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# ENERGY EQUALITY FOR SOLUTIONS OF THREE DIMENSIONAL INCOMPRESSIBLE MAGNETOHYDRODYNAMIC EQUATIONS

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**Abstract**

In this paper, we prove that, like in the case of Navier-Stokes equations, energy equality remains valid for solution of three dimensional incompressible magnetohydrodynamic equations with solution belonging to an appropriate function space.

## 1. Introduction

Magnetohydrodynamics (MHD) is the study of flows of fluids which are electrically conducting and move in a magnetic field. The simplest example of an electrically conducting fluid is a liquid metal like mercury or liquid sodium. MHD treats, in particular, conducting fluids either in liquid form or gaseous form. The equations describing the motion of a viscous incompressible conducting fluid moving in a magnetic field are derived by coupling Navier-Stokes equations with Maxwell equations together with the expression for the Lorentz force. The domain $\Omega$ in which the fluid is moving is either a

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bounded subset of $R^3$ or the whole space $R^3$. In this paper we restrict our considerations to a bounded domain $\Omega$.

During past four or five decades, there has been an extensive study of qualitative properties such as existence, uniqueness, regularity and stability of solutions of the MHD equations. This is evident from the work of Duvaut and Lions [1], Sanchez Palencia E. [2], Sermange and Temam [3] and other researchers working in the field. The methods from nonlinear functional analysis such as Galerkin approximation, fixed point theorems, monotone and coercive operators, semi group theory etc have been applied to establish many a qualitative properties for compressible as well as incompressible MHD flows. The function spaces used are either Holder spaces or Sobolev spaces which are the appropriate function spaces for using these methods and the theory of elliptic operators. In proving the existence of solutions of MHD equations, energy inequality plays a significant role. The energy inequality is given as:

$$|(u,B)(t)|_2^2 + 2\nu\int_{t_0}^{t} |(\nabla u, \nabla B)(s)|_2^2 ds \leq |(u,B)(t_0)|_2^2 + \int_{t_0}^{t} (f(s), u(s))ds,$$

for all $0 \leq t_0 \leq t < T$.

We explain the notations below.

The natural question which then arises is that under what conditions on the initial and boundary data, this energy inequality becomes an equality. For three dimensional incompressible Navier-Stokes equations, such an equality has been derived under different conditions on $s$ and $q$ where a solution $u \in L^s(0,T;L^q(\Omega))$. The first paper in the series was published by J. L. Lions in 1960 and the last one is by Leslie and Shvydkoy in 2016. We refer the reader to the references [4] to [13] for results under various conditions on $s$ and $q$. In the present paper, following some of these works, we prove that energy equality holds for solution of three dimensional incompressible MHD equations when solution belongs to an appropriate function space.

Thus, in Section 2, we formulate the problem and describe appropriate function spaces that will be used in the proof. We also state our main theorem.

In Section 3, we prove a preliminary lemma, and then give the proof of the theorem. We make some comments on the main result as our concluding remarks. The paper ends with the list of references.

## 2. Formulation of Problem and Statement of Main Theorem

We consider the viscous incompressible three-dimensional magnetohydrodynamic (MHD) equations:

$$\left.\begin{aligned} &\partial_t u - \nu\Delta u + u\cdot\nabla u - B\cdot\nabla B + \nabla p = f \\ &\partial_t B - \lambda\Delta B + u\cdot\nabla B - B\cdot\nabla u = 0 \\ &\nabla\cdot u = 0 \ \text{ and } \ \nabla\cdot B = 0 \\ &u(x,t) = 0,\ B(x,t) = 0, \quad x\in\partial\Omega \\ &(u,B)|_{t=0} = (u_0, B_0) \end{aligned}\right\} \tag{1}$$

where $\Omega\subset R^3$ is a smooth bounded domain, $u=(x,t)$ is the velocity field, $B=B(x,t)$ is the magnetic field, $\nu>0$ is the kinematic coefficient of viscosity, $\lambda>0$ is the coefficient of magnetic diffusivity, $p=p(x,t)$ is the pressure, $f=f(x,t)$ is an external force. We assume $f\in L^1([0,T);L^2(\Omega))$.

