{n+1}) + G(x_n, x_{n+1}, x_{n+1})\}. \end{aligned}$$

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) \leq\ & k \max\{G(x_{n-1}, x_n, x_n) + 2G(x_n, x_{n+1}, x_{n+1}), G(x_{n-1}, x_n, x_n), \\ & G(x_{n-1}, x_{n+1}, x_{n+1}) + G(x_n, x_{n+1}, x_{n+1})\}. \end{aligned}$$

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) \leq\ & k \max\{G(x_{n-1}, x_n, x_n) + 2G(x_n, x_{n+1}, x_{n+1}), \\ & G(x_{n-1}, x_{n+1}, x_{n+1}) + G(x_n, x_{n+1}, x_{n+1})\}. \end{aligned}$$

By (G5) in Definition 2.1, we have

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) \leq\ & k \max\{G(x_{n-1}, x_n, x_n) + 2G(x_n, x_{n+1}, x_{n+1}), \\ & G(x_{n-1}, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1}) + G(x_n, x_{n+1}, x_{n+1})\}. \end{aligned}$$

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) \leq\ & k \max\{G(x_{n-1}, x_n, x_n) + 2G(x_n, x_{n+1}, x_{n+1}), \\ & G(x_{n-1}, x_n, x_n) + 2G(x_n, x_{n+1}, x_{n+1})\}. \end{aligned}$$

$$G(x_n, x_{n+1}, x_{n+1}) \leq k\{G(x_{n-1}, x_n, x_n) + 2G(x_n, x_{n+1}, x_{n+1})\}.$$

$$G(x_n, x_{n+1}, x_{n+1}) \leq kG(x_{n-1}, x_n, x_n) + 2kG(x_n, x_{n+1}, x_{n+1}),$$

$$(1-2k)G(x_n, x_{n+1}, x_{n+1}) \leq kG(x_{n-1}, x_n, x_n),$$

$$G(x_n, x_{n+1}, x_{n+1}) \leq \frac{k}{1-2k} G(x_{n-1}, x_n, x_n),$$

$$G(x_n, x_{n+1}, x_{n+1}) \leq qG(x_{n-1}, x_n, x_n). \tag{3.2}$$

where $q = \frac{k}{1-2k} < 1$, since $0 \leq k < \frac{1}{3}$.
Repeated application of inequality (3.2), we obtain

$$G(x_n, x_{n+1}, x_{n+1}) \leq q^n G(x_0, x_1, x_1). \tag{3.3}$$

Then, for all $m, n \in N, m > n$, we have by repeated use of rectangular inequality $(G5)$,

$$G(x_n, x_m, x_m) \leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + \cdots + G(x_{m-1}, x_m, x_m).$$

By (3), we get,

$$G(x_n, x_m, x_m) \leq q^n G(x_0, x_1, x_1) + q^{n+1} G(x_0, x_1, x_1) + \cdots + q^{m-1} G(x_0, x_1, x_1)$$

$$G(x_n, x_m, x_m) \leq (q^n + q^{n+1} + \cdots + q^{m-1}) G(x_0, x_1, x_1).$$

$$G(x_n, x_m, x_m) \leq q^n (1 + q + q^2 + \cdots) G(x_0, x_1, x_1)$$

$$G(x_n, x_m, x_m) \leq \frac{q^n}{1-q} G(x_0, x_1, x_1). \tag{3.4}$$

Then $G(x_n, x_m, x_m) \to 0$ as $n, m \to \infty$.
For $n, m, l \in N$, by rectangular inequality of G-metric space implies that

$$G(x_n, x_m, x_l) \leq G(x_n, x_m, x_m) + G(x_m, x_m, x_l).$$

Taking limit as $n, m, l \to \infty$, we get $G(x_n, x_m, x_l) \to 0$.
So $\{x_n\}$ is G-cauchy sequence. By completeness of (X, G), there exist $u \in X$ such that $\{x_n\}$ is G-converges to u.
Suppose that $Tu \neq u$, $G(x_n, Tu, Tu) = G(Tx_{n-1}, Tu, Tu)$.
Then by (3.1), we have,

$$\begin{aligned} G(x_n, Tu, Tu) \leq\ & k \max\{G(x_{n-1}, Tx_{n-1}, Tx_{n-1}) + G(u, Tu, Tu) + G(u, Tu, Tu), \\ & G(x_{n-1}, Tx_{n-1}, Tx_{n-1}) + G(u, Tx_{n-1}, Tx_{n-1}) + G(u, Tx_{n-1}, Tx_{n-1}), \\ & G(x_{n-1}, Tu, Tu) + G(u, Tu, Tu), G(x_{n-1}, Tu, Tu) + G(u, Tu, Tu)\}. \end{aligned}$$

