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SOME APPLICATION OF DIFFERENTIAL OPERATOR TO ANALYTIC AND MULTIVALENT FUNCTION WITH NEGATIVE COEFFICIENTS

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Abstract

By means of certain differential operator we introduce and investigate two subclasses $\Gamma^*(p, \lambda, \phi, \delta, \alpha)$ and $K\Gamma^*(p, \lambda, \phi, \delta, \alpha)$ of p-valently analytic functions. The various results obtained here for each of these classes .we have attempted to obtain coefficient estimate, distortion theorem, radius of starlikeness, convexity and closure theorem for the classes $\Gamma^*(p, \lambda, \phi, \delta, \alpha)$ and $K\Gamma^*(p, \lambda, \phi, \delta, \alpha)$.

1. Introduction

This paper devoted to study of multivalent functions and its various properties. With the help of differential operator we have introduced and investigated two subclasses $\Gamma^*(p, \lambda, \phi, \delta, \alpha)$ and $K\Gamma^*(p, \lambda, \phi, \delta, \alpha)$ of p-valent and analytic functions. We have discussed properties of these two classes.

Key Words and Phrases : *Multivalent function, Coefficient estimate, Distortion theorem, Radius of star likeness, Differential operator.*

Let $A(p)$ be the class which contains the functions $h(w)$ where

$$h(w) = w^p + \sum_{x=p+1}^{\infty} a_x w^x \tag{1.1}$$

which areregular and p-valent in E for $p \in \mathbb{N}$.

Definition 1 : $h(w) \in A(p)$ is in the subclass $\Gamma(p, \lambda, \phi, \delta, \alpha)$ if

$$\left| \frac{\delta w(D_w^{q+1}(\Omega_p(r,p)h(w))) + \lambda w^2(D_w^{q+2}(\Omega_p(r,p)h(w)))}{(1-\lambda)(D_w^{q}(\Omega_p(r,p)h(w))) + w(D_w^{q+1}(\Omega_p(r,p)h(w)))} - (\delta - \phi) \right| < \alpha$$

$w \in E, q \in \mathbb{N} \cup \{0\}, 0 < \alpha \leq 1, \phi \in \mathbb{R}, \phi < 1, \quad p > q, \gamma, \delta \leq 1.$

Further more a function $h(w) \in A(p)$ is said to be in the subclass $K\Gamma(p, \lambda, \phi, \delta, \alpha)$ if $wh'(w) \in \Gamma(p, \lambda, \phi, \delta, \alpha)$.

Let $T(p)$ be the subclass of $A(p)$ containing the functions of the form

$$h(w) = w^p - \sum_{x=p+1}^{\infty} a_x w^x, \quad a_x \geq 0 \tag{1.2}$$

$\Gamma^*(p, \lambda, \phi, \delta, \alpha)$ and $K\Gamma^*(p, \lambda, \phi, \delta, \alpha)$ are the classes which are the intersection of the classes $\Gamma(p, \lambda, \phi, \delta, \alpha)$ and $K\Gamma(p, \lambda, \phi, \delta, \alpha)$ respectively with $T(p)$.

Define $\Omega_p(a, p)$ as an operator on $h(w)$ as follows

$$\Omega_p(r,p)h(w) = w^p - \sum_{x=p+1}^{\infty} \left(\frac{x+\gamma}{p+\gamma} \right)^r a_x w^x.$$

The operator $\Omega_p(r, p)$ is the operator which closely comparable to the Salagean derivative operator.