When initial data $(u_0,B_0)\in L^2(\Omega)\times L^2(\Omega)$ is divergence free and $T>0$, it is well-known that there exists a weak solution $(u,B)$ to the system (1) in the class

$$\mathcal{LH} = L^\infty_{loc}([0,T);L^2(\Omega)\times L^2(\Omega))\cap L^2_{loc}([0,T);H^1(\Omega)\times H^1(\Omega)) \tag{2}$$

satisfying the following equations simultaneously:

$$\int_0^T\{-(u,\partial_t\varphi)+\nu(\nabla u,\nabla\varphi)+(u.\nabla u,\varphi)-(B.\nabla B,\varphi)\}dt = (u_0,\varphi(0))+\int_0^T(f,\varphi)dt \tag{3}$$

$$\int_0^T\{-(B,\partial_t\psi)+\lambda(\nabla B,\nabla\psi)+(u\cdot\nabla B,\psi)-(B\cdot\nabla u,\psi)\}dt = (B_0,\psi(0)) \tag{4}$$

where, $\varphi,\psi\in C_0^\infty([0,T)\times\Omega)$ are the test functions such that $\nabla\cdot\varphi=0$ and $\nabla\cdot\psi=0$. See, for example, ref. [3].

Now, equations (3) and (4) in their stronger forms can be written as,

$$\begin{aligned} &(u(t),\varphi(t)) + \textstyle\int_0^t\{-(u,\partial_s\varphi)+\nu(\nabla u,\nabla\varphi)+(u\cdot\nabla u,\varphi)-(B.\nabla B,\varphi)\}ds \\ &= (u_0,\varphi(0)) + \textstyle\int_0^t(f,\varphi)ds \end{aligned} \tag{5}$$

and

$$(B(t),\psi(t))+\int_0^t\{-(B,\partial_s\psi)+\lambda(\nabla B,\nabla\psi)+(u.\nabla B,\psi)-(B.\nabla u,\psi)\}ds = (B_0,\psi(0)) \tag{6}$$

for all $t \in [0,T)$ and $\varphi, \psi$ as before.

To proceed further, we recall some standard definitions:

Let us define

$$H = \{u \in L^2(\Omega) : \nabla \cdot u = 0, u \cdot n|_{\partial\Omega} = 0\}. \tag{7}$$

Let $\mathbf{p} : L^2(\Omega) \to H$ be the $L^2$-orthogonal projection.

Let the Stokes operator $A$ be defined as

$$Au = -\mathbf{p}\Delta u. \tag{8}$$

We denote $V^s = \mathcal{D}(A^{s/2}), s > 0$, the domain of the fractional power of $A$ and let

$$V = \{u \in H^1(\Omega) : \nabla \cdot u = 0, u|_{\partial\Omega} = 0\}. \tag{9}$$

The $L^2(\Omega)$ inner product is denoted by $(\cdot,\cdot)$ and the corresponding $L^2(\Omega)$ norm by $|\cdot|$. We endow $V$ with the norm $\|u\| = |\nabla u|$. Here, $H^1(\Omega)$ is the standard Sobolev space.

We now state our main theorem:

**Theorem** : Every weakly continuous weak solution $(u(t), B(t))$ of (1) satisfying conditions (5) and (6) simultaneously on $[0,T)$ with $(u,B) \in \mathcal{LH} \cap L^3([0,T); \mathcal{D}(A^{5/12}) \times \mathcal{D}(A^{5/12}))$ satisfies the energy equality,

$$|(u,B)(t)|_2^2 + 2\nu \int_{t_0}^{t} |(\nabla u, \nabla B)(s)|_2^2 ds = |(u,B)(t_0)|_2^2 + \int_{t_0}^{t} (f(s), u(s))ds,$$

for all $0 \le t_0 \le t < T$.

Here, for simplicity, we have chosen $\lambda = \nu$.