$$\begin{aligned} G(x_n, Tu, Tu) \leq\ & k \max\{G(x_{n-1}, x_n, x_n) + 2G(u, Tu, Tu), \\ & G(x_{n-1}, x_n, x_n) + 2G(u, x_n, x_n), \\ & G(x_{n-1}, Tu, Tu) + G(u, Tu, Tu)\}. \end{aligned} \tag{3.5}$$

Taking the limit as $n \to \infty$ in (3.5), and using the fact that G is continuous in its variables, we get

$$\begin{aligned} G(u, Tu, Tu) \leq\ & k \max\{G(u, u, u) + 2G(u, Tu, Tu), G(u, u, u) + 2G(u, u, u), \\ & G(u, Tu, Tu) + G(u, Tu, Tu)\}. \end{aligned}$$

$$G(u, Tu, Tu) \leq k \max\{2G(u, Tu, Tu), 2G(u, Tu, Tu)\}.$$

This implies that

$$G(u,Tu,Tu) \leq 2kG(u,Tu,Tu). \tag{3.6}$$

The inequality (3.6) is contradiction since $2k < 1$.
This implies that $Tu = u$.
Therefore u is a fixed point of T.
To prove the uniqueness of the fixed point, suppose that $v \neq u$ such that $Tv = v$, then,

$$G(u,v,v) = G(Tu,Tv,Tv),$$

then by (3.1), we have,

$$\begin{aligned} G(u,v,v) \leq\ & k\max\{G(u,Tu,Tu)+G(v,Tv,Tv)+G(v,Tv,Tv),\\ & G(u,Tu,Tu)+G(v,Tu,Tu)+G(v,Tu,Tu), G(u,Tv,Tv)+G(v,Tv,Tv),\\ & G(u,Tv,Tv)+G(v,Tv,Tv)\}. \end{aligned}$$

$$\begin{aligned} G(u,v,v) \leq\ & k\max\{G(u,u,u)+G(v,v,v)+G(v,v,v),\\ & G(u,,u,u)+G(v,u,u)+G(v,u,u),\\ & G(u,v,v)+G(v,v,v), G(u,v,v)+G(v,v,v)\}. \end{aligned}$$

$$\begin{aligned} G(u,v,v) &\leq k\max\{0, 2G(v,u,u), G(u,v,v), G(u,v,v)\}.\\ G(u,v,v) &\leq k\max\{2G(v,u,u), G(u,v,v)\},\\ G(u,v,v) &\leq k\max\{2G(v,u,u), 2G(v,u,u)\}\\ &\qquad (\text{Since } \ G(x,y,y) \leq 2G(y,x,x)). \end{aligned}$$

$$G(u,v,v) \leq 2kG(v,u,u). \tag{3.7}$$

By repeated use of same argument in right side of (3.7), we obtain

$$G(u,v,v) \leq (2k)(2k)G(u,v,v)$$

$$G(u,v,v) \leq 4k^2G(u,v,v) \tag{3.8}$$

which is a contradiction since $0 \leq k < \frac{1}{3} \Rightarrow 0 \leq k^2 < \frac{1}{9} \Rightarrow 0 \leq 4k^2 < \frac{4}{9} < 1 \Rightarrow 4k^2 < 1$.
Therefore u is a unique fixed point of T.

Now, we show that, T is G-continuous at u. Let $\{y_n\}$ be a sequence in X, by completeness of X, the sequence $\{y_n\}$ converges to u in X. That is

$$\lim_{n\to\infty} y_n = u, \tag{3.9}$$

then by (3.1), we have,

$$\begin{aligned} G(Tu, Ty_n, Ty_n) \quad \leq \quad & k \max\{G(u, Tu, Tu) + G(y_n, Ty_n, Ty_n) + G(y_n, Ty_n, Ty_n), \\ & G(u, Tu, Tu) + G(y_n, Tu, Tu) + G(y_n, Tu, Tu), \\ & G(u, Ty_n, Ty_n) + G(y_n, Ty_n, Ty_n), \\ & G(u, Ty_n, Ty_n){+}G(y_n, Ty_n, Ty_n)\}. \end{aligned}$$

$$\begin{aligned} G(u, Ty_n, Ty_n) \quad \leq \quad & k \max\{G(u, u, u) + G(y_n, Ty_n, Ty_n) + G(y_n, Ty_n, Ty_n), \\ & G(u, u, u) + G(y_n, u, u) + G(y_n, u, .u), G(u, Ty_n, Ty_n) + G(y_n, Ty_n, Ty_n)\}. \end{aligned}$$

$$G(u, Ty_n, Ty_n) \leq k \max\{2G(y_n, Ty_n, Ty_n), 2G(y_n, u, u), G(u, Ty_n, Ty_n){+}G(y_n, Ty_n, Ty_n)\}.$$