$D_w^q h(w)$ is the order differential operator for $h(w) \in A(p)$ defined in (1.1)

$$D_w^q(\Omega_p(r,p)h(w)) = \frac{p!}{(p-q)!} w^{p-q} - \sum_{x=p+1}^{\infty} \frac{x^{"\prime}}{(x-q)!} \left(\frac{x+\gamma}{p+\gamma} \right)^r a_x w^{x-q}, \quad p > q.$$

2. Coefficient Estimates

Theorem 1 : A function $h(w)$ given by (1.2) is in $\Gamma^*(p, \lambda, \phi, \delta, \alpha)$ if and only if

$$\sum_{x=p+1}^{\infty} \frac{1}{\Psi(x)} \, a_x \leq 1$$

where

$$\Psi(x) = \frac{\frac{p!}{(p-q)!}[\alpha(1-\lambda+p-q) - \lambda(p-q)(p-q-1) - \phi(p-q) + (\delta-\phi)(1-\lambda)]}{\frac{x!}{(x-q)!}\left(\frac{x+\gamma}{p+\gamma}\right)^r [\lambda(x-q)(x-q-1) + \phi(x-q) - (\delta-\phi)(1-\lambda) + \alpha(1-\lambda+x-q)]}.$$

Proof : Suppose $h(w)$ is in $\Gamma^*(p, \lambda, \phi, \delta, \alpha)$ Thus

$$\left| \frac{\delta w(D_w^{q+1}(\Omega_p(r,p)h(w))) + \lambda w^2(D_w^{q+2}(\Omega_p(r,p)h(w)))}{(1-\lambda)(D_w^{q}(\Omega_p(r,p)h(w))) + w(D_w^{q+1}(\Omega_p(r,p)h(w)))} - (\delta-\phi) \right| < \alpha \tag{2.2}$$

we have

$$\left| \frac{\frac{p!}{(p-q)!}[\lambda(p-q)(p-q-1) + \phi(p-q) - (\delta-\phi)(1-\lambda)]w^{p-q} - \sum_{x=p+1}^{\infty} \frac{x!}{x-q!}\left(\frac{x+\gamma}{p+\gamma}\right)^r [\lambda(x-q)(x-q-1) + \phi(x-q) - (\delta-\phi)(1-\lambda)]a_x w^{x-q}}{\frac{p!}{(p-q)!}[1-\lambda+p-q]w^{p-q} - \sum_{x=p+1}^{\infty} \frac{x!}{(x-q)!}\left(\frac{x+\gamma}{p+\gamma}\right)^r [1-\lambda+x-q]a_x w^{x-q}} \right| \leq \alpha \tag{2.3}$$

We know that $|Re(w)| \leq |w|$, so choosing values of w on real axis.
In (2.3) $w \to 1$ allowing through real axis, it follows that

$$\sum_{x=p+1}^{\infty} \frac{x!}{(x-q)!}\left(\frac{x+\gamma}{p+\gamma}\right)^r [\lambda(x-q)(x-q-1) + \phi(x-q)$$
$$-(\delta-\phi)(1-\lambda) + \alpha(1-\lambda+x-q)]a_x$$
$$\leq \frac{p!}{(p-q)!}[\alpha(1-\lambda+p-q) - \lambda(p-q)(p-q-1) - \phi(p-q) + (\delta-\phi)(1-\lambda)].$$

Define

$$\Psi(x) = \frac{\frac{p!}{(p-q)!}[\alpha(1-\lambda+p-q) - \lambda(p-q)(p-q-1) - \phi(p-q) + (\delta-\phi)(1-\lambda)]}{\frac{x!}{(x-q)!}\left(\frac{x+\gamma}{p+\gamma}\right)^r [\lambda(x-q)(x-q-1) + \phi(x-q) - (\delta-\phi)(1-\lambda) + \alpha(1-\lambda+x-q)]}.$$

We get

$$\sum_{x=p+1}^{\infty} \frac{1}{\Psi(x)} a_x \leq 1.$$

Corollary 1 : If the function $((w)$ is in $\Gamma^*(p, \lambda, \phi, \delta, \alpha)$ then

$$a_x \leq \Psi(x)$$

for $x = 1 + p, 2 + p, \cdots$.