Now, we introduce some more notations which will be useful to prove the main theorem. We define an orthonormal basis of eigenvectors $\{w_n\}$ in $H$ and a sequence of positive eigenvalues $\{\lambda_n\}$, such that

$$Aw_n = \lambda_n w_n, \quad w_n \in D(A) \tag{10}$$

and

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \cdots, \qquad \lim_{n\to\infty} \lambda_n = \infty. \tag{11}$$

Then we can write,

$$u = \sum_{n=1}^{\infty} (u, w_n) w_n, \quad \text{for any} \quad u \in H. \tag{12}$$

We denote $u_n = (u, w_n)$ and define the operator $A^s$ by,

$$A^s u = \sum_{n=1}^{\infty} \lambda_n^s u_n w_n, \quad \text{for} \quad s > 0 \tag{13}$$

and the space

$$V^s = \left\{ u \in H : u = \sum_{n=1}^{\infty} u_n w_n, \|u\|_s^2 = \sum_{n=1}^{\infty} \lambda_n^s |u_n|^2 < \infty \right\}. \tag{14}$$

So, $V^s = \mathcal{D}(A^{s/2})$. We denote $V^1$ by $V$.
Now, let us denote $\mathcal{B}(u,v) = \mathbf{p}(u \cdot \nabla v) \in V'$, for $u, v \in V$ where, $V'$ is the dual of $V$. Hence, we rewrite (1) as the following differential equations in $V'$ :

$$\partial_t u + \nu A u + \mathcal{B}(u,u) - \mathcal{B}(B,B) = g \tag{15}$$

$$\partial_t B + \lambda A B + \mathcal{B}(u,B) - \mathcal{B}(B,u) = 0 \tag{16}$$

where $g = \mathbf{p} f$.
We denote the trilinear form $b(u,v,w) = \langle \mathcal{B}(u,v), w \rangle$, which has the following properties:

$$b(u,v,w) = -b(u,w,v), \quad u, v, w \in V$$

and $b(u,v,v) = 0$, for all $u, v \in V$.
Let us define

$$P_k u = \sum_{n:\lambda_n \le k} u_n w_n, \quad u \in H. \tag{17$a$}$$

Let $u \in V^\beta$ and we denote $u^1 = P_k u, u^h = u - u^1$. Thus, as $k \to \infty, u^h \to 0$ in the norm.
Similarly, we define

$$P_k B = \sum_{n:\lambda_n \le k} B_n w_n, \quad B \in H \tag{17$b$}$$

where $B_n = (B, w_n)$.
Let $B \in V^\beta$ and we denote $B^1 = P_k B, B^h = B - B^1$. Thus, as $k \to \infty, B^h \to 0$ in the norm.
We use the following inequalities:

$$\left.\begin{aligned} \|u^1\|_\beta &\le k^{\beta-\alpha} \|u^1\|_\alpha \\ \|u^h\|_\alpha &\le k^{\alpha-\beta} \|u^h\|_\beta \end{aligned}\right\} \tag{18}$$

and

$$\left.\begin{array}{rcl} \|B^1\|_\beta & \leq & k^{\beta-\alpha}\|B^1\|_\alpha \\ \|B^h\|_\alpha & \leq & k^{\alpha-\beta}\|B^h\|_\beta \end{array}\right\} \tag{19}$$

whenever $\beta > \alpha$.

## 3. Proof of the Main Theorem

We need the following lemma to prove the main Theorem:

**Lemma** : Let $(u(t), B(t))$ be a weakly continuous weak solution of (1) on $[0, T)$. Then,

$$|(u,B)(t)|^2 + 2\nu \int_{t_0}^t \|(u,B)\|^2 ds = |(u,B)(t_0)|^2 + 2\int_{t_0}^t (g,u)ds$$

$$2\lim_{k\to\infty} \int_{t_0}^t [b(u,u^1,u) + b(B,B,u^1) + b(u,B^1,B) + b(B,u,B^1)]ds$$

for all $0 \leq t_0 \leq t < T$. (20)

**Proof** : We can see that $(u^1, B^1) \in C([0,T); V\times V)$ and $(\partial_t u^1, \partial_t B^1) \in L^2([0,T); V\times V)$. If we put $u^1$ as a test function in (5), then we get

$$\begin{aligned} &|u^1(t)|^2 - |u^1(t_0)|^2 + 2\nu \int_{t_0}^t \|u^1\|^2 ds - 2\int_{t_0}^t (g,u^1)ds \\ &= 2\int_{t_0}^t b(u,u^1,u)ds + 2\int_{t_0}^t b(B,B,u^1)ds. \end{aligned} \tag{21}$$

Similarly, if we put $B^1$ as a test function in (6), we get

$$|B^1(t)|^2 - |B^1(t_0)|^2 + 2\lambda \int_{t_0}^t \|B^1\|^2 ds = 2\int_{t_0}^t b(u,B^1,B)ds + 2\int_{t_0}^t b(B,u,B^1)ds. \tag{22}$$

Here, the limit of the right hand side of both the equations exists as $k \to \infty$.