By $(G5)$ of Definition 2.1, we have,

$$\begin{aligned} G(u, Ty_n, Ty_n) \quad \leq \quad & k \max\{2(G(y_n, u, u) + G(u, Ty_n, Ty_n)), 2G(y_n, u, u), \\ & G(u, Ty_n, Ty_n) + (G(y_n, u, u) + G(u, Ty_n, Ty_n)\}. \end{aligned}$$

$$\begin{aligned} G(u, Ty_n, Ty_n) \quad \leq \quad & k \max\{2G(y_n, u, u) + 2G(u, Ty_n, Ty_n), 2G(y_n, u, u), \\ & G(y_n, u, u) + 2G(u, Ty_n, Ty_n)\}. \end{aligned}$$

$$G(u, Ty_n, Ty_n) \leq k\{2G(y_n, u, u) + 2G(u, Ty_n, Ty_n)\}.$$

$$G(u, Ty_n, Ty_n) \leq 2kG(y_n, u, u) + 2kG(u, Ty_n, Ty_n),$$

$$(1 - 2k)G(u, Ty_n, Ty_n) \leq 2kG(y_n, u, u),$$

$$G(u, Ty_n, Ty_n) \leq \frac{2k}{1 - 2k} G(y_n, u, u). \tag{3.10}$$

Taking the limit as $n \to \infty$ in (3.10), we obtain that

$$G(u, Ty_n, Ty_n) \to 0 \;\text{ as }\; n \to \infty.$$

Therefore $Ty_n \to Tu = u$ as $n \to \infty$.

This implies that T is G-continuous at u.

Corollary 3.2 : Let (X, G) be a complete G-metric space and let $T : X \to X$ be a self mapping which satisfies the following condition for some $m \in N$ and for all $x, y, z \in X$,

$$\begin{aligned} G(T^m x, T^m y, T^m z) \leq\ & k \max\{G(x, T^m x, T^m x) + G(y, T^m y, T^m y) \\ & +G(z, T^m z, T^m z), G(x, T^m x, T^m x) + G(y, T^m x, T^m x) \\ & +G(z, T^m x, T^m x), G(x, T^m y, T^m y) + G(z, T^m y, T^m y), \\ & G(x, T^m z, T^m z) + G(y, T^m z, T^m z)\}. \qquad (3.11) \end{aligned}$$

where $0 \leq k < \frac{1}{3}$, then T has a unique fixed point (say u), and T is G-continuous at u.

Proof : Given that $T : X \to X$ is self mapping, then for all $m \in N, T^m : X \to X$. Therefore (X, G) be a complete G-metric space and $T^m : X \to X$ be a mapping which satisfies the given condition (3.11), then by Theorem 3.1, T^m has a unique fixed point (say u), and T^m is G-continuous.

Now we shall prove that u is a unique fixed point T^m.

Consider $Tu = T(T^m u) = T^{m+1}u = T^m(Tu)$.

Therefore $T^m(Tu) = Tu$. Tu is a fixed point of T^m.

Since T^m has a unique fixed point u, $Tu = u$.

Therefore u is a unique fixed point of T^m.

Theorem 3.3 : Let (X, G) be a complete G-metric space and $T : X \to X$ be a mapping which satisfies the following condition for all $x, y, z \in X$,

$$\begin{aligned} G(Tx, Ty, Ty) \leq\ & k \max\{G(x, Tx, Tx) + 2G(y, Ty, Ty), G(x, Tx, Tx) + 2G(y, Tx, Tx), \\ & G(x, Ty, Ty) + G(y, Ty, Ty)\}. \qquad (3.12) \end{aligned}$$

where $0 \leq k < \frac{1}{3}$, then T has a unique fixed point (say u), and T is G-continuous at u.

Proof : Setting $z = y$ in condition (3.1), then it reduced to condition (3.12), and the proof follows the theorem (3.1).

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