Theorem : The function $h(w)$ given by (1.2) is in $K\Gamma^*(p, \lambda, \phi, \delta, \alpha)$ if and only if

$$\sum_{x=p+1}^{\infty} \frac{x}{\Psi(x)} a_x \leq p.$$

Proof : Suppose the function $h(w)$ is in $K\Gamma^*(p, \lambda, \phi, \delta, \alpha)$. Therefore $wf'(w) \in \Gamma^*(p, \lambda, \phi, \delta, \alpha)$. Let $g(w) = wh'(w)$

$$g(w) = pw^p - \sum_{x=p+1}^{\infty} x a_x w^x.$$

Thus

$$\left| \frac{\delta w(D_w^{q+1}(\Omega_p(r,p)g(w))) + \lambda w^2(D_w^{q+2}(\Omega_p(r,p)g(w)))}{(1-\lambda)(D_w^q(\Omega_p(r,p)g(w))) + w(D_w^{q+1}(\Omega_p(r,p)g(w)))} - (\delta - \phi) \right| < \alpha \tag{2.5}$$

Consider

$$\begin{aligned}
&\delta w(D_w^{q+1}(\Omega_p(r,p)g(w))) + \lambda w^2(D_w^{q+2}(\Omega_p(r,p)g(w))) - (\delta - \phi) \\
&[(1-\lambda)(D_w^q(\Omega_p(r,p)g(w))) + w(D_w^{q+1}(\Omega_p(r,p)g(w)))] \\
&= \frac{p!p}{(p-q)!}[\lambda(p-q)(p-q-1) + \phi(p-q) - (\delta-\phi)(1-\lambda)]w^{p-q} \\
&- \sum_{x=p+1}^{\infty} \frac{x!x}{(k-q)!} \left(\frac{x+\gamma}{p+\gamma}\right)^r [\lambda(x-q)(x-q-1) + \phi(x-q) - (\delta-\phi)(1-\lambda)]a_x w^{x-q}.
\end{aligned}$$

Now consider

$$\begin{aligned}
&(1 - \lambda(D_w^q(\Omega_p(r,p)g(w))) + w(D_w^{q+1}(\Omega_p(r,p)g(w))) \\
&= \frac{p!p}{(p-q)!}[1-\lambda+p-q]w^{p-q} - \sum_{x=p+1}^{\infty} \frac{x!x}{(x-q)!} \left(\frac{x+\gamma}{p+\gamma}\right)^r [1-\lambda+x-q]a_x w^{x-q}.
\end{aligned}$$

From (2.5) we have

$$\left| \frac{\frac{p!}{(p-q)!}[\lambda(p-q)(p-q-1) + \phi(p-q) - (\delta-\phi)(1-\lambda)]w^{p-q} - \sum_{x=p+1}^{\infty} \frac{x!}{k-q!} \left(\frac{x+\gamma}{p+\gamma}\right)^r [\lambda(x-q)(x-q-1) + \phi(x-q) - (\delta-\phi)(1-\lambda)]a_x w^{x-q}}{\frac{p!p}{(p-q)!}[1-\lambda+p-q]w^{p-q} - \sum_{x=p+1}^{\infty} \frac{x!x}{(x-q)!} \left(\frac{x+\gamma}{p+\gamma}\right)^r [1-\lambda+x-q]a_x w^{x-q}} \right| \leq \alpha \tag{2.6}$$

We know that $|Re(w)| \leq |w|$, allowing $w \to 1_-$ through real axis, we get

$$\sum_{x=p+1}^{\infty} \frac{x!x}{(x-q)!}\left(\frac{x+\gamma}{p+\gamma}\right)^r [\lambda(x-q)(x-q-1)+\phi(x-q)$$
$$-(\delta-\phi)(1-\lambda)+\alpha(1-\lambda+x-q)]a_x$$
$$\leq \frac{p!p}{(p-q)!}[\alpha(1-\lambda+p-q)-\lambda(p-q)(p-q-1)-\phi(p-q)+(\delta-\phi)(1-\lambda)].$$

Therefore we get

$$\sum_{x=p+1}^{\infty} \frac{x}{\Psi(x)} a_x \leq p.$$

We can prove the converse on the line of Theorem 1.