By taking $\lambda = \nu$, and adding (21) and (22) lemma gets proved.

To prove the theorem, in view of the lemma, it is sufficient to show that,

$$\lim_{k\to\infty} \int_0^T [b(u,u^1,u) + b(B,B,u^1) + b(u,B^1,B) + b(B,u,B^1)]ds = 0. \tag{23}$$

For this, we write $b(u,u^1,u) = b(u^h + u^1, u^1, u^h + u^1)$. Hence,

$$b(u,u^1,u) = b(u^h,u^1,u^h) + b(u^1,u^1,u^h) + b(u^h,u^1,u^1) + b(u^1,u^1,u^1). \tag{24}$$

Similarly we get,

$$b(B,B,u^1) = b(B^h,B^h,u^1) + b(B^1,B^h,u^1) + b(B^h,B^1,u^1) + b(B^1,B^1,u^1). \tag{25}$$

$$b(u, B^1, B) = b(u^h, B^1, B^1) + b(u^h, B^1, B^h) + b(u^1, B^1, B^h) + b(u^1, B^1, B^1). \tag{26}$$

$$b(B, u, B^1) = b(B^h, u^h, B^1) + b(B^1, u^h, B^1) + b(B^h, u^1, B^1) + b(B^1, u^1, B^1). \tag{27}$$

In (24) - (27), using properties of $b(u, v, w)$, we observe that

$$b(u^h, u^1, u^1) = 0, b(u^1, u^1, u^1) = 0, b(u^h, B^1, B^1) = 0 \text{ and } b(u^1, B^1, B^1) = 0.$$

Also, $b(B^1, B^1, u^1) = -b(B^1, u^1, B^1)$.

Hence, we estimate the remaining terms by using the standard estimate

$$|b(u, v, w)| \leq \|u\|_{s_1} \|v\|_{s_2+1} \|w\|_{s_3} \tag{28}$$

where, $s_1 + s_2 + s_3 \geq 3/2$.

Now, let us set $s_1 = s_2 = s_3 = 1/2$.

To estimate the first term of (24), we use (28) and get

$$|b(u^h, u^1, u^h)| \leq \|u^h\|_{1/2}^2 \|u^1\|_{3/2}.$$

By (18) we get,

$$\|u^h\|_{1/2} \leq k^{-1/3} \|u^h\|_{5/6}$$

$$\|u^1\|_{3/2} \leq k^{2/3} \|u^1\|_{5/6}$$

Thus

$$|b(u^h, u^1, u^h)| \leq \|u^h\|_{5/6}^2 \|u^1\|_{5/6}. \tag{29}$$

Similarly,

$$|b(B^h, B^h, u^1)| \leq \|B^h\|_{5/6}^2 \|u^1\|_{5/6} \tag{30}$$

$$|b(u^h, B^1, B^h)| \leq \|u^h\|_{5/6} \|B^1\|_{5/6} \|B^h\|_{5/6} \tag{31}$$

$$|b(B^h, u^h, B^1)| \leq \|B^h\|_{5/6} \|u^h\|_{5/6} \|B^1\|_{5/6} \tag{32}$$

Now, we set $s_1 = 5/6, s_2 = 0, s_3 = 2/3$.

Again using (18), (19) and (28) we obtain

$$|b(u^1, u^1, u^h)| \leq \|u^1\|_{5/6}^2 \|u^h\|_{5/6} \tag{33}$$

$$|b(B^1, B^h, u^1)| \leq \|B^1\|_{5/6} \|B^h\|_{5/6} \|u^1\|_{5/6} \tag{34}$$

$$|b(B^h, B^1, u^1)| \leq \|B^h\|_{5/6} \|B^1\|_{5/6} \|u^1\|_{5/6} \tag{35}$$

$$|b(u^1, B^1, B^h)| \leq \|u^1\|_{5/6}\|B^1\|_{5/6}\|B^h\|_{5/6} \tag{36}$$

$$|b(B^1, u^h, B^1)| \leq \|B^1\|^2_{5/6}\|u^h\|_{5/6} \tag{37}$$

$$|b(B^h, u^1, B^1)| \leq \|B^h\|_{5/6}\|u^1\|_{5/6}\|B^1\|_{5/6} \tag{38}$$

In addition, we can write,

$$|b(u^h, u^1, u^h)| \leq \|u\|^3_{5/6} \tag{39}$$

$$|b(B^h, B^h, u^1)| \leq \|B\|^2_{5/6}\|u\|_{5/6} \tag{40}$$

$$|b(u^h, B^1, B^h)| \leq \|u\|_{5/6}\|B\|^2_{5/6} \tag{41}$$

$$|b(B^h, u^h, B^1)| \leq \|B\|^2_{5/6}\|u\|_{5/6} \tag{42}$$

$$|b(u^1, u^1, u^h)| \leq \|u\|^3_{5/6} \tag{43}$$

$$|b(B^1, B^h, u^1)| \leq \|u\|_{5/6}\|B\|^2_{5/6} \tag{44}$$

$$|b(B^h, B^1, u^1)| \leq \|u\|_{5/6}\|B\|^2_{5/6} \tag{45}$$

$$|b(u^1, B^1, B^h)| \leq \|u\|_{5/6}\|B\|^2_{5/6} \tag{46}$$

$$|b(B^1, u^h, B^1)| \leq \|u\|_{5/6}\|B\|^2_{5/6} \tag{47}$$

$$|b(B^h, u^1, B^1)| \leq \|u\|_{5/6}\|B\|^2_{5/6}. \tag{48}$$

Now, combining the estimates from (29) to (38), we get

$$\begin{aligned}
&\lim_{k\to\infty}\int_0^T [|b(u, u^1, u)| + |b(B, B, u^1)| + |b(u, B^1, B)| + |b(B, u, B^1)|]ds \\
&\leq \lim_{k\to\infty}\int_0^T [\|u^h\|^2_{5/6}\|u^1\|_{5/6} + \|B^h\|^2_{5/6}\|u^1\|_{5/6} \\
&+2\|B^1\|_{5/6}\|B^h\|_{5/6}\|u^h\|_{5/6} + \|u^1\|^2_{5/6}\|u^h\|_{5/6} + \|B^1\|^2_{5/6}\|u^h\|_{5/6} \\
&+4\|B^1\|_{5/6}\|B^h\|_{5/6}\|u^1\|_{5/6}]ds.
\end{aligned}$$

Using the standard product norm and Young's inequality we get

$$\begin{aligned}
&\lim_{k\to\infty}\int_0^T [|b(u, u^1, u)| + |b(B, B, u^1)| + |b(u, B^1, B)| + |b(B, u, B^1)|]ds \\
&\leq \lim_{k\to\infty}\int_0^T [2\|(u^1, B^1)\|_{5/6}\|(u^h, B^h)\|^2_{5/6} + 3\|(u^h, B^h)\|_{5/6}\|(u^1, B^1)\|^2_{5/6}]ds.
\end{aligned}$$

By virtue of the inequalities (39) - (48), we can apply the Dominated Convergence Theorem and conclude that the limit on RHS exists.

Moreover, since $\|(u^h, B^h)\|_{5/6} \to 0$ in the limit, $RHS \to 0$ in $L^1([0,T))$ as $k \to \infty$. Thus, by taking $\lambda = \nu$, we get

$$|(u,B)(t)|^2 + 2\nu \int_{t_0}^{t} \|(u,B)\|^2 ds = |(u,B)(t_0)|^2 + 2\int_{t_0}^{t} (g,u) ds,$$

which is the desired energy equality.

## 4. Concluding Remarks

In this paper, we have proved that for a smooth bounded domain $\Omega \subset R^3$, if a weak solution $(u,B) \in \mathcal{LH} \cap L^3([0,T); \mathcal{D}(A^{5/12})) \times \mathcal{D}(A^{5/12})$ then the solution $(u,B)$ satisfies the energy equality. In a recent paper [13], Leslie and Shvydkoy found new $L^q L^p$ conditions on solutions of the $3D$ incompressible Navier-Stokes equations which guarantee that the energy equality survives a one-point singularity. It would be interesting to see if this result can be extended to the present MHD case.

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