Corollary 2 : If the function $h(w)$ given by (1.2) is in $K\Gamma^*(p, \lambda, \phi, \delta, \alpha)$ then

$$a_x \leq \frac{p}{x}\Psi(x)$$

for $x = 1+p, w+p, \cdots$.

3. Growth, Distortion Theorem

Theorem 3 : If the function $h(w)$ given by (1.2) is in $\Gamma^*(p, \lambda, \phi, \delta, \alpha)$ then

$$|w|^p - \Psi(p+1)|w|^{1+p} \leq |h(w)| \leq |w|^p + \Psi(p+1)|w|^{1+p}.$$

Proof : The function $h(w)$ is in $\Gamma^*(p, \lambda, \phi, \delta, \alpha)$ if and only if

$$\sum_{x=p+1}^{\infty} \frac{x}{\Psi(x)} a_x \leq p.$$

$$|h(w)| \leq |w^p| + \left|\sum_{x=p+1}^{\infty} a_x w^x\right| \leq |w|^p + \Psi(p+1)|w|^{1+p}. \tag{3.1}$$

Similarly

$$|h(w)| \geq |w^p| - \left|\sum_{x=p+1}^{\infty} a_x w^x\right| \leq |w|^p - \Psi(p+1)|w|^{1+p}. \tag{3.2}$$

From (3.1) and (3.2) we have

$$|w|^p - \Psi(p+1)|w|^{1+p} \leq |h(w)| \leq |w|^p + \Psi(p+1)|w|^{1+p}.$$

Hence the result.

Theorem 4 : If $h(w)$ is in $K\Gamma^*(p,\lambda,\phi,\delta,\alpha)$ then

$$|w|^p - \left(\frac{p}{p+1}\right)\Psi(p+1)|w|^{1+p} \le |h(w)| \le |w|^p + \left(\frac{p}{p+1}\right)\Psi(p+1)|w|^{1+p}.$$

Proof : $h(w)$ given by (1.2) is in $K\Gamma^*(p,\lambda,\phi,\delta,\alpha)$ then if and only if

$$\sum_{x=p+1}^{\infty} \frac{x}{\Psi(x)} a_x \le p.$$

$$|h(w)| \le |w^p| + \left|\sum_{x=p+1}^{\infty} a_x w^x\right| \le |w|^p + \left(\frac{p}{p+1}\right)\Psi(p+1)|w|^{1+p}. \tag{3.3}$$

Similarly

$$|h(w)| \ge |w^p| - \left|\sum_{x=p+1}^{\infty} a_x w^x\right| \ge |w|^p - \left(\frac{p}{p+1}\right)\Psi(p+1)|w|^{1+p}. \tag{3.4}$$

From (3.3) and (3.4) we have

$$|w|^p - \left(\frac{p}{p+1}\right)\Psi(p+1)|w|^{1+p} \le |h(w)| \le |w|^p + \left(\frac{p}{p+1}\right)\Psi(p+1)|w|^{1+p}.$$

Hence the result.

Theorem 5 : If the function $h(w)$ given by (1.2) is in $\Gamma^*(p,\lambda,\phi,\delta,\alpha)$ then

$$p|w|^{p-1} + (p+1)\Psi(p+1)|w|^p \le |h'(w)| \le p|w|^{p-1} + (p+1)\Psi(p+1)|w|^p.$$

Proof : The function $h(w)$ is in $\Gamma^*(p,\lambda,\phi,\delta,\alpha)$ therefore

$$\sum_{x=p+1}^{\infty} \frac{1}{\Psi(x)} a_x \le 1.$$

$$h'(w) = pw^{p-1} - \sum_{x=p+1}^{\infty} x a_x w^{x-1}$$

$$|h'(w)| \le |pw^{p-1}| + \left|\sum_{x=p+1}^{\infty} x a_x w^{x-1}\right|$$

$$|h'(w)| \le p|w|^{p-1} + (p+1)\Psi(p+1)|w|^p. \tag{3.5}$$

Similarly

$$|h'(w)| \ge p|w|^{p-1} - (p+1)\Psi(p+1)|w|^p. \tag{3.6}$$

From (3.5) and (3.6) we have

$$p|w|^{p-1}+(p+1)\Psi(p+1)|w|^{p}\leq |h'(w)|\leq p|w|^{p-1}+(p+1)\Psi(p+1)|w|^{p}.$$

Hence the result.

Theorem 6 : If the function $h(w)$ given by (1.2) is in $K\Gamma^{*}(p,\lambda,\phi,\delta,\alpha)$ then

$$p|w|^{p-1}-\left(\frac{p}{p+1}\right)\Psi(p+1)|w|^{p}\leq |h'(w)|\leq p|w|^{p-1}+\left(\frac{p}{p+1}\right)\Psi(p+1)|w|^{p}.$$

Proof : The function $h(w)$ is in $K\Gamma^{*}(p,\lambda,\phi,\delta,\alpha)$ therefore

$$\sum_{x=p+1}^{\infty}\frac{x}{\Psi(x)}a_x\leq p.$$

$$h'(w)=pw^{p-1}-\sum_{x=p+1}^{\infty}xa_xw^{x-1}$$

$$|h'(w)|\leq |pw^{p-1}|+\left|\sum_{x=p+1}^{\infty}xa_xw^{x-1}\right|$$

$$|h'(w)|\leq p|w|^{p-1}+\left(\frac{p}{p+1}\right)\Psi(p+1)|w|^{p}. \tag{3.7}$$

Similarly

$$|h'(w)|\geq p|w|^{p-1}-\left(\frac{p}{p+1}\right)\Psi(p+1)|w|^{p}. \tag{3.8}$$

From (3.7) and (3.8) we have

$$p|w|^{p-1}\left(\frac{p}{p+1}\right)\Psi(p+1)|w|^{p}\leq |h'(w)|\leq p|w|^{p-1}+\left(\frac{p}{p+1}\right)\Psi(p+1)|w|^{p}.$$

Hence the result.

4. Radius of Convexity

Theorem 7 : Let $h(w)$ given by (1.2) is in $\Gamma^{*}(p,\lambda,\phi,\delta,\alpha)$. Then h is in $\mathcal{K}(\xi)$ in $|w|<R_1$, where

$$R_1=\inf_{x\geq 1+p}\left\{\left\{\frac{(p-\xi)}{x\Psi(x)}\right\}^{\frac{1}{x+1-p}}\right\}.$$

Proof : It is sufficient to show that

$$\left|\frac{h'(w)}{w^{p-1}}-p\right|\leq p-\xi \quad \text{for} \quad |w|<R_1.$$

We have

$$\left|\frac{h'(w)}{w^{p-1}}-p\right|=\left|-\sum_{x=p+1}^{\infty}xa_xw^{x-p+1}\right|\leq\sum_{x=p+1}^{\infty}x|a_x||w|^{x-p+1}.$$

Thus

$$\sum_{x=p+1}^{\infty}\frac{x}{(p-\xi)}|a_x||w|^{x-p+1}\leq 1. \tag{4.1}$$

Theorem 1 conforms that

$$\sum_{x=p+1}^{\infty}\frac{1}{\Psi(x)}|a_x|\leq 1. \tag{4.2}$$

Hence (4.1) will be true if

$$\frac{x}{(p-\xi)}|w|^{x-p+1}\leq\frac{1}{\Psi(x)}.$$

We obtain

$$|w|\leq\left\{\frac{(p-\xi)}{x\Psi(x)}\right\}^{\frac{1}{x+1-p}}$$

as required.

Theorem 8 : Let the function $h(w)$ given by (1.2) is in $\Gamma^*(p,\lambda,\phi,\delta,\alpha)$. Then h is in $\mathbf{S}^*(\xi)$ in $|w|<R_2$, where

$$R_2=\inf_{x\geq 1+p}\left\{\left\{\frac{(p-\xi)}{x-\xi}\right\}^{\frac{1}{\Psi(x)}}\right\}^{\frac{1}{x-p}}.$$

Proof : We must show that

$$\left|\frac{wh'(w)}{h(w)}-p\right|\leq p-\xi.$$

We have

$$\left|\frac{wh(w)}{h(w)}-p\right|=\frac{\sum\limits_{x=p+1}^{\infty}(x-p)|a_x||w|^{x-p}}{1-\sum\limits_{x=p+1}^{\infty}|a_x||w|^{x-p}}\leq p-\xi. \tag{4.3}$$

Hence (4.3) holds true if

$$\sum_{x=p+1}^{\infty}\frac{(x-\xi)}{(p-\xi)}|a_x||w|^{x-p}\leq 1. \tag{4.4}$$

Theorem 1 conforms that

$$\sum_{x=p+1}^{\infty}\frac{1}{\Psi(x)}|a_x|\leq 1. \tag{4.5}$$

Hence by using (4.4) and (4.5) will be true if

$$\frac{(x-\xi)}{(p-\xi)}|w|^{x-p} \leq \frac{1}{\Psi(x)}$$

$$|w|^{x-p} \leq \left(\frac{p-\xi}{x-\xi}\right)\frac{1}{\Psi(x)}$$

as required.

Theorem 9 : Let the function $h(w)$ given by (1.2) is in $\Gamma^*(p,\lambda,\phi,\delta,\alpha)$. Then h is in $C(\xi)$ in $|W| < R_3$, where

$$R_3 = \inf_{x\geq 1+p}\left\{\left\{\left(\frac{p(p-\xi)}{x(x-\xi)}\right)\right\}\frac{1}{\Psi(x)}\right\}^{\frac{1}{x-p}}.$$

Proof : We know that h is convex if and only if wh' is starlike.

We must show that

$$\left|\frac{wg'(w)}{g(w)} - p\right| \leq p-\xi$$

where $g(w) = wf'(w)$

$$\left|\frac{wg'(w)}{g(w)} - p\right| = \left|\frac{-\sum\limits_{x=p+1}^{\infty} x(x-p)a_x w^x}{pw^p - \sum\limits_{x=p+1}^{\infty} x a_x w^x}\right| \leq \frac{\sum\limits_{x=p+1}^{\infty} x(x-p)|a_x||w|^{x-p}}{p - \sum\limits_{x=p+1}^{\infty} x|a_x||w|^{x-p}} \leq p-\xi.$$

Therefore we have

$$\sum_{x=p+1}^{\infty}\frac{x(x-\xi)}{p(p-\xi)}|a_x||w|^{x-p} \leq 1. \tag{4.6}$$

Theorem 1 conforms that

$$\sum_{x=p+1}^{\infty}\frac{1}{\Psi(x)}|a_x| \leq 1. \tag{4.7}$$

Hence by using (4.6) and (4.7) will be true if

$$\frac{x(x-\xi)}{p(p-\xi)}|w|^{x-p} \leq \frac{1}{\Psi(x)}$$

$$|w| \leq \left\{\left(\frac{p(p-\xi)}{x(x-\xi)}\right)\frac{1}{\Psi(x)}\right\}^{\frac{1}{x-p}}$$

as required.

5. Closure Theorem

Theorem 10 : Let $h_1(w) = w^p$ and $h_z(w) = w^r - \Psi(x)w^x$ for $x \geq 1 + p$. Then $h(w) \in \Gamma^*(p, \lambda, \phi, \delta, \alpha)$ if and only if $h(w)$ can be expressed in the form $h(w) = \lambda_1 h_1(w) + \sum_{x=p+1}^{\infty} \lambda_x h_x(w)$ where $\lambda_x \geq 0$ and $\lambda_1 + \sum_{x=p+1}^{\infty} \lambda_x = 1$.

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